



FIG. 1: Experiment with the ‘figure-eight’ stirring protocol, showing an advected blob of dye (India ink) in sugar syrup. The coordinate system used here is also indicated, as is the distance $d(t)$ between the dark mixing pattern and the wall.

I. THE ROLE OF WALLS

Consider the experimental setup shown in Fig. 1. A dark blob of ink in a white fluid has been stretched and folded repeatedly by the periodic movement of a rod. The movement of the rod defines a stirring protocol with period T . The mixing pattern has a kidney shape, and it slowly grows and approaches the wall. The distance of closest approach is $d(t)$, where t is time.

Inside the central mixing region, we assume the action of the flow is that of a simple chaotic mixer. By this we mean that fluid elements are stretched, on average, at a given rate λ . Hence, after a time t a blob of initial size δ will have length $\delta e^{\lambda t}$. However, because of diffusion, its width will stabilise at an equilibrium between compression and diffusion [1–3] at a scale

$$\ell = \sqrt{\kappa/\lambda}, \quad (1)$$

where κ is the molecular diffusivity. The length ℓ is called the Batchelor length.

At every period, the pattern gets progressively closer to the wall. Assuming molecular diffusion can be neglected, because of area preservation some white fluid must have entered the central mixing region. It does so in the form of white strips, clearly visible as layers inside the pattern of Fig. 1. In fact, if we assume that the mixing pattern grows uniformly, we can write the width $\Delta(t)$ of a strip injected at period t/T as

$$\Delta(t) = d(t) - d(t+T) \simeq -T \dot{d}(t) \quad (2)$$

where we also assumed that $d(t)$ changes slowly in time, consistent with experimental observations. This is positive since d is a decreasing function of time.

Now, if a white strip is injected at time $\tau < t$, how long does it last before it is wiped out by diffusion? The answer is the solution to the equation

$$\Delta(\tau) e^{-\lambda(t-\tau)} = \ell. \quad (3)$$

We interpret this formula as follows: The strip initially had width $\Delta(\tau)$ when it was injected; it gets compressed by the flow in the central mixing region by a factor $e^{-\lambda(t-\tau)}$ depending on its age, $t - \tau$; and once it is compressed to the Batchelor length ℓ it quickly diffuses away. Thus, we can solve (3) to find the age the strip has when it gets wiped out by diffusion,

$$t - \tau = \lambda^{-1} \log(\Delta(\tau)/\ell). \quad (4)$$

Eventually, at time t_B , a newly-injected filament will have width equal to the Batchelor length. This occurs when

$$\Delta(t_B) = \ell, \quad (5)$$

which can be solved for t_B given a form for $\Delta(t)$. After this time it makes no sense to speak of newly-injected filaments as ‘white,’ since they are already dominated by diffusion at their birth. Hence, the description we present here is valid only for times earlier than t_B , but large enough that the edge of the mixing pattern has reached the vicinity of the wall.

In the experiment we measure the intensity of pixels in the central mixing region. We observe for $1 \ll t/T \lesssim t_B/T$ that the concentration variance is dominated by the amount of still-white strips in the central region [4, 5]. Because of area conservation, the total area of injected white material that is still visible at time t is proportional to

$$\mathcal{A}_w(t) = d(\tau(t)) - d(t). \quad (6)$$

where we use (4) to solve for $\tau(t)$, the injection time of the oldest strip that is still white at time t . Hence, our goal is to estimate $\mathcal{A}_w(t)$ for times $1 \ll t/T \lesssim t_B/T$, since \mathcal{A}_w is directly proportional to the concentration variance. To do this we need $\tau(t)$, which requires specifying $\Delta(t)$. We look at two possible forms.

A. Exponential approach to a free-slip wall

Consider first the case where $d(t) = d(0)e^{-\mu t}$, for some positive constant μ . We have $\Delta(t) = -Td\dot{t} = \mu Td(0)e^{-\mu t} = \Delta(0)e^{-\mu t}$. From (5), we have $t_B = \mu^{-1} \log(\Delta(0)/\ell)$, and from (4),

$$t - \tau = \frac{\mu}{\lambda - \mu} (t_B - t). \quad (7)$$

By assumption, $\tau < t < t_B$, so for consistency we require $\mu < \lambda$, i.e., the rate of approach toward the wall is slower than the natural decay rate of the chaotic mixer. The area of white material in the mixing region is then obtained from (6),

$$\mathcal{A}_w(t) = d(0)e^{-\mu t} \left(\exp\left(\frac{\mu^2}{\lambda - \mu}(t_B - t)\right) - 1 \right), \quad (8)$$

which in the wall-dominated regime ($\lambda/\mu \gg 1$) can be approximated by

$$\mathcal{A}_w(t) \sim d(0)\lambda^{-1}\mu^2(t_B - t)e^{-\mu t}, \quad t \lesssim t_B. \quad (9)$$

The decay rate of the ‘white’ area is completely dominated by the walls. The central mixing process is potentially more efficient ($\lambda > \mu$), but it is starved by the boundaries.

If $\mu > \lambda$, we have $t > t_B$ in (7), since newly injected strips reach the Batchelor length ℓ before strips that were injected previously. This violates our assumptions, and we conclude

that in that case the white strips can be neglected; the decay rate of the concentration variance is then given by the natural decay rate λ .

As an example of an exponential approach to the wall, consider the velocity field near a free-slip boundary,

$$u(x, y) = u_0(x) + O(y), \quad v(x, y) = -u'_0(x)y + O(y^2), \quad (10)$$

which satisfies the incompressibility constraint. (Since the dynamics near the wall are slow, we can use a steady flow here to model the time- T Poincaré map.) Along a separatrix at $x = x_s$, we have $u_0(x_s) = 0$ since the velocity field changes sign. The rate of approach along the separatrix is thus given by $\dot{d} = v(x_s, d) = -u'_0(x_s)d$, so that $\mu = u'_0(x_s)$. Hence, if $u'_0(x_s)$ is large enough, the rate of decay of concentration variance will not be limited by wall effects.

B. Algebraic approach to a no-slip wall

If the fluid at the wall is subject to no-slip boundary conditions, the Taylor expansion (10) is modified to become

$$u(x, y) = u_1(x)y + O(y^2), \quad v(x, y) = -\frac{1}{2}u'_1(x)y^2 + O(y^3). \quad (11)$$

The rate of approach along the separatrix at $x = x_s$ is given by $\dot{d} = v(x_s, d) = -\frac{1}{2}u'_1(x_s)d^2$, with asymptotic solution $d(t) \sim 2/(u'_1(x_s)t)$, for $d(0)u'_1(x_s)t \gg 1$. This is independent of the initial condition $d(0)$: asymptotically, a fluid particle forgets its initial position; this explains why material lines bunch up against each other faster than they approach the wall, as reflected by the front in the upper part of Fig. 1. The total area of remaining white strips at time t as given by (6) is proportional to

$$\mathcal{A}_w(t) = \frac{2}{u'_1(x_s)\tau} - \frac{2}{u'_1(x_s)t} = \frac{2}{u'_1(x_s)} \frac{t - \tau}{\tau t}. \quad (12)$$

The width of injected strips is $\Delta(t) = -T\dot{d} = 2T/(u'_1(x_s)t^2)$. Equation (4) cannot be solved exactly, but since $\tau(t)$ is algebraic the right-hand side of (4) is not large, implying that $t/\tau \simeq 1$ for large t . We can thus replace τ by t in (4) and the denominator of (12), and find

$$\mathcal{A}_w(t) \simeq \frac{2}{u'_1(x_s)} \frac{\log(\Delta(t)/\ell)}{\lambda t^2}, \quad 1/(d(0)u'_1(x_s)) \ll t \ll t_B. \quad (13)$$

Compare this to the exponential case, Eq. (9): the decay of concentration variance is now algebraic ($1/t^2$), with a logarithmic correction. The form (13) has been verified in experiments and using a simple map model [4, 5].

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