Topological Chaos in Spatially Periodic Domains

Matthew Finn

Jean-Luc Thiffeault

Emmanuelle Gouillart

http://www.ma.imperial.ac.uk/~jeanluc

Department of Mathematics

Imperial College London

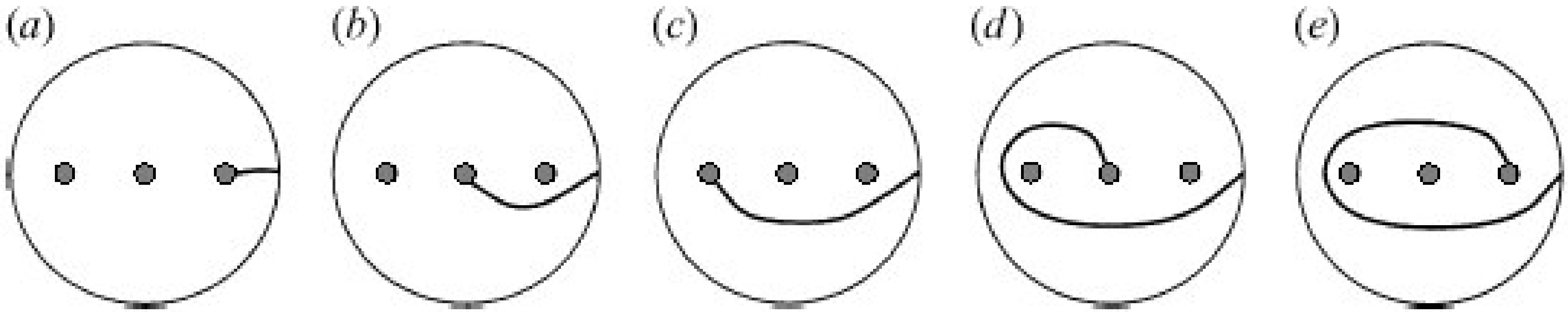
Experiment of Boyland, Aref, & Stremler

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

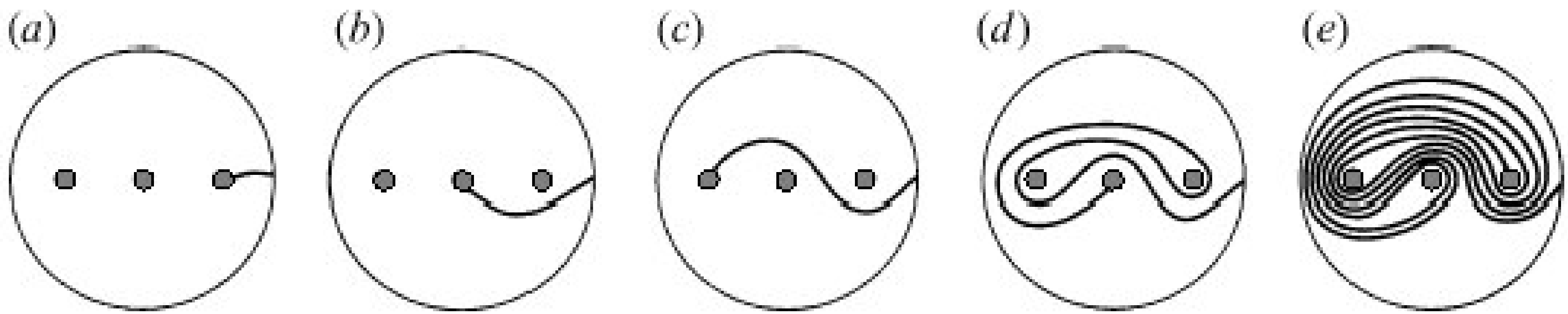
[P. L. Boyland, M. A. Stremler, and H. Aref, *Physica D* **175**, 69 (2003)]

Two Stirring Protocols

$\sigma_1\sigma_2$ protocol

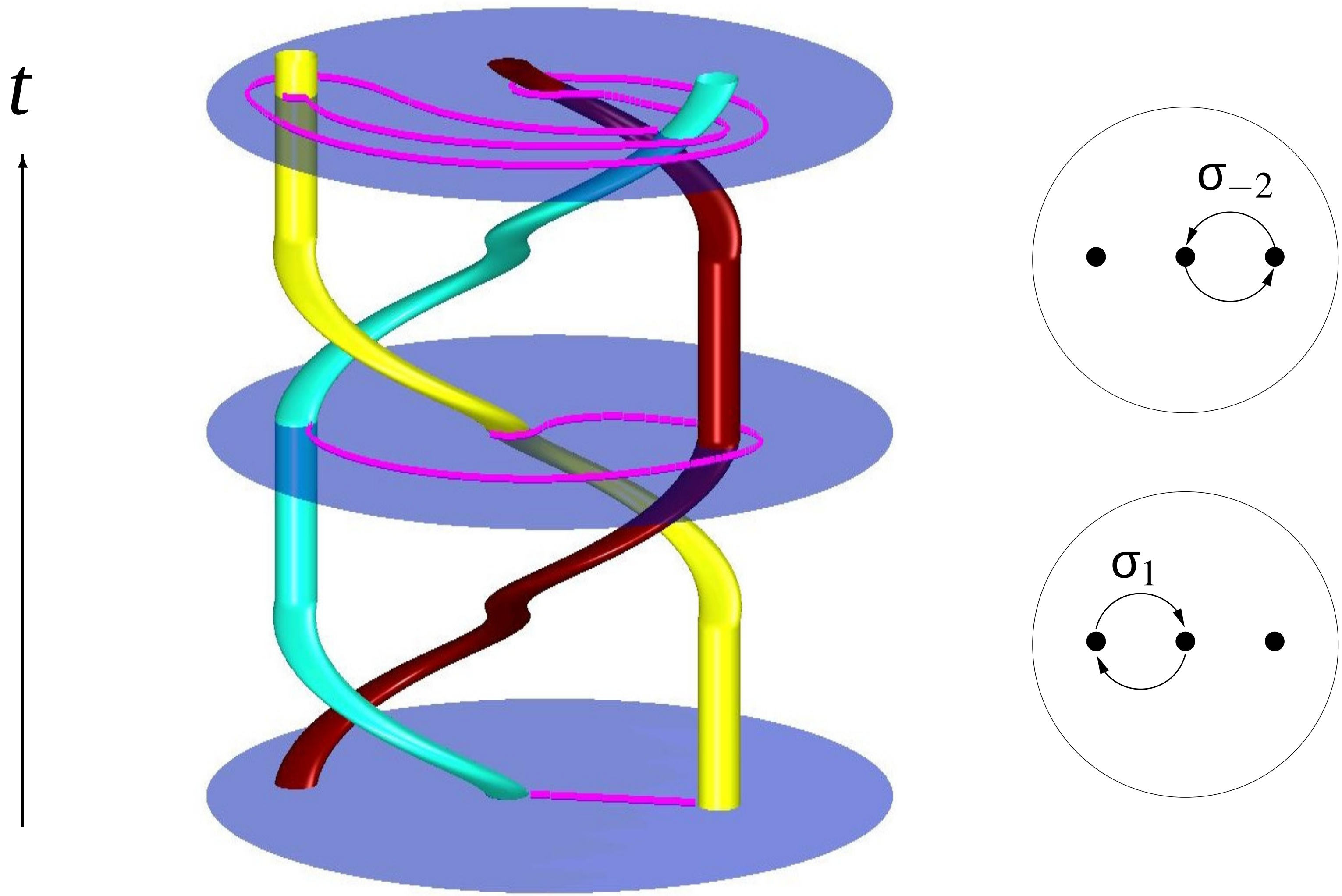


$\sigma_1^{-1}\sigma_2$ protocol

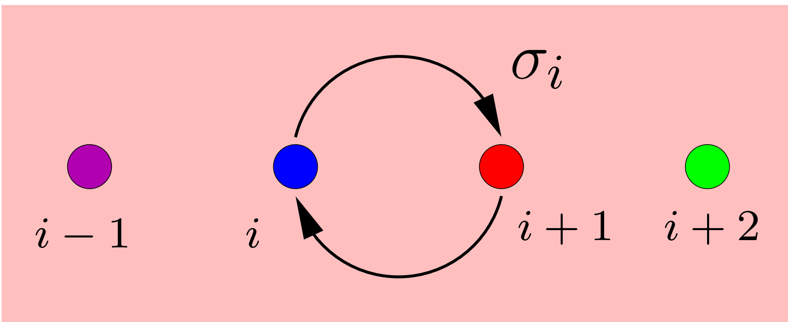


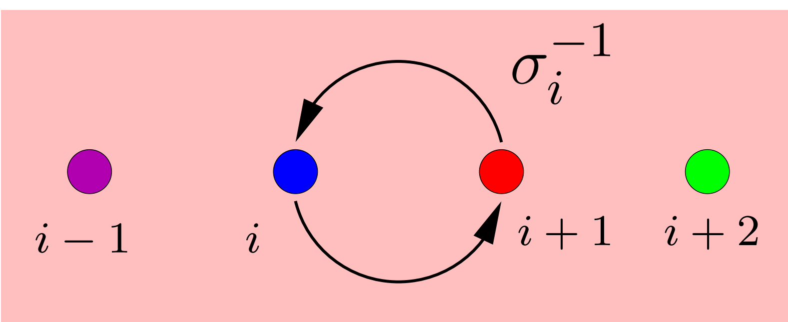
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

The Connection with Braiding



Generators of the n-Braid Group





A generator

$$\sigma_i\,, \quad i = 1, \ldots, n-1$$

is the clockwise interchange of the i th and $(i+1)$th rod.

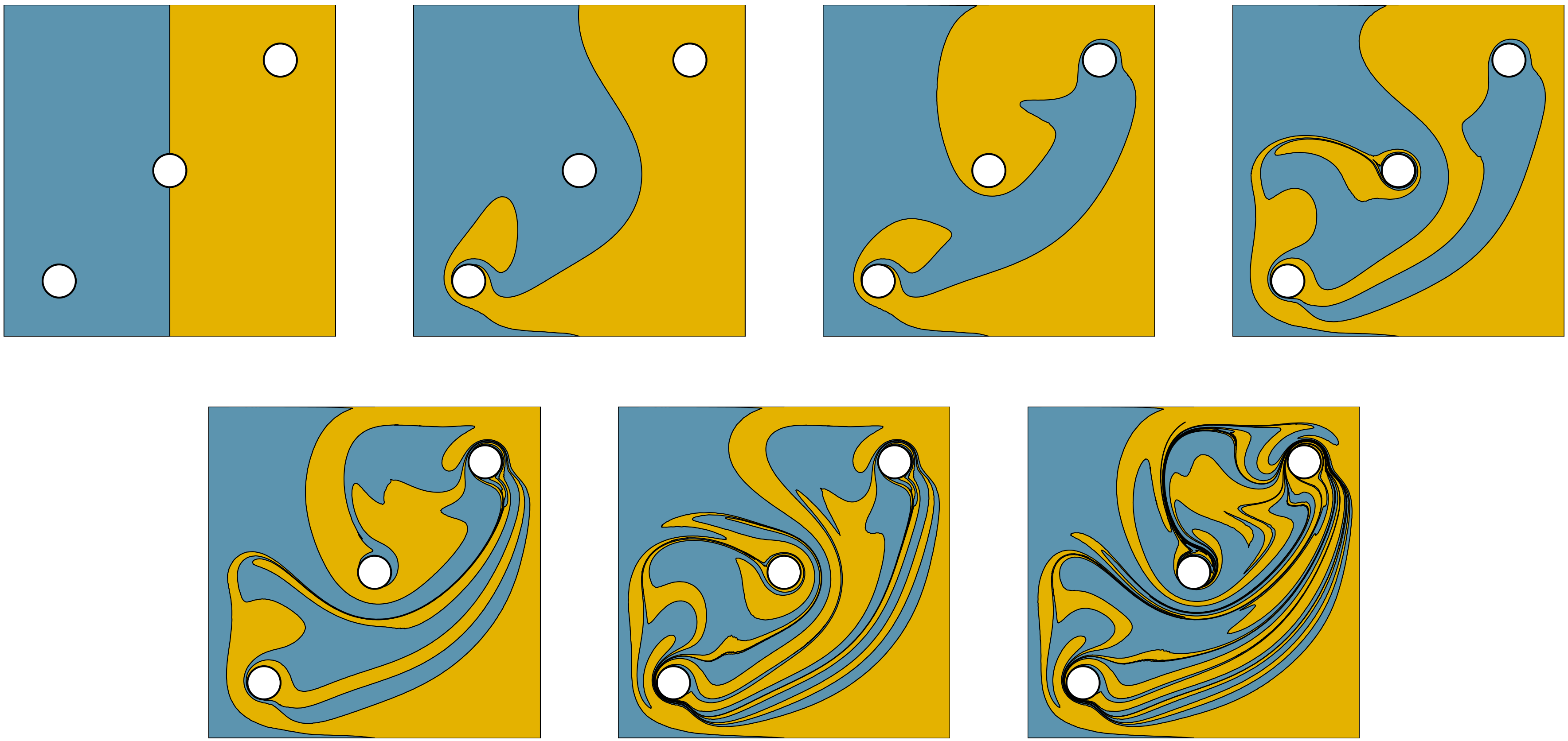
The generators obey the presentation

$$\sigma_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\sigma_i$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| > 1$$

These generators are used to characterise the motion of the rods.

Three-rod Mixer in a Bounded Domain



[M. D. Finn, S. M. Cox, and H. M. Byrne, *J. Fluid Mech.*. **493**, 345 (2003)]

[A. Vikhansky, *Chaos.* **14**, 14 (2004)]

Three-rod Mixer in a Bounded Domain

[movie 1: bounded.mpg]

Computing the Line-stretching from a Braid

- How much are lines stretched by a given braid? What is the exponential rate? (could be zero)
- This rate is referred to as the braid's topological entropy.
- This is a lower bound on the flow's topological entropy! (line-stretching exponent)
- The T.E. of a braid is found from variations on "train-tracks" algorithms.
- The T.E. is obtained from a transition matrix.

Computing the Line-stretching from a Braid

- How much are lines stretched by a given braid? What is the exponential rate? (could be zero)
- This rate is referred to as the braid's topological entropy.
- This is a lower bound on the flow's topological entropy! (line-stretching exponent)
- The T.E. of a braid is found from variations on "train-tracks" algorithms.
- The T.E. is obtained from a transition matrix.
- What about periodic boundary conditions?
- Cylinders occur in theory and experiments (The Ring of Solomon).
- Tori certainly popular with theory, and maybe even in experiments (data analysis).

Conformal Map from the Cylinder to the Plane

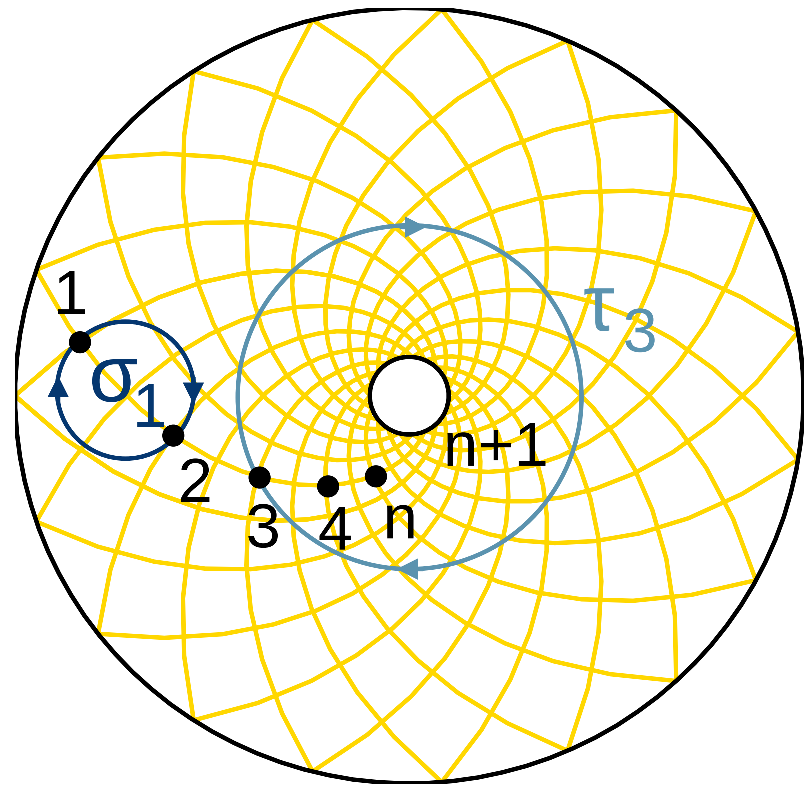
An interesting problem: what about singly-periodic boundary conditions?

Conformal map from cylinder to punctured plane:

$$w = \exp(2\pi i z)$$

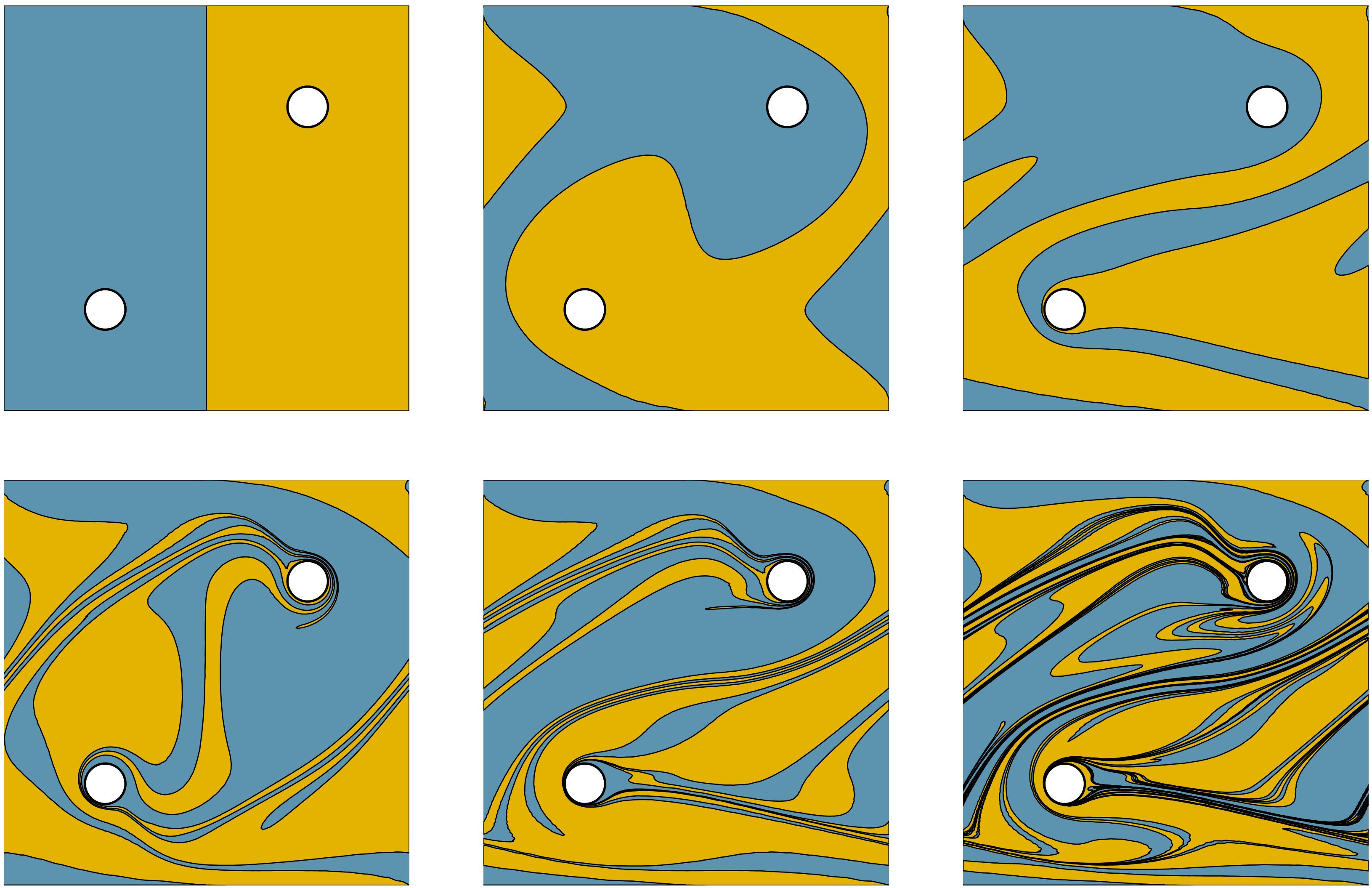
The origin in the w-plane acts as an extra rod!

So it should be possible to make a nontrivial braid with just two rods!



Suggested in [P. L. Boyland, M. A. Stremler, and H. Aref, *Physica D* **175**, 69 (2003)]

Two-rod Mixer on a Cylinder

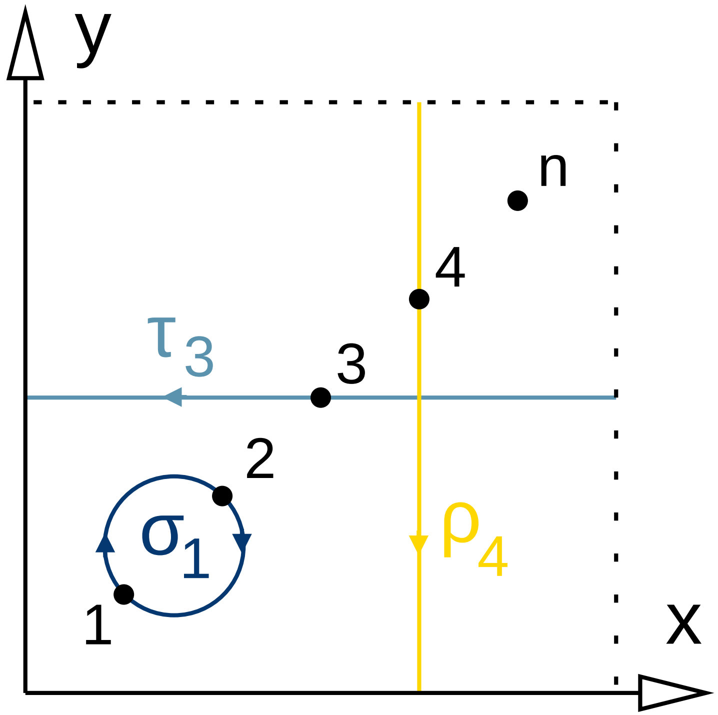


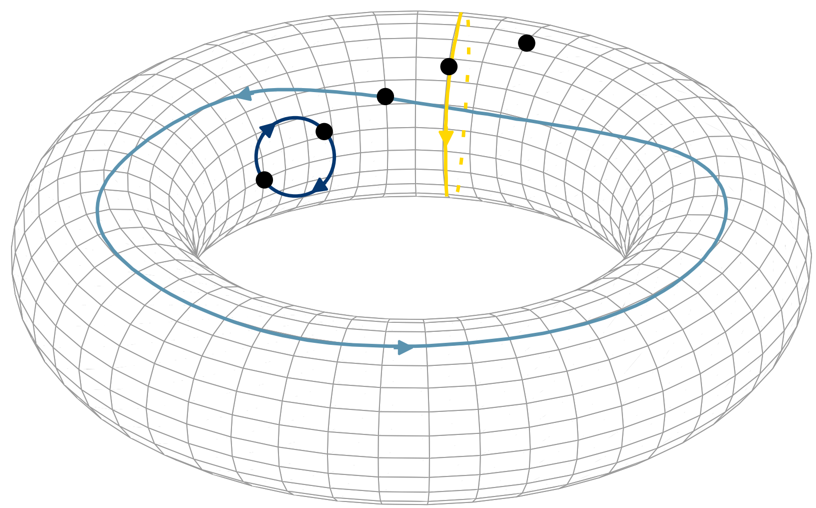
Two-rod Mixer on a Cylinder

[movie 2: singly.mpg]

The Torus: Need New Braid Operations

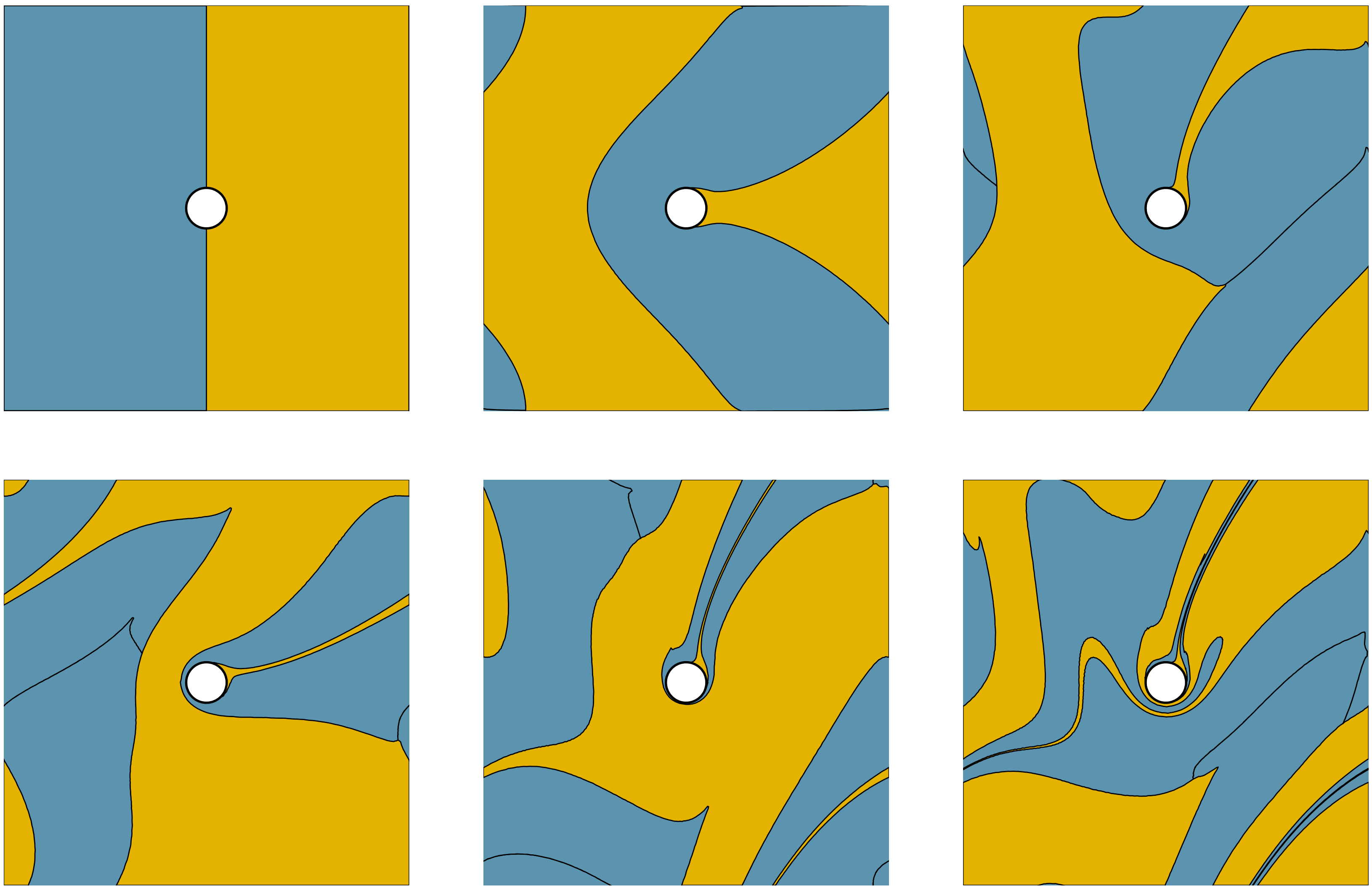
There is no corresponding conformal map for the torus.





So how do we compute T.E.? Many chaotic systems live on doubly-periodic domains...

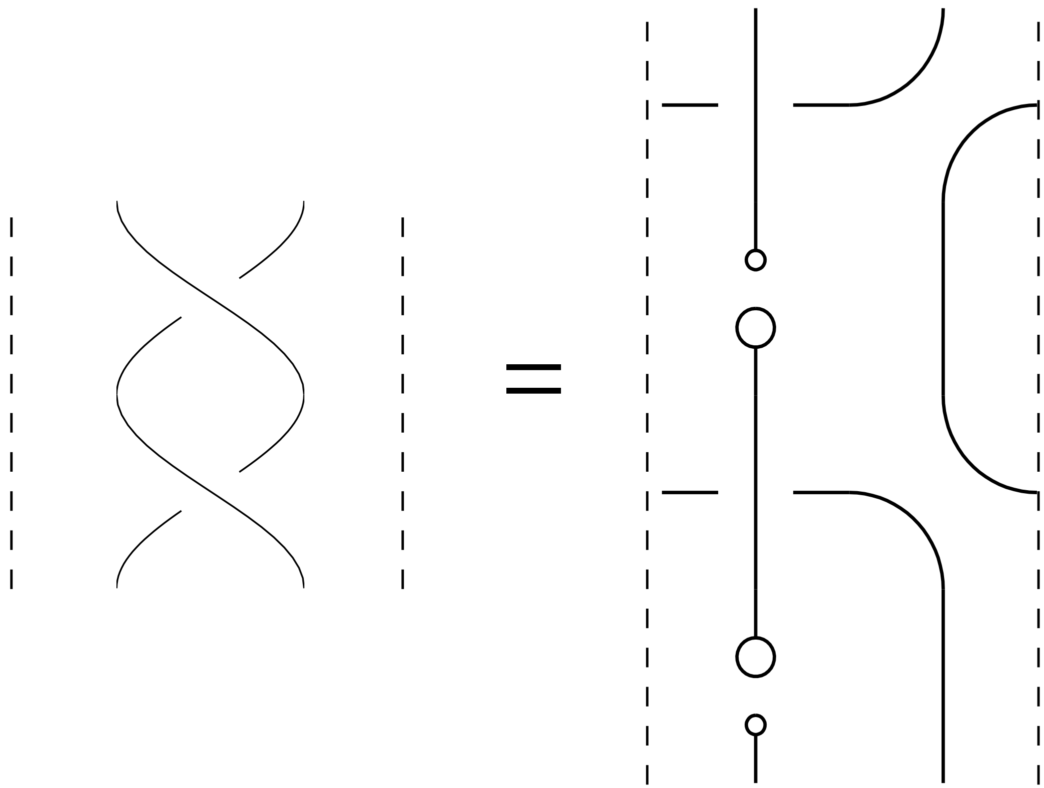
One-rod Mixer on a Torus: No Entropy



One-rod Mixer on a Torus: No Entropy

[movie 3: doubly.mpg]

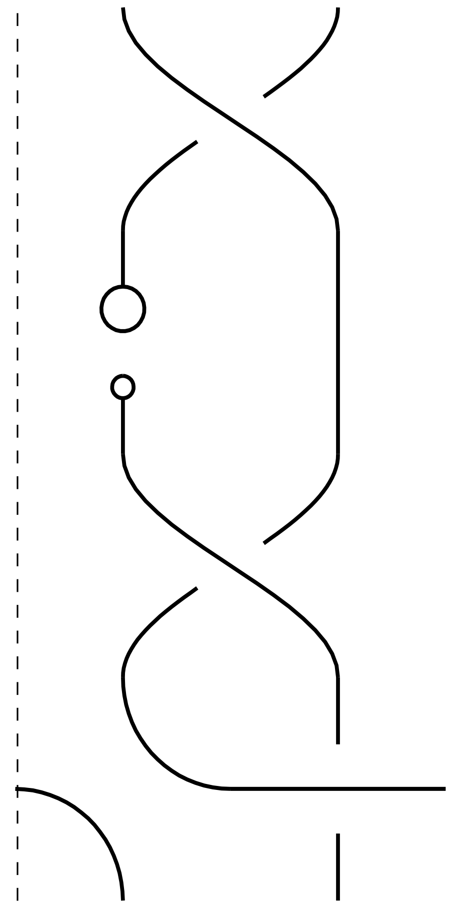
Torus with Two Rods: Presentation



$$\sigma_1^2 = \rho_1^{-1}\tau_2\rho_1\tau_2^{-1}$$

[J. S. Birman, *Comm. Pure Appl. Math.*, **22**, 41 (1969)]

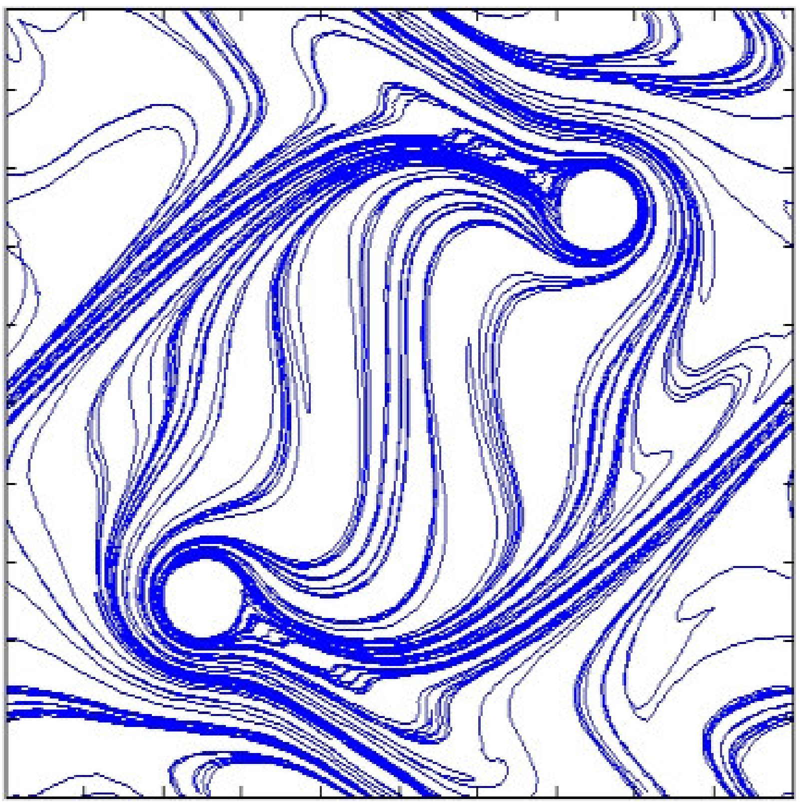
The Braid $\tau_1\sigma_1\rho_1^{-1}\sigma_1$

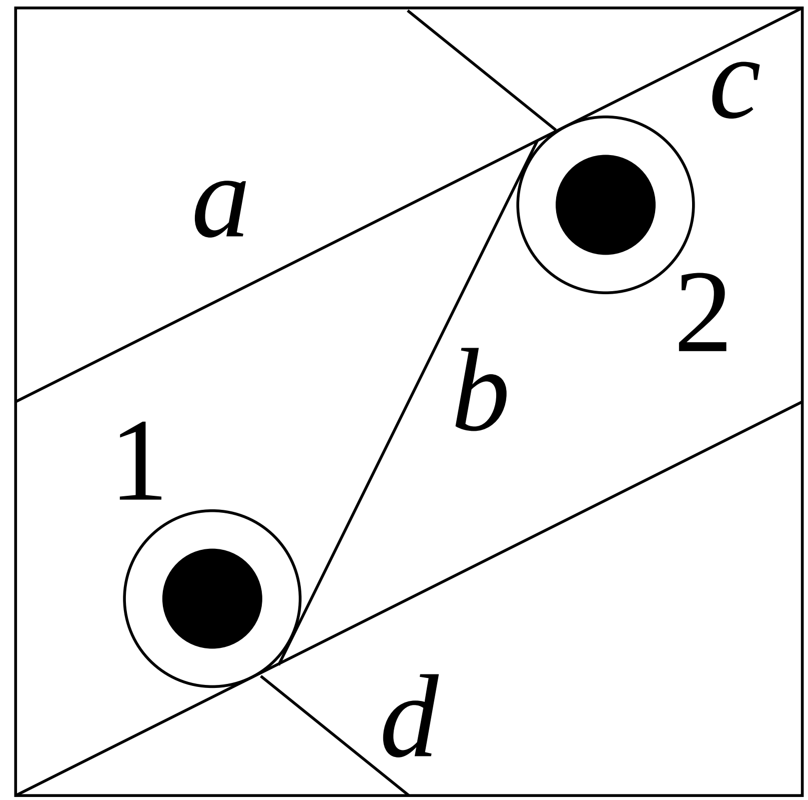


Two-rod Mixer on a Torus: $\tau_1\sigma_1\rho_1^{-1}\sigma_1$

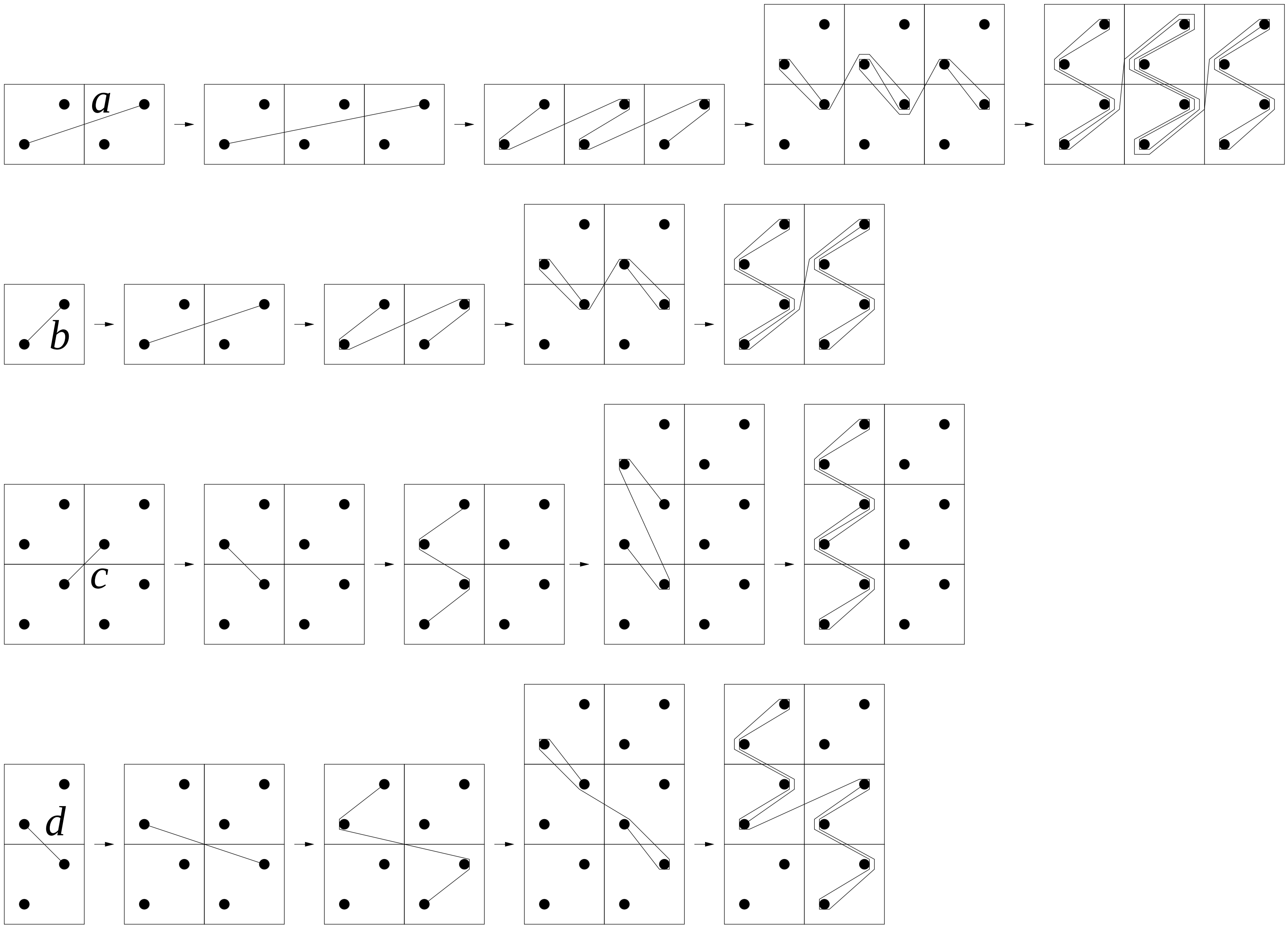
[movie 4: periodic.mpg]

The Torus Braid $\tau_1\sigma_1\rho_1^{-1}\sigma_1$: Train Tracks!





Evolution of Invariant Graph for $\tau_1\sigma_1\rho_1^{-1}\sigma_1$



Transition Matrix for $\tau_1\sigma_1\rho_1^{-1}\sigma_1$

Careful inspection reveals edges are mapped to edge-paths as

$$a \mapsto a2c1a2a1c2a1a2d1a2a1c2a1a2c1a,$$

$$b \mapsto a2c1a2a1c2a1a2d1a2a1c2a1a2c1a2a1c2a1a2d1a2a1c2a1a2c1a,$$

$$c \mapsto a2c1a2a1c2a1b2a1c2a1a2c1a,$$

$$d \mapsto a2c1a2a1c2a1c2a1a2c1a,$$

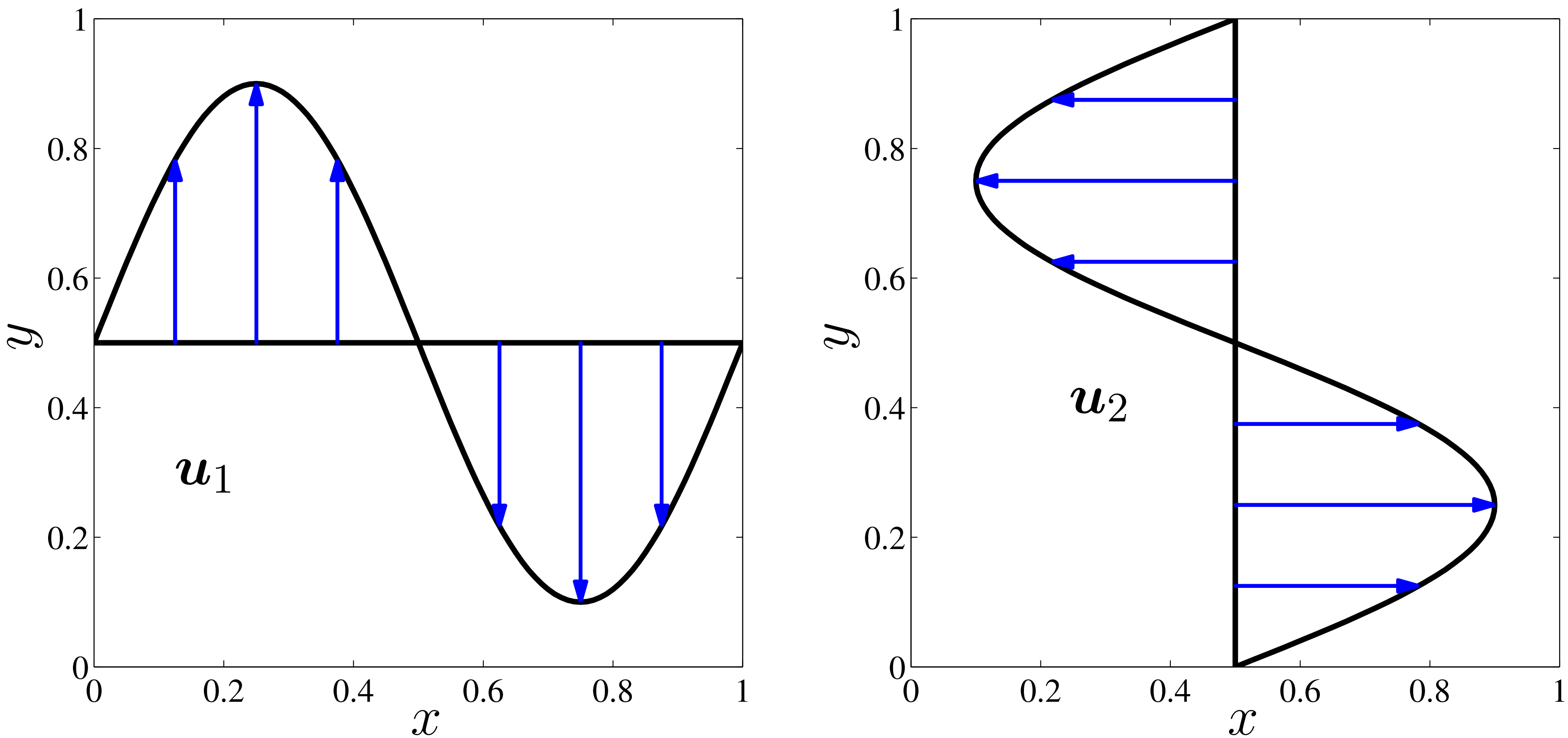
$$1 \mapsto 1, \qquad 2 \mapsto 2.$$

Edges alternate with a loops (good). The transition matrix is then

$$\begin{bmatrix} 10 & 18 & 8 & 7 \\ 0 & 0 & 1 & 0 \\ 4 & 7 & 4 & 4 \\ 1 & 2 & 0 & 0 \end{bmatrix}$$

with largest eigenvalue 14.48, so the braid has a T.E. of 2.67.

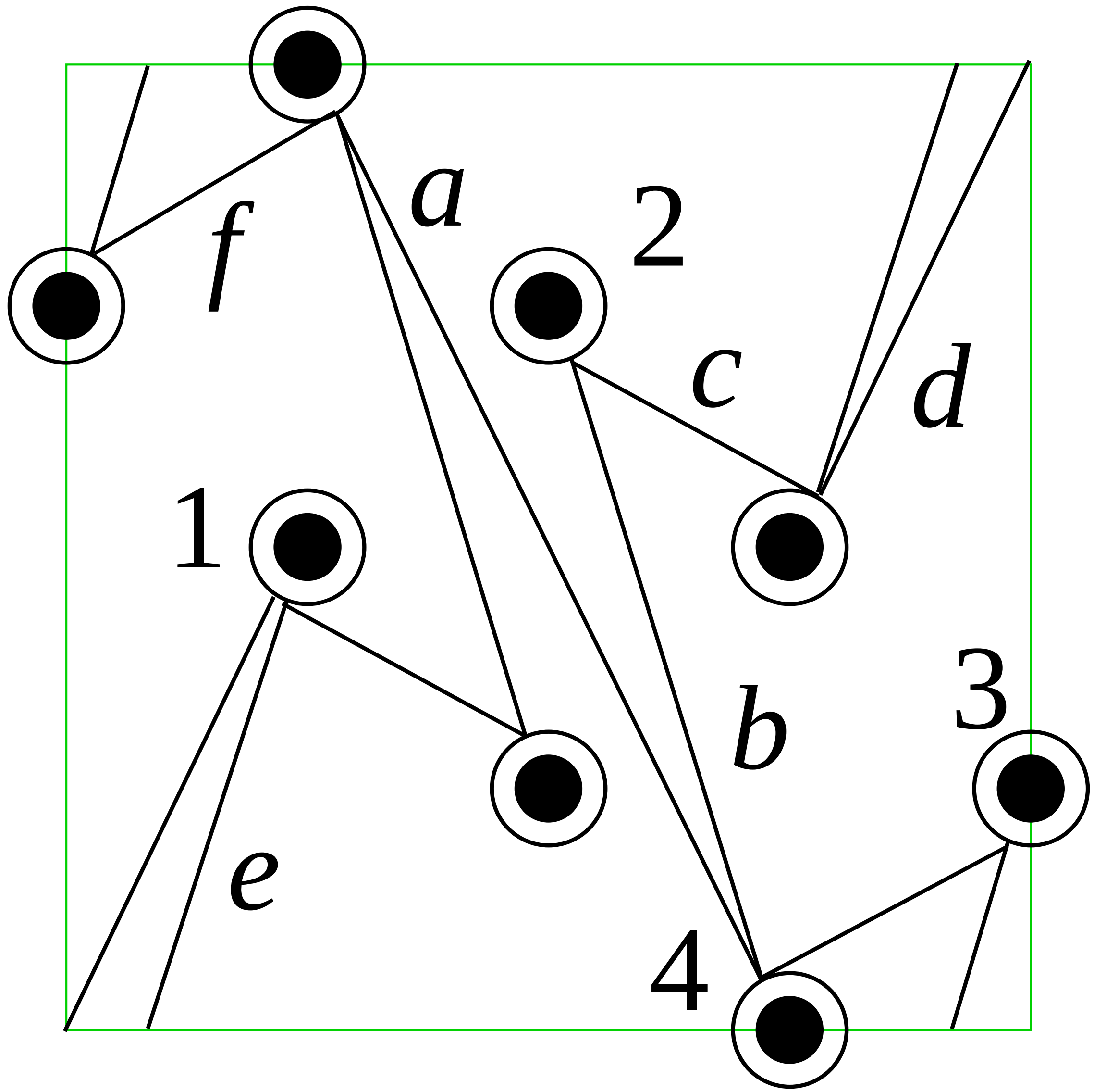
The Sine Flow



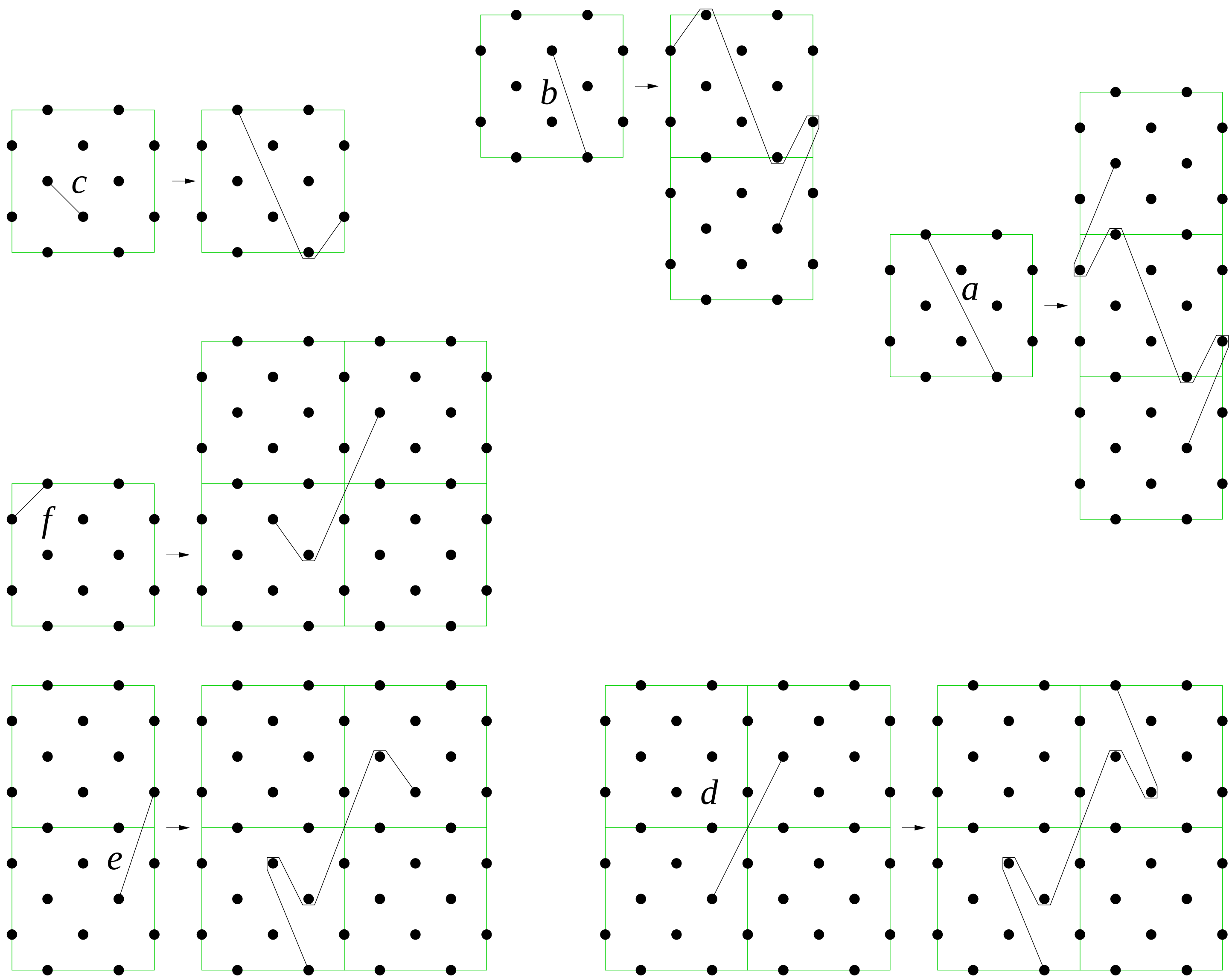
The Sine Flow: Animated Poincaré Map

[movie 5: sf_poincare.mpg]

Sine Flow: Train Track



Sine Flow: Evolution of Invariant Graph



Sine Flow: Transition Matrix

Edges are mapped as

$$
\begin{aligned}
a &\mapsto e4f3a3f4e, \\
b &\mapsto f3a3f4e, \\
c &\mapsto a3f, \\
d &\mapsto b2c1d1c2b, \\
e &\mapsto b2c1e1c, \\
f &\mapsto c1d, \\
1 &\mapsto 3, \quad 2 \mapsto 4, \quad 3 \mapsto 1, \quad 4 \mapsto 2
\end{aligned}
$$

Transition matrix:

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & 2 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
2 & 1 & 0 & 0 & 1 & 0 \\
2 & 2 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

with spectral radius 3.32 and topological entropy 1.20.

This proves that there is chaos in the sine flow for this particular parameter value, but of course it says nothing about the measure of the chaotic set.

Conclusions

- Topological chaos is a nice way to (at least partially) “explain” the growth of material lines.
- Periodic boundary conditions gives rise to more complexity, especially doubly-periodic (torus).
- Train tracks (invariant graph) harder to find in the case of the torus.
- You can often glean the invariant graph from a picture of the flow. In that case the lower bound for the T.E. should do pretty well.
- Part of a more general programme to inject topological ideas into the study of the kinematics of mixing.