The Evolution of Finite-time Lyapunov Exponents in Chaotic Flows

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with Allen Boozer

Overview

We are interested in the advection-diffusion equation:

$$\frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = \nabla\cdot(D\nabla\phi)$$

where the Eulerian velocity field $\mathbf{v}(\boldsymbol{x},t)$ is some prescribed time-dependent flow, which may or may not be be chaotic. The quantity ϕ represents the concentration of some passive scalar, ρ is the density, and D is the diffusion coefficient.

We assume that the Lagrangian dynamics are strongly chaotic ($\lambda L^2/D \gg 1$).

Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates $\boldsymbol{x}$ satisfies

$$\frac{d\boldsymbol{x}}{dt}(\boldsymbol{\xi}, t) = \mathbf{v}(\boldsymbol{x}(\boldsymbol{\xi}, t), t),$$

where $\boldsymbol{\xi}$ are Lagrangian coordinates which label fluid elements. The usual choice is to take as initial condition $\boldsymbol{x}(\boldsymbol{\xi}, t = 0) = \boldsymbol{\xi}$, which says that fluid elements are labeled by their initial position.

$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, t)$ is thus the transformation from Lagrangian ($\boldsymbol{\xi}$) to Eulerian ($\boldsymbol{x}$) coordinates.

This transformation gets horrendously complicated as time evolves.

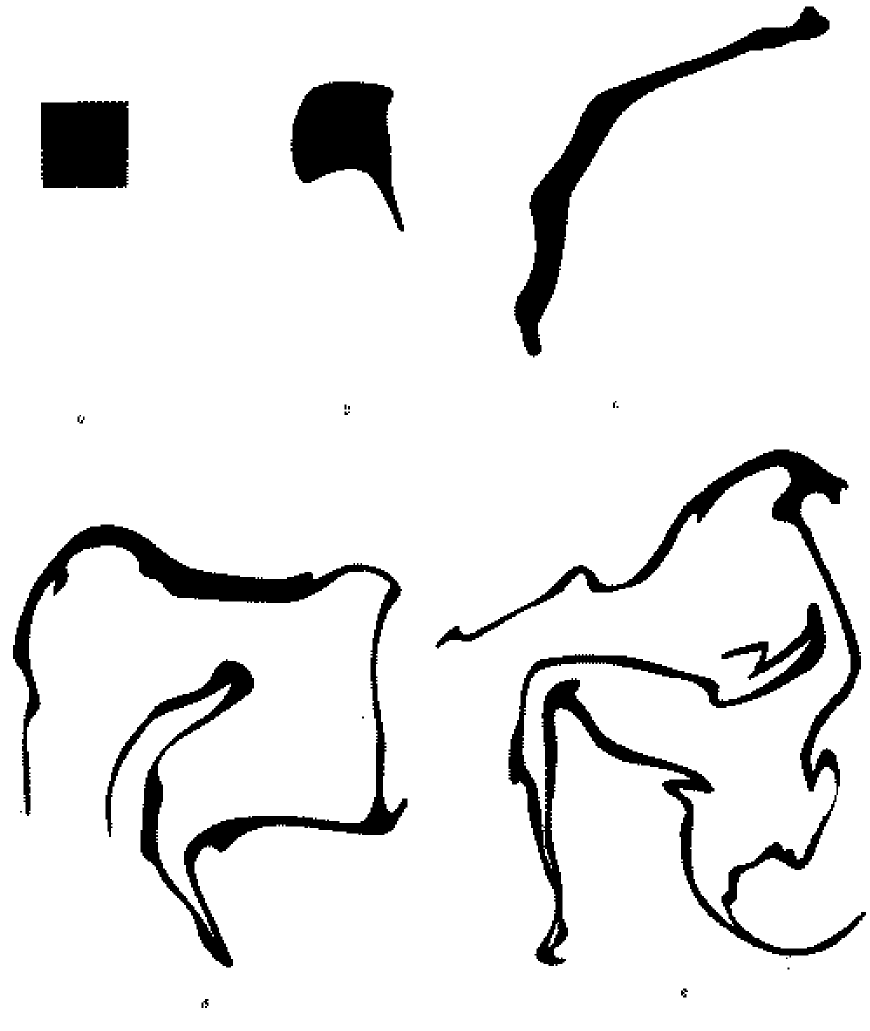
Lyapunov Exponents

The rate of exponential separation of neighbouring Lagrangian trajectories is measured by Lyapunov exponents

$$\lambda_\infty = \lim_{t\to\infty} \frac{1}{t} \ln \left\| (T_{\boldsymbol{x}}^t \mathbf{v}) \mathbf{w}_0 \right\|,$$

where $T_{\boldsymbol{x}}^t \mathbf{v}$ is the time-evolved tangent mapping of the velocity field (the matrix $\partial \mathbf{v}/\partial \boldsymbol{x}$) and $\mathbf{w}_0$ is some constant vector.

Lyapunov exponents converge very slowly. So, for practical purposes we are always dealing with finite-time Lyapunov exponents.



(Welander, 1955)

The Idea

- Can we characterize the spatial and temporal evolution of finite-time Lyapunov exponents in a generic manner?
- Can we quantify the impact of these exponents on diffusion?

Tang and Boozer (1996) brought the tools of differential geometry to bear on this problem.

Results: a generic functional form for the time evolution of finite-time Lyapunov exponents, and a relation between their spatial dependence and the shape of the stable manifolds.

A little differential geometry...

The Jacobian of the transformation from Lagrangian (ξ) to Eulerian ($\boldsymbol{x}$) coordinates

$$J^i{}_j \equiv \frac{\partial x^i}{\partial \xi^j}$$

The Jacobian tells us how tensors transform:

- Covariant:

$$\tilde{V}_j = J^k{}_j \, V_k ,$$

- Contravariant:

$$\tilde{W}^i = J^i{}_k \, W^k .$$

Measuring distances

The distance between two infinitesimally separated points in Eulerian space is given by

$$ds^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = \delta_{ij}\, dx^i dx^j \,.$$

Therefore, in Lagrangian coordinates distances are given by

$$ds^2 = \delta_{ij} \left(\frac{dx^i}{d\xi^k} d\xi^k \right) \left(\frac{dx^j}{d\xi^\ell} d\xi^\ell \right) = (J^i{}_k\, \delta_{ij}\, J^j{}_\ell)\, d\xi^k d\xi^\ell \,.$$

The distance function now depends on the Lagrangian coordinate ξ through the Jacobian J.

The Metric Tensor

The tensor δ_{ij} is a metric in the Eulerian (Euclidean) space. The tensor

$$g_{k\ell}(\boldsymbol{\xi}, t) \equiv \sum_i {J^i}_k \, {J^i}_\ell = \left(J^T J\right)_{k\ell}$$

is the same metric tensor but in the Lagrangian coordinate system.

Since the metric tells us about the distance between two neighbouring Lagrangian trajectories, its eigenvalues are related to the finite-time Lyapunov exponents.

2-D Incompressible Flow

We will now restrict ourselves to a 2-D, incompressible velocity field $\mathbf{v}$. This means that

$$\det g = (\det J)^2 = 1.$$

Now, g is a positive-definite symmetric matrix, which implies that it has real positive eigenvalues, $\Lambda(\boldsymbol{\xi}, t) \geq 1$ and $\Lambda^{-1}(\boldsymbol{\xi}, t) \leq 1$, and orthonormal eigenvectors $\hat{\mathbf{e}}(\boldsymbol{\xi}, t)$ and $\hat{\mathbf{s}}(\boldsymbol{\xi}, t)$:

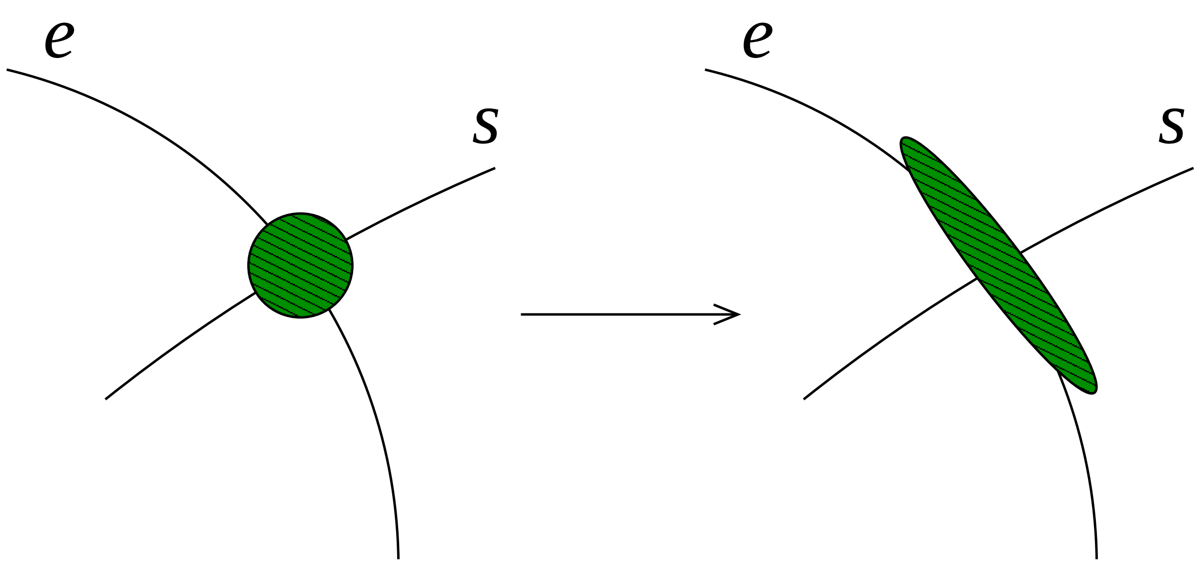
$$\boxed{g_{k\ell}(\boldsymbol{\xi}, t) = \Lambda\, e_k\, e_\ell + \Lambda^{-1}\, s_k\, s_\ell}$$

The finite-time Lyapunov exponents are given by

$$\boxed{\lambda(\boldsymbol{\xi}, t) = \ln \Lambda(\boldsymbol{\xi}, t)/2\, t}$$

Stable and Unstable Directions

At a fixed coordinate $\boldsymbol{\xi}$:

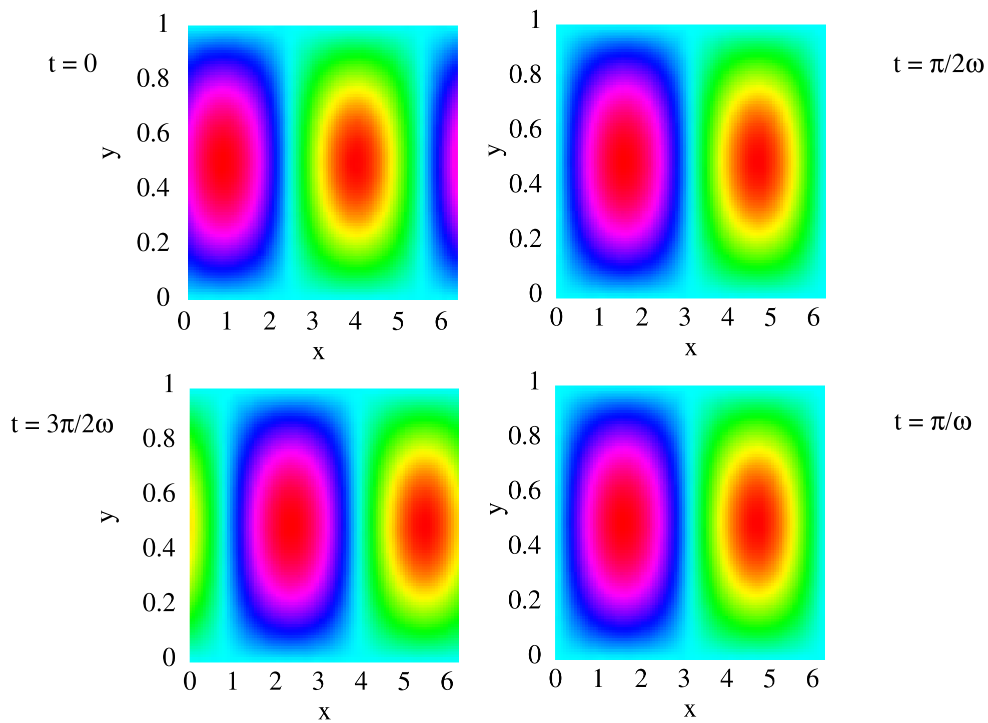


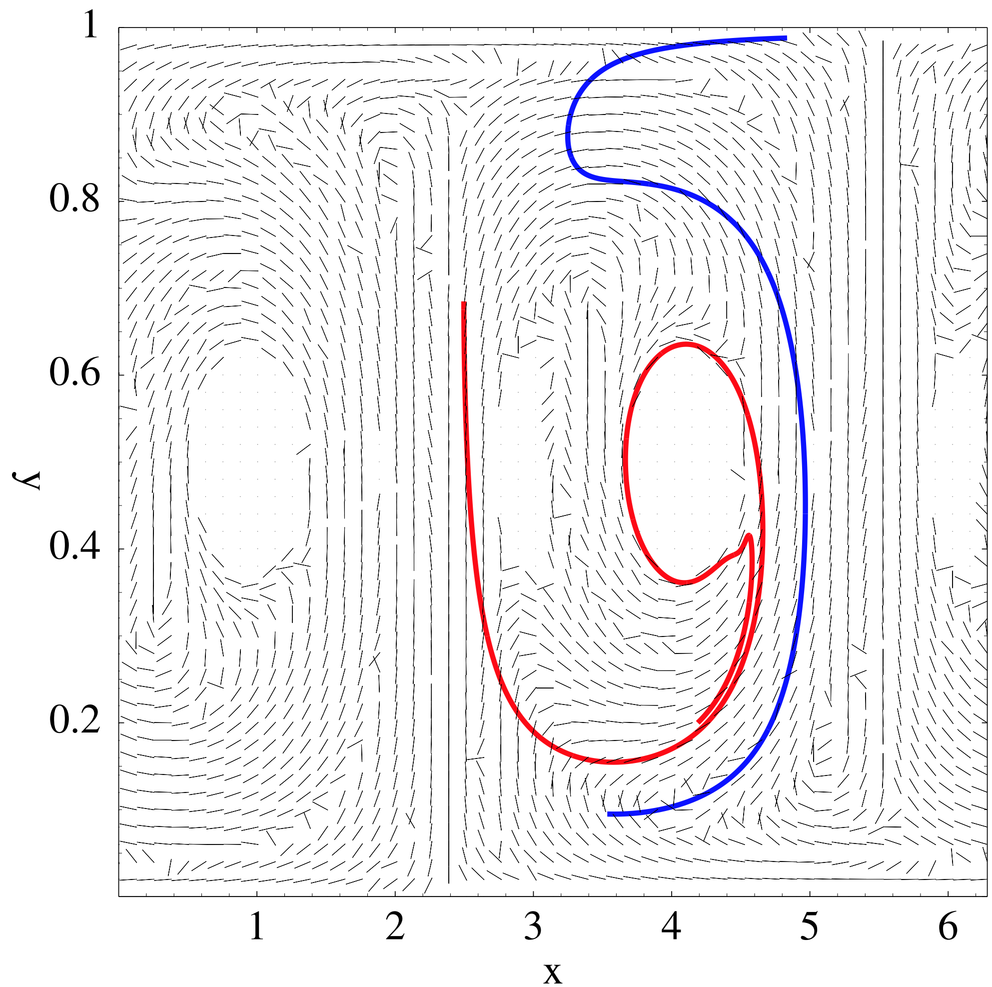
The stable and unstable manifolds $\hat{\mathbf{e}}(\xi, t)$ and $\hat{\mathbf{s}}(\xi, t)$ converge exponentially to their asymptotic values $\hat{\mathbf{e}}_\infty(\xi)$ and $\hat{\mathbf{s}}_\infty(\xi)$, whereas Lyapunov exponents converge logarithmically.

Model System

Oscillating convection rolls: $\mathbf{v} = (-\partial_y \psi, \partial_x \psi)$, with

$$\psi(\boldsymbol{x}, t) = Ak^{-1}(\sin kx \sin \pi y + \epsilon \cos \omega t \cos kx \cos \pi y)$$





$\hat{\mathbf{s}}_\infty$ field for oscillating rolls with $A = k = \epsilon = \omega = 1$, with two typical portions of the stable manifold in red and blue.

The Advection-diffusion Equation

Under the coordinate change to Lagrangian variables the diffusion term becomes

$$\nabla \cdot (D\nabla\phi) = \frac{\partial}{\partial x^i}(D\delta^{ij}\frac{\partial\phi}{\partial x^j}) = \frac{\partial}{\partial \xi^i}(Dg^{ij}\frac{\partial\phi}{\partial \xi^j}).$$

In Lagrangian coordinates the diffusivity becomes Dg^{ij}: it is no longer isotropic.

The advection-diffusion equation is thus just the diffusion equation,

$$\boxed{\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial \xi^i}(Dg^{ij}\frac{\partial\phi}{\partial \xi^j}),}$$

because by construction the advection term drops out.

Diffusion along $\hat{\mathrm{s}}_\infty$ and $\hat{\mathrm{e}}_\infty$

The diffusion coefficients along the $\hat{\mathbf{s}}_\infty$ and $\hat{\mathbf{e}}_\infty$ lines are

$$D^{ss} = s_{\infty i}(Dg^{ij})s_{\infty j} = D\exp(2\lambda\, t),$$

$$D^{ee} = e_{\infty i}(Dg^{ij})e_{\infty j} = D\exp(-2\lambda\, t).$$

We see that D^{ee} goes to zero exponentially quickly, while D^{ss} grows exponentially.

Hence, essentially all the diffusion occurs along the $\hat{\mathbf{s}}_\infty$-line.

Spatial Dependence of $\lambda(\xi, t)$

Differential geometry tells us if a metric describes a flat space, then its Riemann curvature tensor must vanish in every coordinate system.

After some tedious algebra, we find this implies that the quantity

$$\boxed{\hat{\mathbf{s}}_\infty \cdot \nabla_0 \lambda(\xi, t)\, t + \nabla_0 \cdot \hat{\mathbf{s}}_\infty}$$

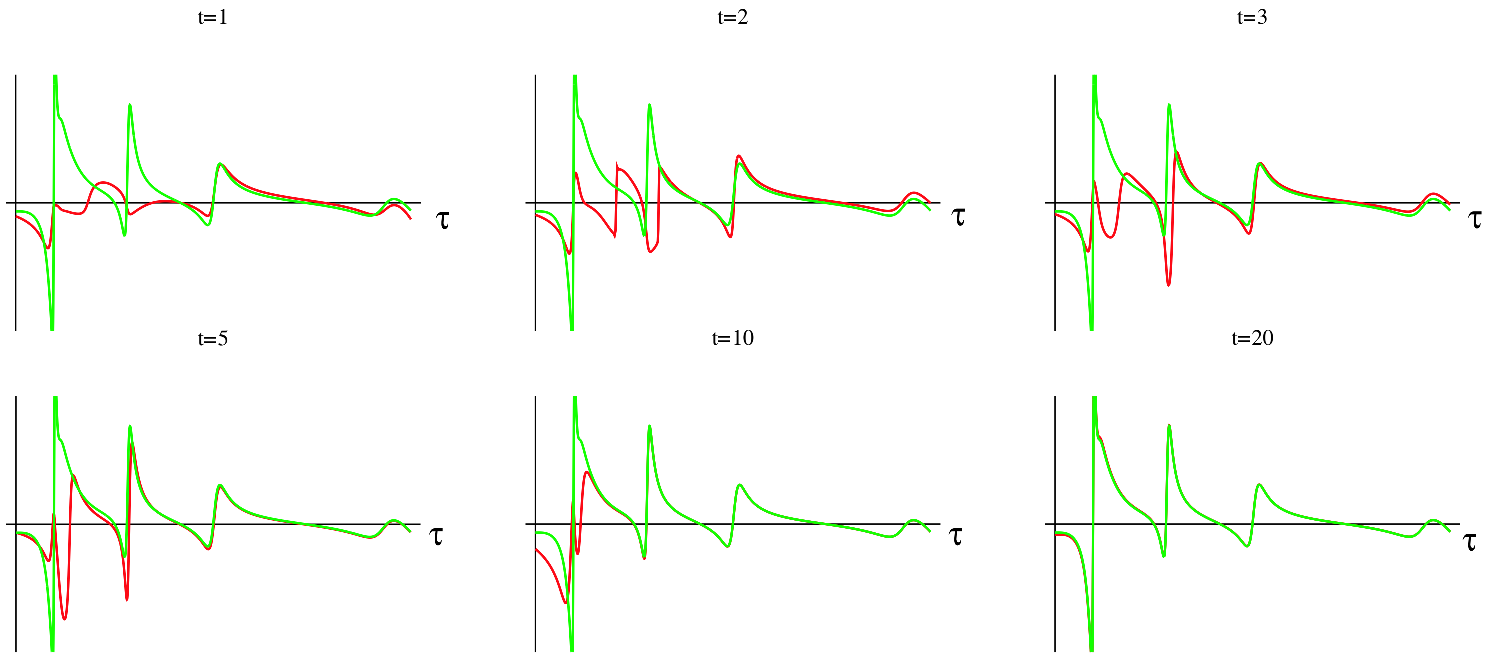
converges to 0 exponentially. Hence, it can be shown that the finite-time Lyapunov exponents must have the form

$$\boxed{\lambda(\xi, t) = \frac{\tilde{\lambda}(\xi)}{t} + \frac{f(\xi, t)}{\sqrt{t}} + \lambda_\infty,}$$

where $\hat{\mathbf{s}}_\infty \cdot \nabla_0 f = 0$ (the $1/\sqrt{t}$ factor comes from known results on the variance of the exponents).

Convergence on the $\hat{\mathbf{s}}_\infty$-line

$\nabla_0 \cdot \hat{\mathbf{s}}_\infty + (\hat{\mathbf{s}}_\infty \cdot \nabla_0)\lambda\, t$ evaluated on an $\hat{\mathbf{s}}_\infty$-line.



τ is the distance along the $\hat{\mathbf{s}}_\infty$-line.

Green: $-\nabla_0 \cdot \hat{\mathbf{s}}_\infty$

Red: $(\hat{\mathbf{s}}_\infty \cdot \nabla_0)\lambda\, t.$

Evolution of the Distribution

The time evolution of the probability distribution function of finite-time Lyapunov exponents is given by

$$P(\lambda, t) = \sqrt{\frac{1}{2\pi t\, G''(\lambda_\infty)}}\, \exp(-t\, G(\lambda)),$$

where $G(\lambda_\infty) = G'(\lambda_\infty) = 0$. This is the probability distribution for a random variable that is the average of many independent, identically distributed variables.

The width of the distribution sharpens as time evolves, and becomes a delta function as $t \to \infty$, peaked at λ_∞.

If the range of λ of interest is small compared to the standard deviation, we can approximate G by expanding around λ_∞,

$$G(\lambda) \simeq \frac{1}{2}\, G''(\lambda_\infty)(\lambda - \lambda_\infty)^2.$$

so that the probability distribution becomes Gaussian:

$$P(\lambda, t) = \sqrt{\frac{1}{2\pi t\, G''(\lambda_\infty)}}\, \exp\left(-\frac{1}{2}\, t\, G''(\lambda_\infty)(\lambda - \lambda_\infty)^2\right),$$

Note the the standard deviation is $\sigma = 1/\sqrt{G''(\lambda_\infty)\, t}$.

(The Gaussian approximation works best for strongly chaotic flows.)

If we take the spatial average $\langle\cdot\rangle$ of our expression for the finite-time Lyapunov exponents,

$$\langle\lambda\rangle(t) = \frac{\langle\tilde{\lambda}\rangle}{t} + \frac{\langle f(\xi,t)\rangle}{\sqrt{t}} + \lambda_\infty,$$

we find that the dominant contribution to the standard deviation for large t is

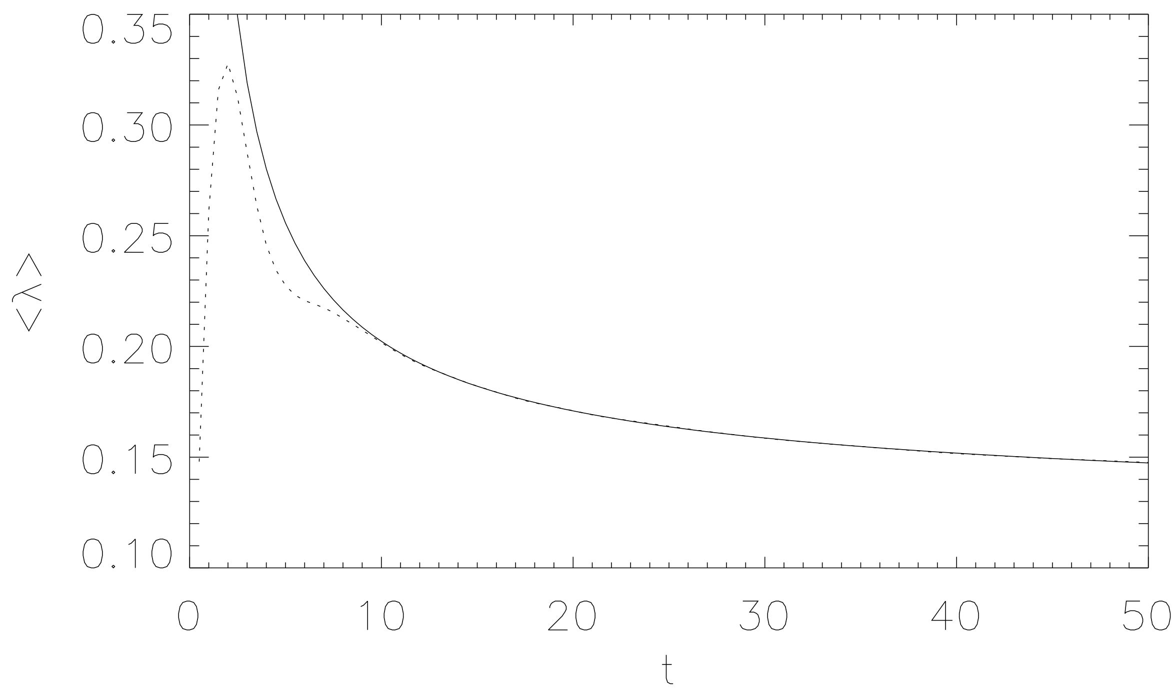
$$\sigma = \frac{\sqrt{\langle\lambda^2\rangle - \langle\lambda\rangle^2}}{\langle\lambda\rangle} \sim \frac{\sqrt{\langle f(\xi,t)^2\rangle - \langle f(\xi,t)\rangle^2}}{\lambda_\infty\sqrt{t}}.$$

To agree with the Gaussian result of $\sigma \sim 1/\sqrt{t}$, we require

$$\lim_{t\to\infty}\langle f(\xi,t)\rangle = f_0, \quad \lim_{t\to\infty}\left\langle f(\xi,t)^2\right\rangle = f_1^2,$$

i.e., the first two moments of f become independent of time for large t.

Example:

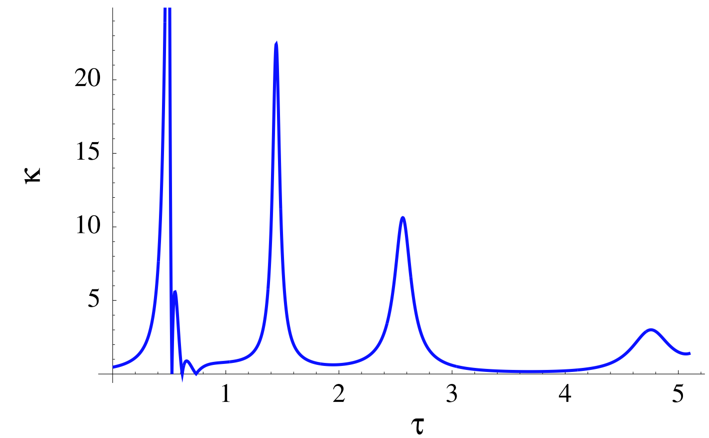


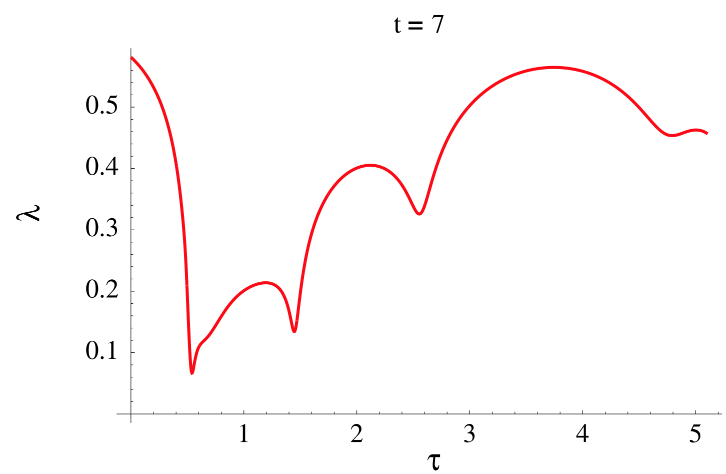
Dotted: Numerical

Solid: $0.305/t + 0.175/\sqrt{t} + 0.117$

Allows us to determine $\lambda_\infty = 0.117$ rapidly and accurately.

Curvature and Lyapunov Exponents





Finite-time Lyapunov exponent $\lambda(\xi(\tau), t)$ has local minima near high-curvature $\kappa \equiv (\hat{\mathbf{s}}_\infty \cdot \nabla_0)\hat{\mathbf{s}}_\infty$ regions of $\hat{\mathbf{s}}_\infty$-line.

Conclusions

- Diffusion occurs overwhelmingly along the stable direction.
- The spatial dependence of Lyapunov exponents along $\hat{\mathbf{s}}$ lines is contained in the smooth function $\tilde{\lambda}(\xi)$, which decays as $1/t$.
- The notoriously slow convergence of Lyapunov exponents is embodied in the nonsmooth function $f(\xi, t)$, which is constant on $\hat{\mathbf{s}}$ lines and decays as $1/\sqrt{t}$.
- Relationship between $\hat{\mathbf{s}}_\infty(\xi)$, $\kappa \equiv (\hat{\mathbf{s}}_\infty \cdot \nabla_0)\hat{\mathbf{s}}_\infty$, and $\tilde{\lambda}(\xi)$.
- Sharp bends in the $\hat{\mathbf{s}}$ line lead to locally small finite-time Lyapunov exponents (diffusion is hindered).
- Tested directly on oscillating-rolls flow.