the mathematics of taffy pulling

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the standard 3-pronged taffy puller

Taffy is a type of candy.

Needs to be pulled: this aerates it and makes it lighter and chewier.

We can assign a growth: length multiplier per period.

[movie by M. D. Finn]

play movie

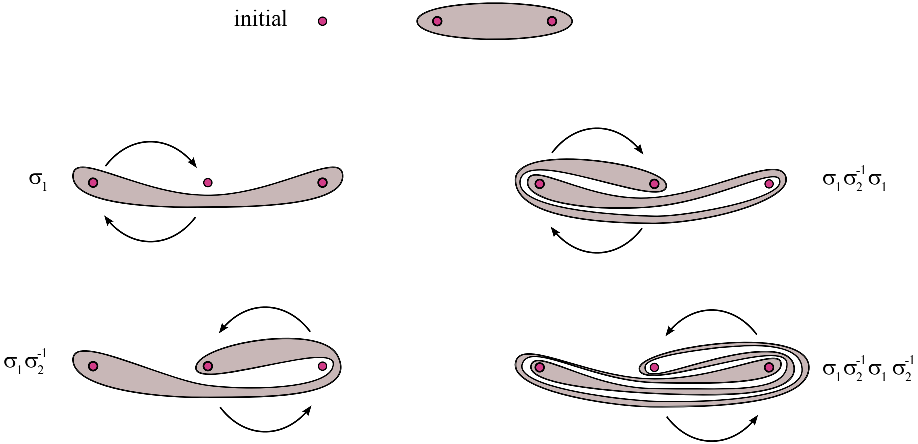
standard 4-pronged taffy puller

play movie

http://www.youtube.com/watch?v=Y7tlHDsquVM

[MacKay (2001); Halbert & Yorke (2014)]

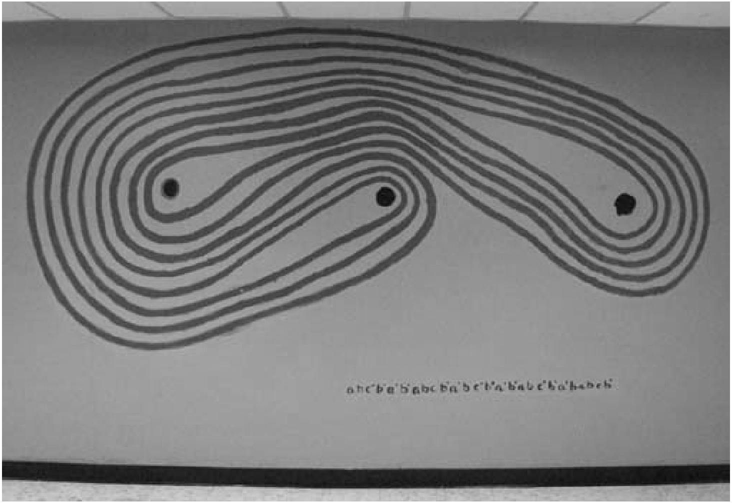
a simple taffy puller



[Remark for later: each prong moves in a ‘figure-eight’ orbit.]

the famous mural

This is the same action as in the famous mural painted at Berkeley by Thurston and Sullivan in the Fall of 1971:



linear maps on the torus

The simple taffy puller has a growth factor equal to

$$\phi^2 = \phi + 1 = 2.6180\ldots$$

where ϕ is the Golden Ratio.

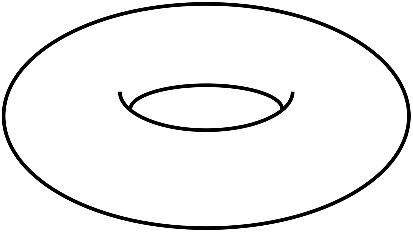
Such quadratic numbers also arise for linear maps on the torus T^2, such as Arnold's Cat Map:

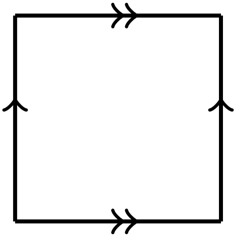
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1, \quad x, y \in [0,1]^2$$

The largest eigenvalue of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is ϕ^2. Coincidence?

What's the connection between taffy pullers and these maps?

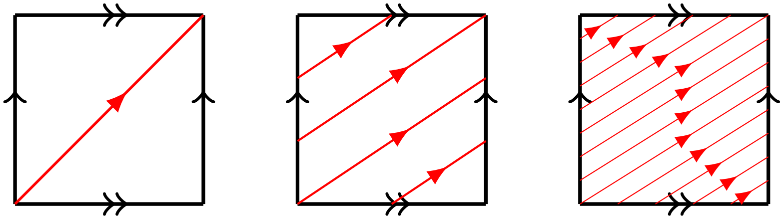
The 'standard model' for the torus is the biperiodic unit square:





action of map on the torus

The Cat Map stretches loops exponentially:



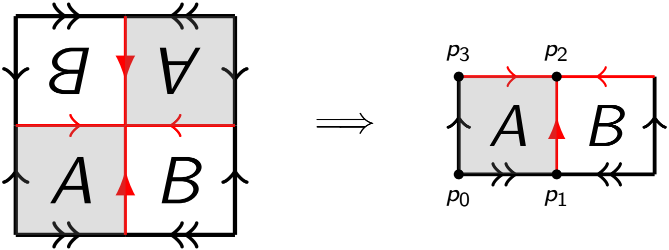
This loop will stand in for a piece of taffy.

hyperelliptic involution

Consider the linear map $\iota(x) = -x \bmod 1$. This map is called the hyperelliptic involution ($\iota^2 = \mathrm{id}$).

Construct the quotient space

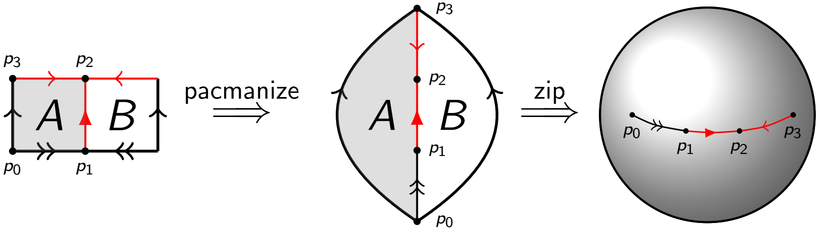
$$S = T^2/\iota$$



Claim: the surface S (right) is a sphere with four punctures!

sphere with four punctures

Here's how we see that S is a sphere:



The punctures $p_{1,2,3}$ are the prongs of our taffy puller.

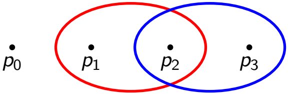
(The fixed puncture p_0 plays no role here, other than acting as a topological obstruction.)

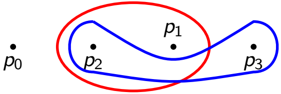
Linear maps commute with ι, so all linear torus maps 'descend' to taffy puller motions.

Any 3-pronged taffy puller motion can be represented as a product of

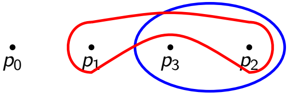
$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and their inverses, known as Dehn twists.





action of σ_1



action of σ_2

These can also be view as generators for the braid group for 3 strings.

the Silver Ratio

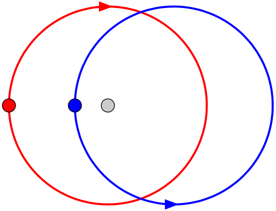
By decomposing taffy puller motions as a product of the σ_1 and σ_2 operations, we can find the growth factor for any 3-pronged taffy puller.

For instance, the simple taffy puller has a motion $\sigma_1\sigma_2^{-1}$ which we already saw gives a growth equal to the Cat Map's, ϕ^2.

The standard 3-pronged taffy puller has a motion $\sigma_1^2\sigma_2^{-2}$, which has matrix representation

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

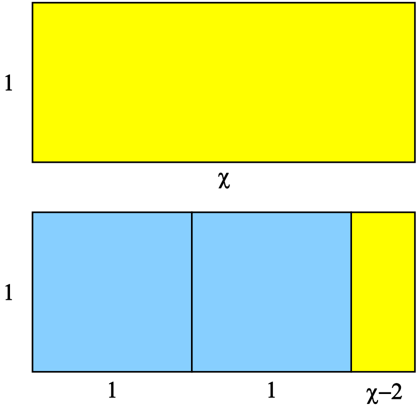
with growth $\chi^2 = (1+\sqrt{2})^2$, where χ is the Silver Ratio.



Surprisingly, the standard 4-pronged taffy puller has exactly the same growth factor.

the Silver Ratio (2)

A rectangle has the proportions of the Silver Ratio if, after taking out two squares, the remaining rectangle has the same proportions as the original.



$$\frac{\chi}{1} = \frac{1}{\chi - 2}$$

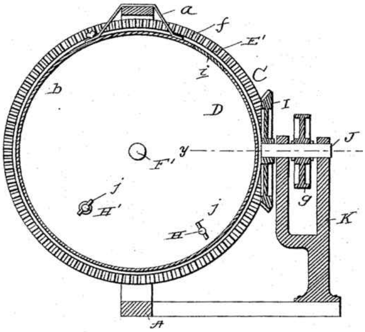
$$\chi = 1+\sqrt{2} = 2.4142\ldots$$

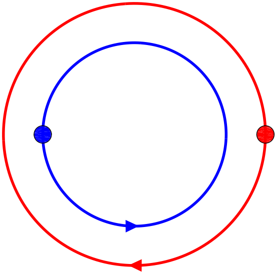
Both major taffy puller designs (3- and 4-pronged) have growth χ^2.

the history of taffy pullers

But who invented the well-known designs for taffy pullers?
Google patents is an awesome resource.

The very first: Firchau (1893)



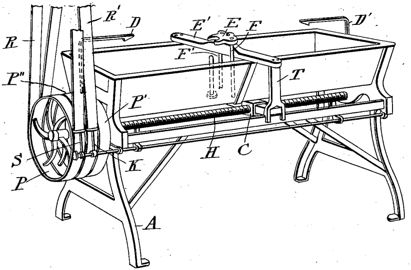


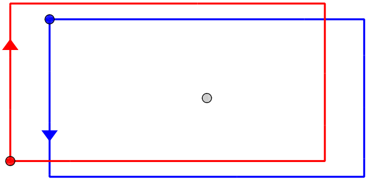
This is a terrible taffy puller. It was likely never built, but plays an important role in the looming. . .

taffy patent wars

the first true taffy puller

I think Herbert M. Dickinson (1906, but filed in 1901) deserves the title of inventor of the first taffy puller:

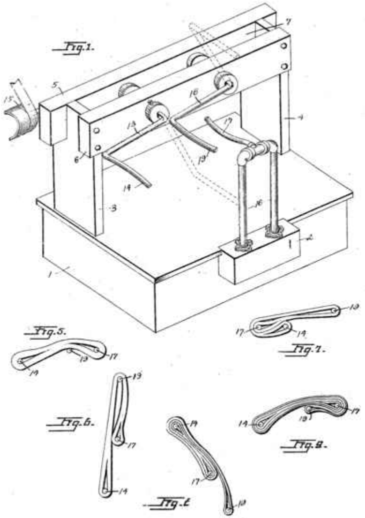


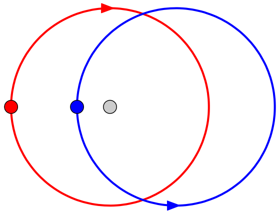


Awkward design: the moving prongs get 'tripped' each cycle. But it is topologically the same as the 3-pronged device still in use today.

There seem to be questions as to whether it ever worked, or if it really pulled taffy rather than mixing candy.

Robinson & Deiter (1908) greatly simplified this design to one still in use today.

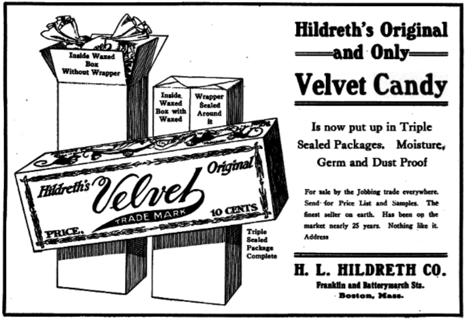




Herbert L. Hildreth

The uncontested taffy magnate of the early 20th century was Herbert L. Hildreth of Maine.

The Hotel Velvet in Old Orchard, Maine

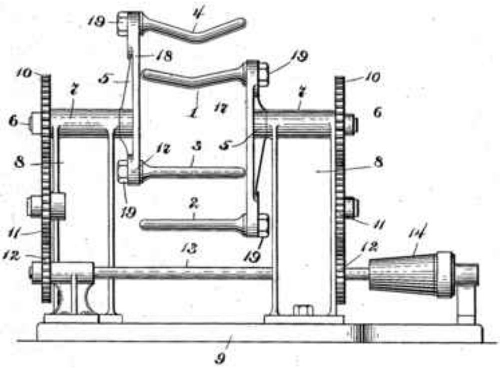


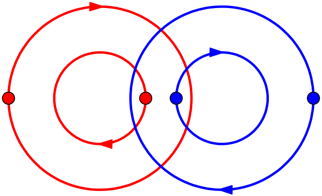
The Confectioners Gazette (1914)

His hotel was on the beach, and taffy was popular at such resorts. He sold it wholesale as well.

the 4-pronged design

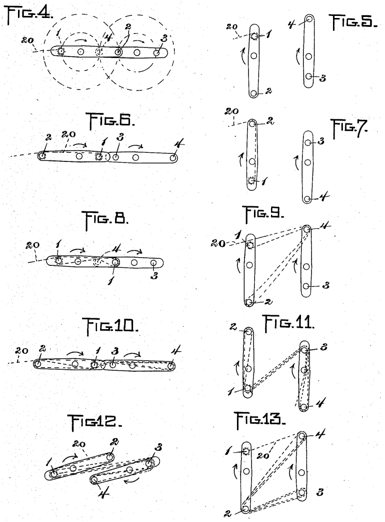
The first 4-pronged design is by Thibodeau (1903, filed 1901), an employee of Hildreth.

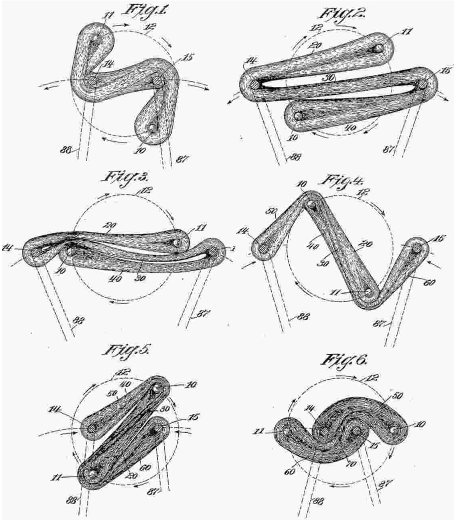




Hildreth was not pleased by this but bought the patent for $75,000 (about two million of today's dollars).

some patents have beautiful diagrams





Thibodeau (1903)

Richards (1905)

So many concurrent patents were filed that lawsuits ensued for more than a decade. Shockingly, the taffy patent wars went all the way to the US Supreme Court. The opinion of the Court was delivered by Chief Justice William Howard Taft (*Hildreth v. Mastoras*, 1921):

> *The machine shown in the Firchau patent [has two pins that] pass each other twice during each revolution [. . .] and move in concentric circles, but do not have the relative in-and-out motion or Figure 8 movement of the Dickinson machine. With only two hooks there could be no lapping of the candy, because there was no third pin to re-engage the candy while it was held between the other two pins. The movement of the two pins in concentric circles might stretch it somewhat and stir it, but it would not pull it in the sense of the art.*

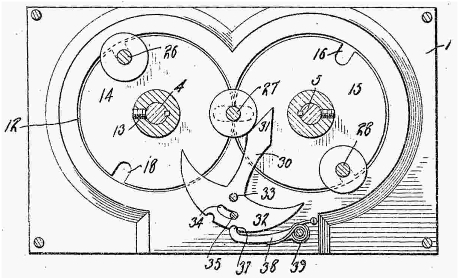
The Supreme Court opinion displays the fundamental insight that at least three prongs are required to produce some sort of rapid growth.

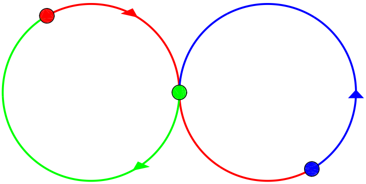
the quest for the Golden ratio

Is it possible to build a device that realizes the simplest taffy puller, with growth ϕ^2?

The problem is that each prong moves in a Figure-eight! This is hard to do mechanically.

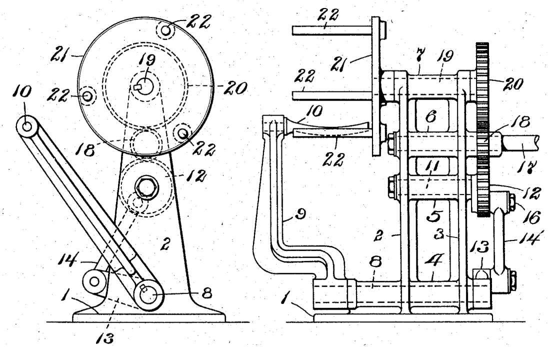
After some digging, found the patent of Nitz (1918):

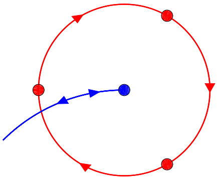




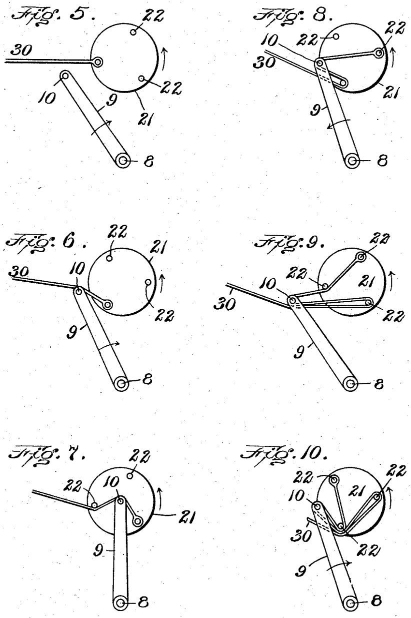
the quest for the Golden ratio (2)

There is actually an earlier 4-pronged design by Thibodeau (1904) which has (Golden ratio)^2 growth:





Since it uses four prongs to get a quadratic growth, the map must involve a branched cover of the torus by a theorem of Franks & Rykken (1999). (The same happens for the 4- vs 3-pronged ‘standard’ taffy pullers.)

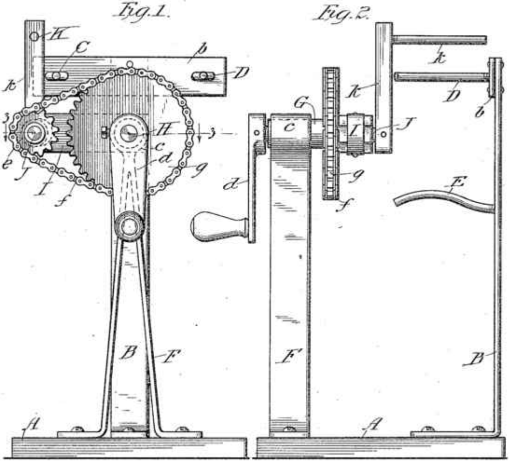


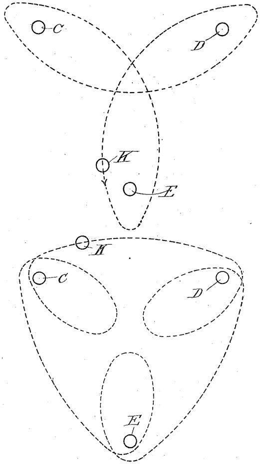
Thibodeau (1904) once again gives very nice diagrams for the action of his taffy puller.

(He has at least 5 patents for taffy pullers.)

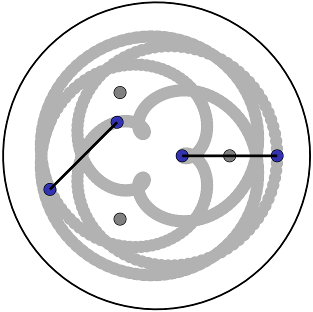
planetary designs

A few designs are based on 'planetary' gears, such as McCarthy (1916):

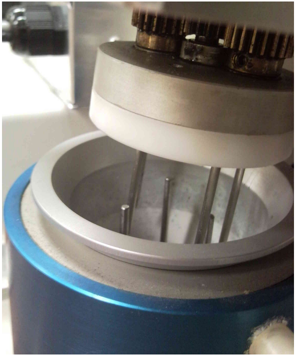


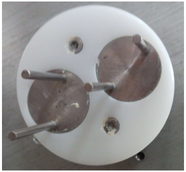


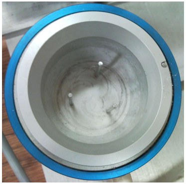
A modern planetary design is the mixograph, a device for measuring the properties of dough:



play movie

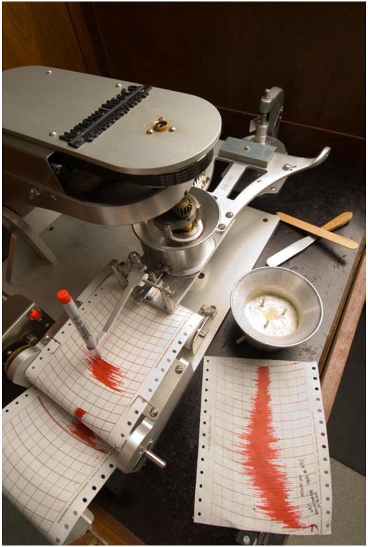






[Department of Food Science, University of Wisconsin. Photos by J-LT.]

the mixograph (2)

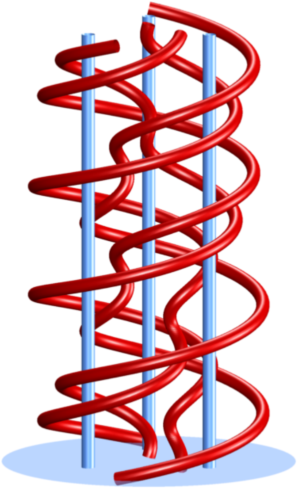


The mixograph measures the resistance of the dough to the pin motion.

This is graphed to determine properties of the dough, such as water absorption and 'peak time.'

[Wheat and Flour Testing Methods: A Guide to Understanding Wheat and Flour Quality]

Encode the topological information as a sequence of generators of the Artin braid group B_n.



Equivalent to the 7-braid

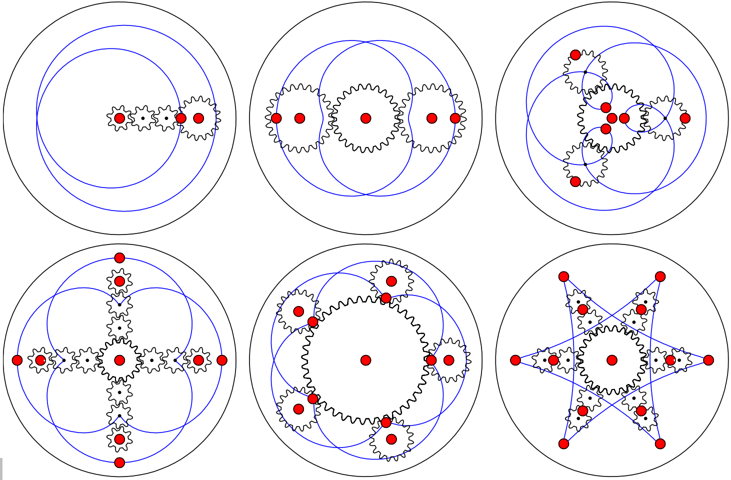
$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

We feed this braid to the Bestvina–Handel algorithm, which determines the Thurston type of the braid (pseudo-Anosov) and finds the growth as the largest root of

$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1$$

$\boxed{\simeq 4.186}$

As part of an optimization procedure, we (Finn & Thiffeault, 2011) designed a family of planetary mixers with silver ratio expansion:

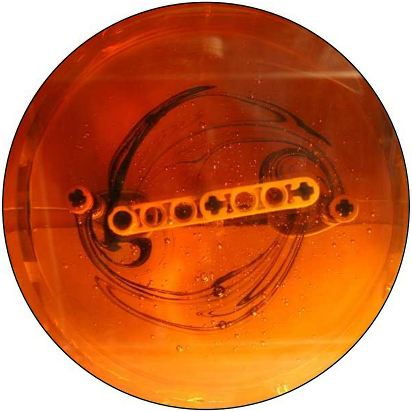


play movie

silver mixers: building one out of Legos

play movie

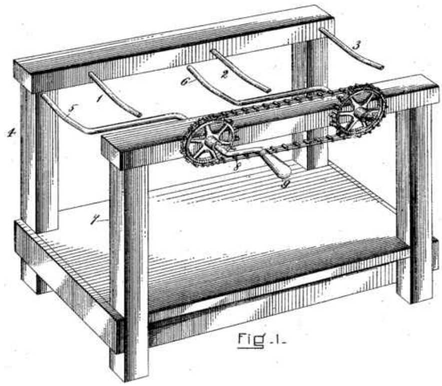
silver mixer in action

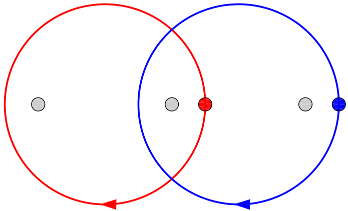
 

play movie

[See Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743 for proofs, heavily influenced by work on π_1-stirrers of Boyland, P. L. & Harrington, J. (2011). *Algeb. Geom. Topology,* **11** (4), 2265–2296.]

There remains many patents that I call 'exotic' which use nonstandard motions: such as Jenner (1905):

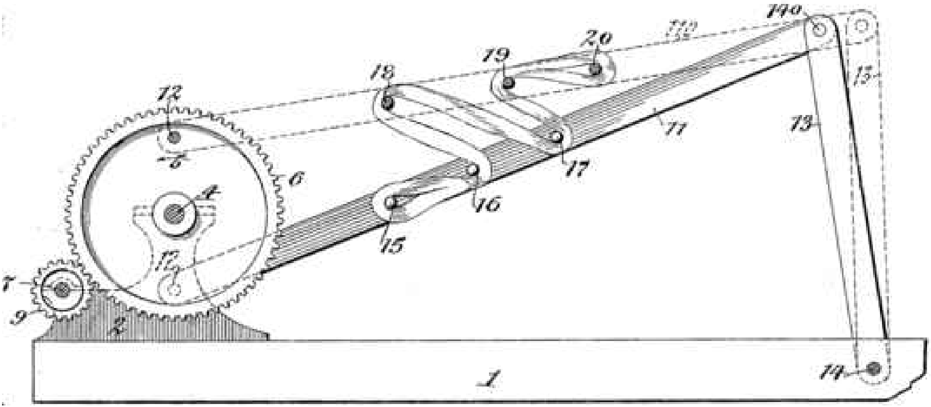




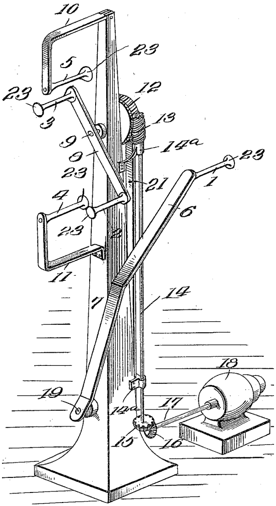
Growth given by $\phi + \sqrt{\phi}$, a peculiar number that magically popped up in Spencer Smith's research on optimal braids on the torus.

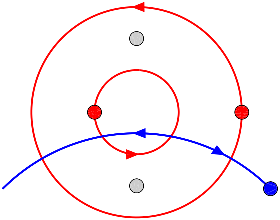
exotic designs (2)

Shean & Schmelz (1914):



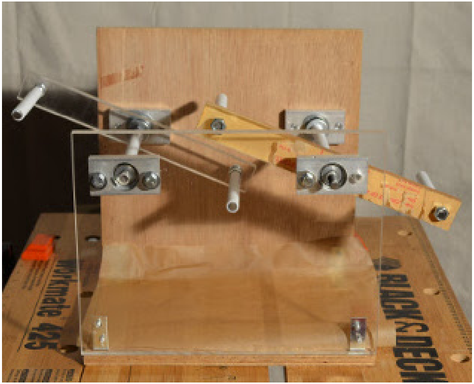
My personal favorite, McCarthy & Wilson (1915):

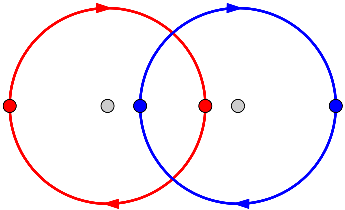




let's try our hand at this

6-pronged design with Alex Flanagan:

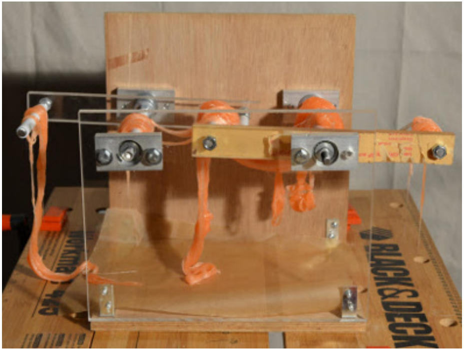


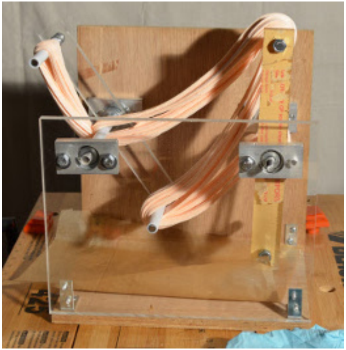


The software tools allow us to rapidly try designs. This one is simple and has huge growth (13.9 vs 5.8 for the standard pullers).

making taffy is hard

Early efforts yielded mixed results: . . . but eventually we got better at it



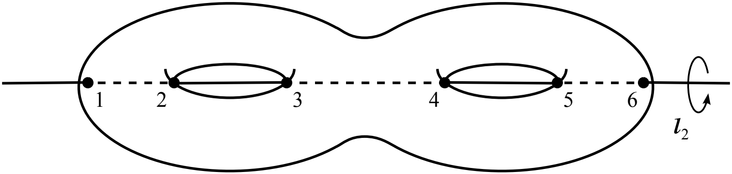


play movie

(BTW: The physics of candy making is fascinating. . .)

six-pronged puller: mathematical construction

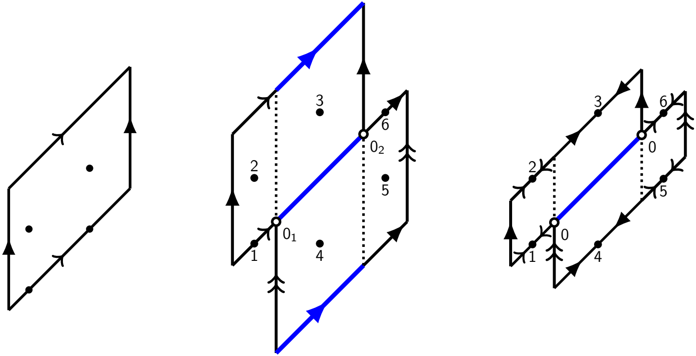
The six prongs are fixed points of a hyperelliptic involution of a genus-two surface:



[See Thiffeault, J.-L. (2018). *Math. Intelligencer,* **40** (1), 26–35. arXiv:1608.00152.]

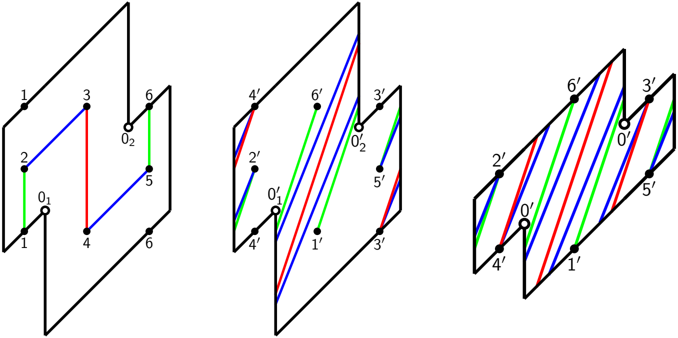
a cut-up genus-two surface

Two tori are glued to make the genus-two surface:

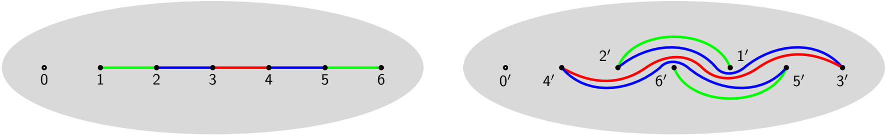


A quotient by the involution then gives a sphere with 6 distinguished points.

map on a genus-two surface



$$\phi(x) = \begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix} \cdot x$$



there is a deeper point here

- My real interest is in fluid mixing, in particular of viscous substances.
- The taffy pullers illustrate that mixing is a combinatorial process, akin to shuffling.
- The taffy designs also pop up in 'serious' chemical mixers.
- The topological dynamics methods pioneered by Thurston allows us to understand these prong motions in great detail.
- For example, in addition to the growth, there is a measure that tells us how taffy is distributed on the prongs.
- pseudo-Anosov maps themselves are still the subject of intense study. The taffy pullers provide a battery of nice examples.

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