Local and Global Aspects of Mixing

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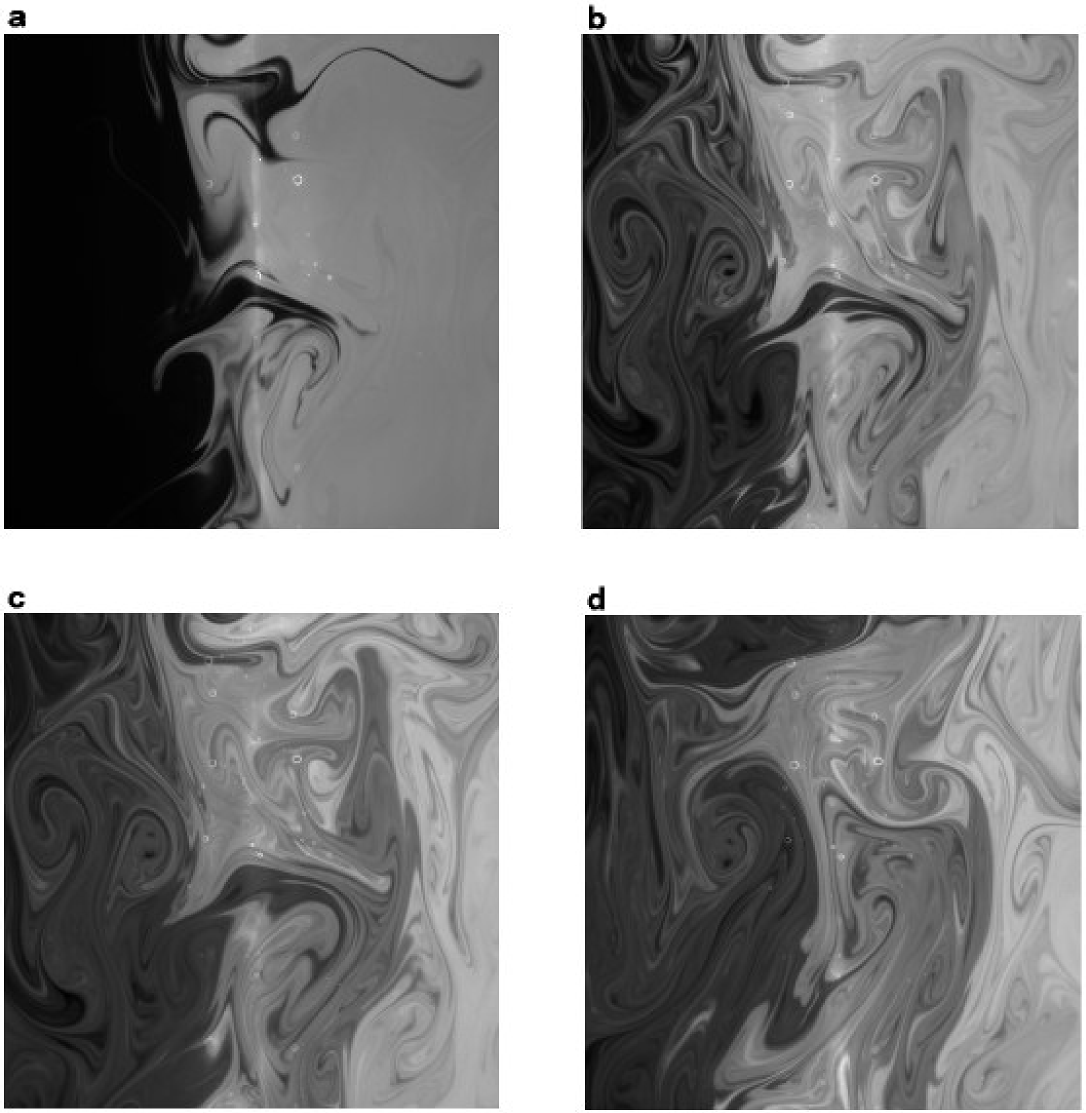
Experiment of Rothstein *et al.*: Persistent Pattern

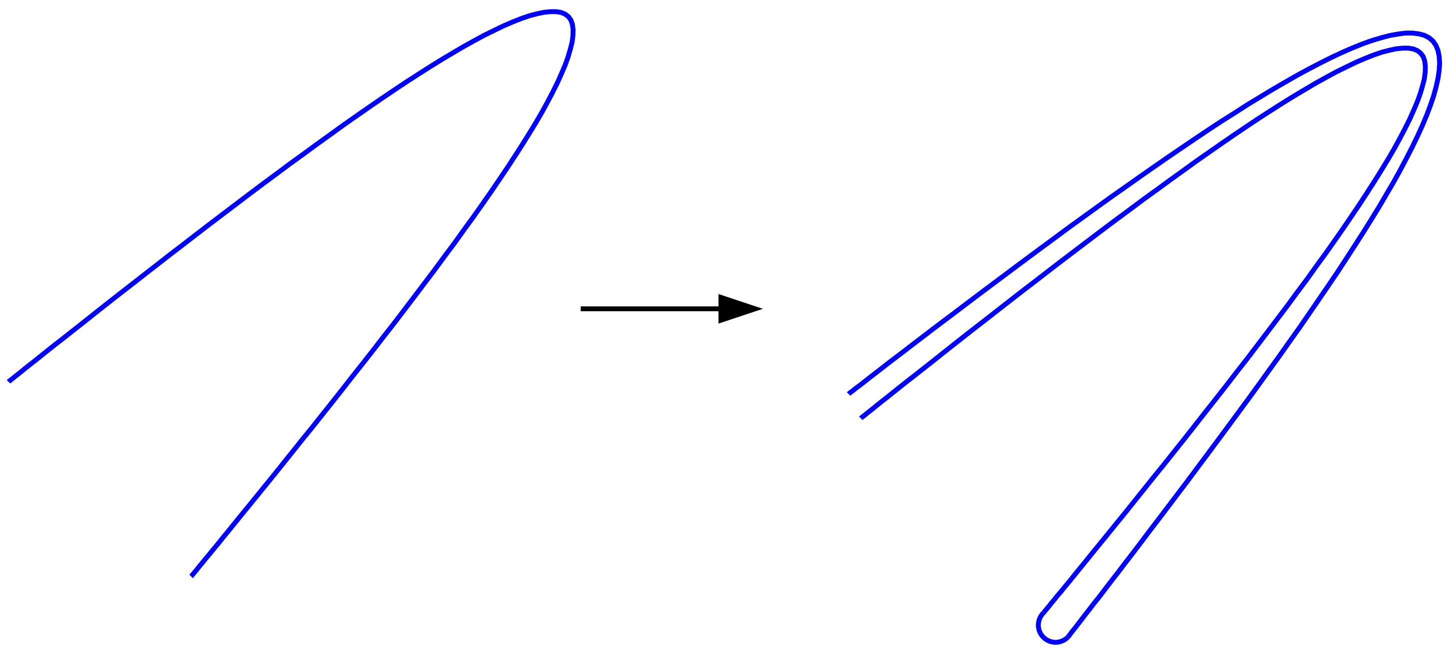


Disordered array of magnets with oscillatory current drive a thin layer of electrolytic solution.

periods $2, 20, 50, 50.5$

[Rothstein, Henry, and Gollub, Nature **401**, 770 (1999)]

Evolution of Pattern



- “Striations”
- Smoothed by diffusion
- Eventually settles into “pattern” (eigenfunction)

Local vs Global Regimes of Mixing

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Global theory:

- Eigenfunction of advection–diffusion operator.
- [Pierrehumbert, Chaos Sol. Frac. (1994)] Strange eigenmode
  [Fereday et al., Wonhas and Vassilicos, PRE (2002)] Baker's map
  [Sukhatme and Pierrehumbert, PRE (2002)]
  [Fereday and Haynes (2003)] Unified description

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- Today: Try to connect the two pictures.
- Cannot often do this! Map allows (mostly) analytical results.

A Bit of History

Eulerian (spatial) coordinates are due to…

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d'Alembert

A Bit of History

…and Lagrangian (material) coordinates to…

d'Alembert

Euler

The people responsible for the confusion…

The people responsible for the confusion…

Lagrange

Dirichlet

(See footnote in Truesdell, *The Kinematics of Vorticity*.)

The Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0,1]^2$,

$$\mathcal{M}(\boldsymbol{x}) = \mathbb{M} \cdot \boldsymbol{x} + \phi(\boldsymbol{x}),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \phi(\boldsymbol{x}) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot \boldsymbol{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $\varepsilon = 0$ the stretching of fluid elements is homogeneous in space.
For small ε the system is still uniformly hyperbolic.

Advection and Diffusion: Eulerian Viewpoint

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution $\theta^{(i-1)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \mathcal{H}_\kappa\, \theta^{(i-1)}(\mathcal{M}^{-1}(\boldsymbol{x}))$$

where κ is the diffusivity, with the heat operator $\mathcal{H}_\kappa$ and kernel h_κ

$$\mathcal{H}_\kappa\theta(\boldsymbol{x}) := \int_{\mathbb{T}^2} h_\kappa(\boldsymbol{x}-\boldsymbol{y})\theta(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y};$$

$$h_\kappa(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}^2\kappa).$$

In other words: advect instantaneously and then diffuse for one unit of time.

Transfer Matrix

Fourier expand $\theta^{(i)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{\theta}^{(i)}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \,.$$

The effect of advection and diffusion becomes

$$\hat{\theta}^{(i)}_{\boldsymbol{k}}(\boldsymbol{x}) = \sum_{\boldsymbol{q}} \mathbb{T}_{\boldsymbol{k}\boldsymbol{q}}\, \hat{\theta}^{(i-1)}_{\boldsymbol{q}},$$

with the transfer matrix,

$$\begin{aligned} \mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} &:= \int_{\mathbb{T}^2} \exp\left(2\pi\mathrm{i}\,(\boldsymbol{q}\cdot\boldsymbol{x} - \boldsymbol{k}\cdot\mathcal{M}(\boldsymbol{x})) - \kappa\,\boldsymbol{q}^2\right)\,\mathrm{d}\boldsymbol{x}, \\ &= \mathrm{e}^{-\kappa\,\boldsymbol{q}^2}\,\delta_{0,Q_2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left((k_1+k_2)\,\varepsilon\right), \qquad \boldsymbol{Q} := \boldsymbol{k}\cdot\mathbb{M} - \boldsymbol{q}, \end{aligned}$$

where the J_Q are the Bessel functions of the first kind.

Variance: A Measure of Mixing

In the absence of diffusion ($\kappa = 0$) the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} \left|\theta^{(i)}(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}} \sigma^{(i)}_{\boldsymbol{k}}, \qquad \sigma^{(i)}_{\boldsymbol{k}} := \left|\hat{\theta}^{(i)}_{\boldsymbol{k}}\right|^2$$

is preserved. (We assume the spatial mean of θ is zero.)

For $\kappa > 0$ the variance decays.

We consider the case $\kappa \ll 1$, of greatest practical interest.

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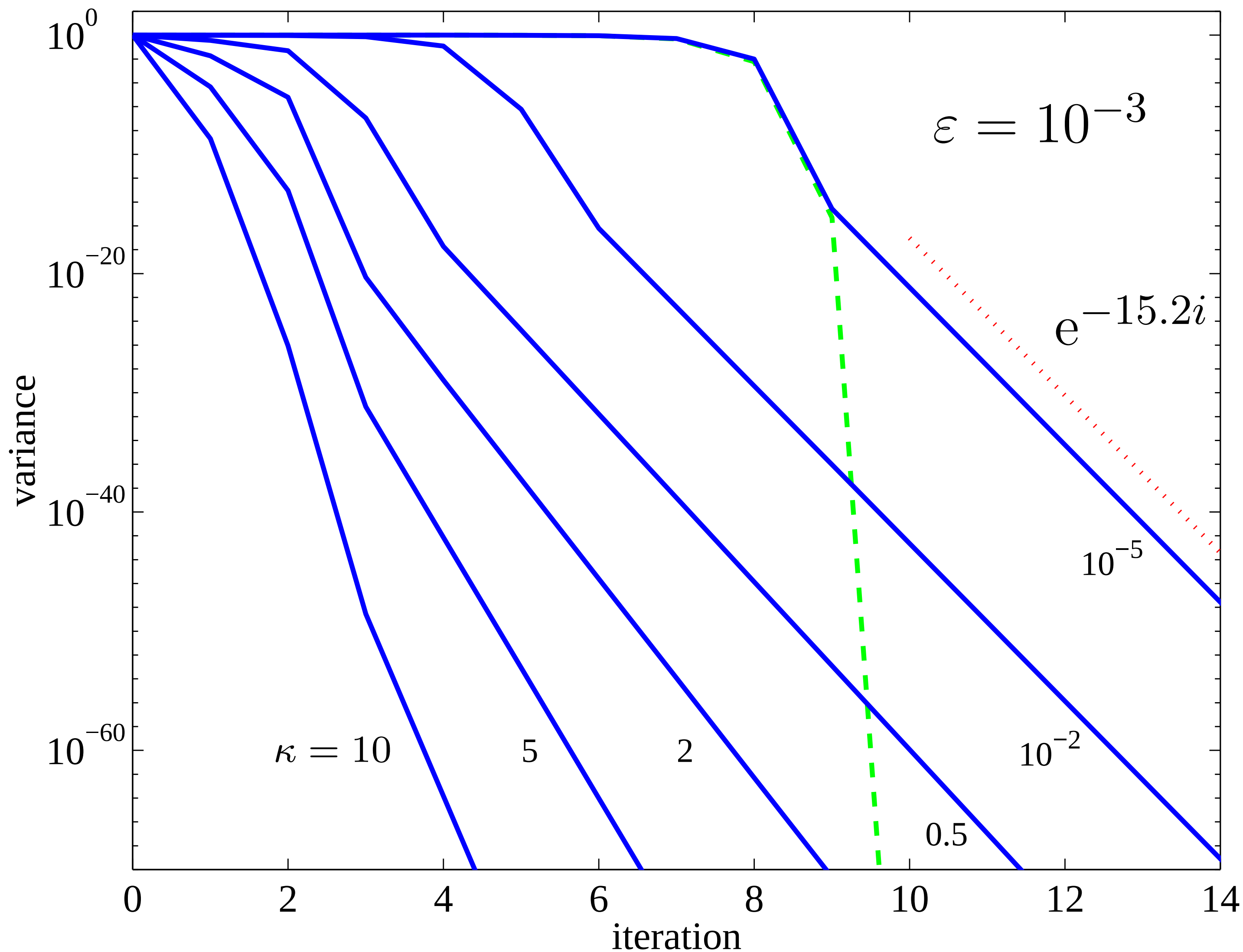
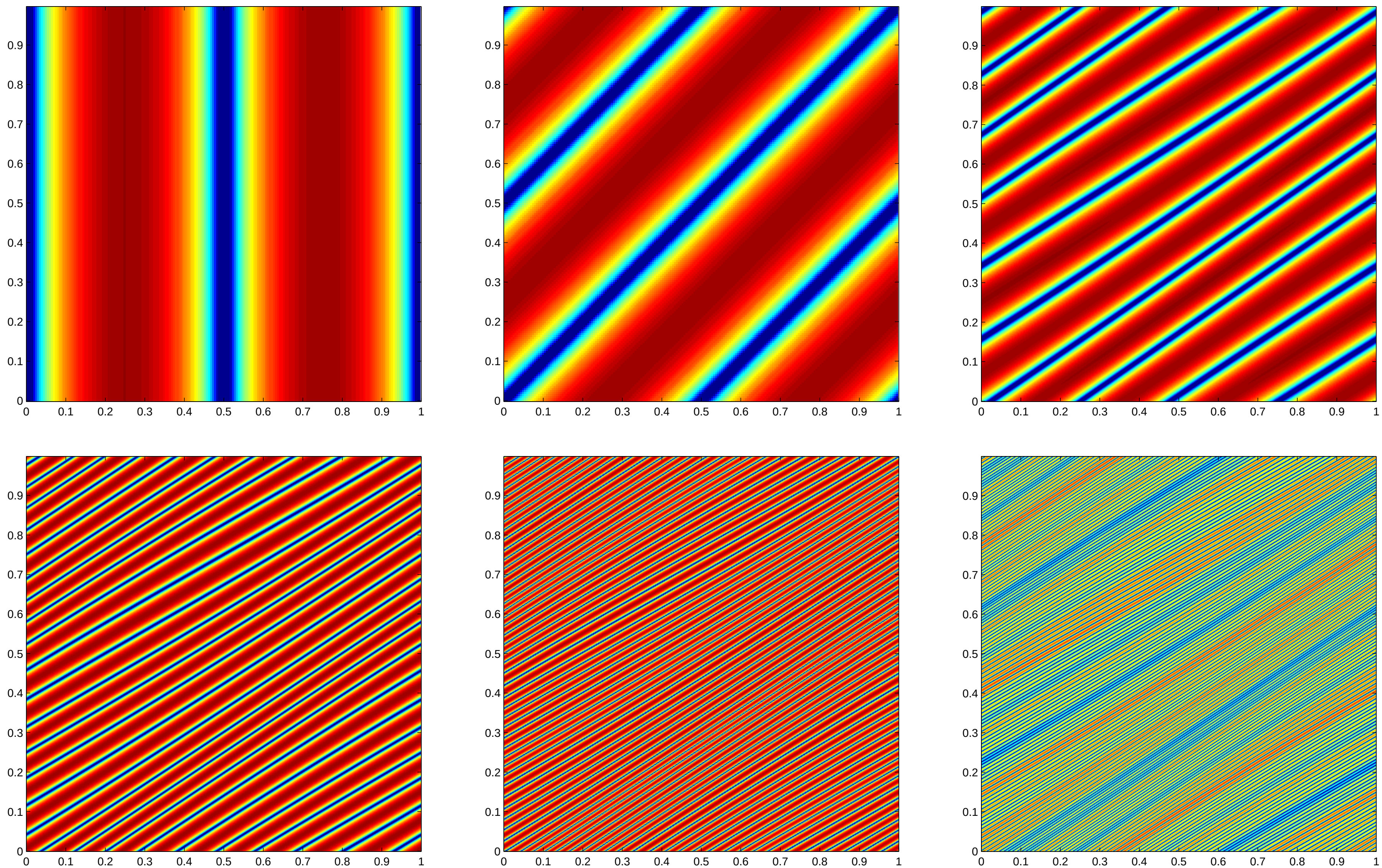
Three phases:

- The variance is initially constant;
- It then undergoes a rapid superexponential decay;
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.

Decay of Variance

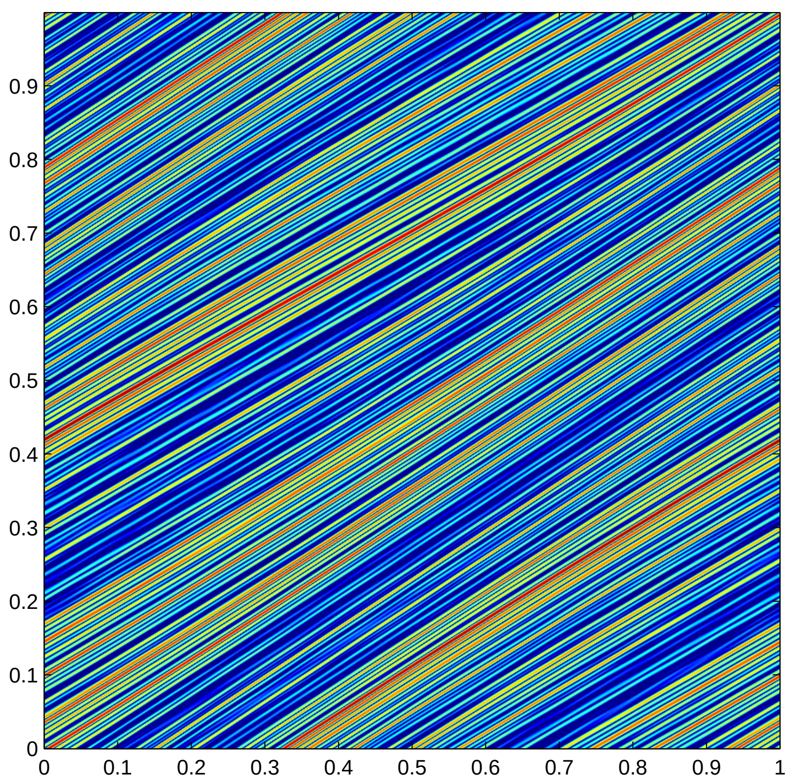


Variance: 5 iterations for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$

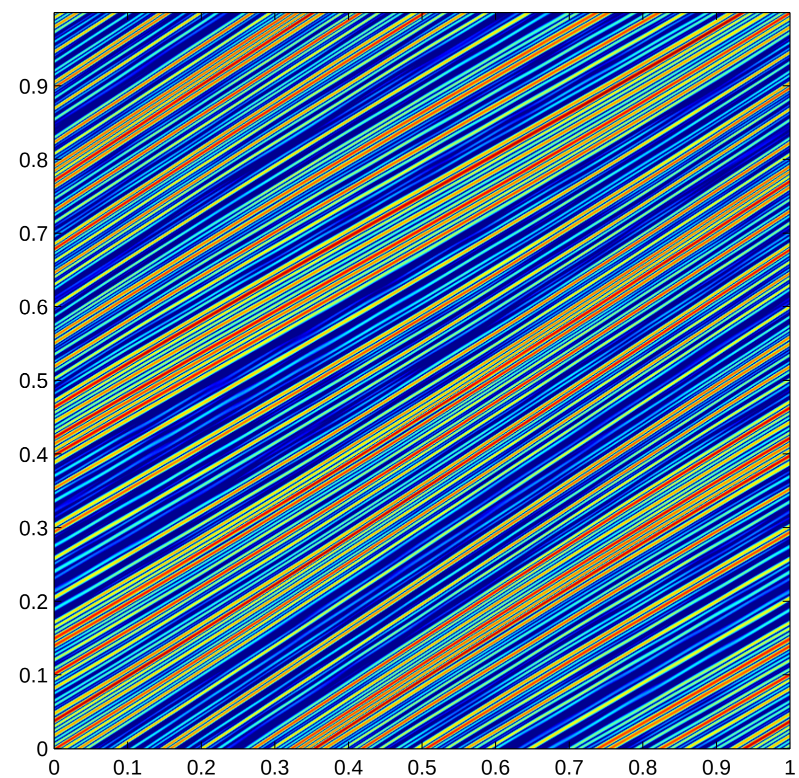


Eigenfunction for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$

(Renormalised by decay rate)



$i = 25$



$i = 30$

Decay Rate

For small ε, the dominant Bessel function is J_1, so the decay factor μ^2 for the variance is given by

$$\mu = \left|\mathbb{T}_{(0\ 1),(0\ 1)}\right| = \mathrm{e}^{-\kappa}\, J_1\,(\varepsilon) = \tfrac{1}{2}\varepsilon + \mathcal{O}\!\left(\kappa\,\varepsilon, \varepsilon^2\right).$$

Hence, for small ε the decay rate is limited by the $(0\ \ 1)$ mode. The decay rate is independent of κ for $\kappa \to 0$.

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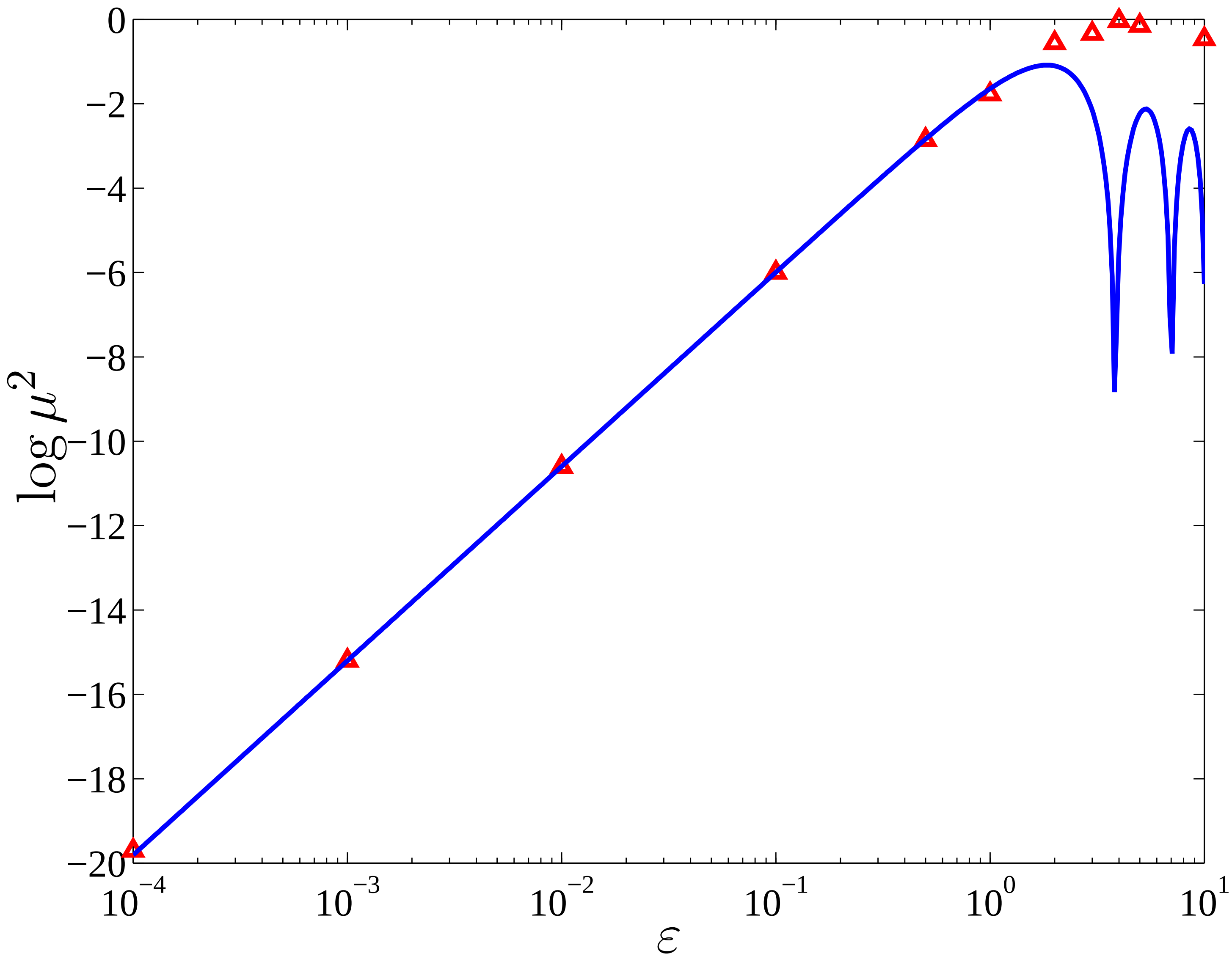
In the baker's map the discontinuity implies a slow convergence of the Fourier modes. However, it is a one-dimensional problem.

Decay Rate as $\kappa \to 0$



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- Why do this? The two viewpoints are a priori unrelated, because they for these highly-chaotic systems they are connected by an extremely convoluted (*i.e.*, inaccessible) transformation!
- But must give same answer for a scalar quantity like the decay rate.

Advection and Diffusion: Eulerian to Lagrangian

Advection-diffusion (A–D) equation:

$$\partial_t\theta + \boldsymbol{v}\cdot\partial_{\boldsymbol{x}}\theta = \widetilde{\kappa}\,\partial_{\boldsymbol{x}}^2\theta.$$

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$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{X} .$$

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Transform A–D equation to Lagrangian coordinates,

$$\dot{\theta} = \partial_{\boldsymbol{X}} (\mathbb{D} \cdot \partial_{\boldsymbol{X}} \theta).$$

Anisotropic diffusion tensor, in terms of metric or Cauchy–Green strain tensor:

$$\mathbb{D} := \widetilde{\kappa}\, g^{-1}; \qquad g_{pq} := \sum_i \frac{\partial x^i}{\partial X^p} \frac{\partial x^i}{\partial X^q}\,.$$

From Flow to Map

Velocity field doesn't enter the Lagrangian equation directly: regard the time dependence in $\mathbb{D}$ as given by map rather than flow.

The solution of the A–D equation in Fourier space is then

$$\hat{\theta}_{\boldsymbol{k}}^{(i)} = \sum_{\boldsymbol{\ell}} \exp\bigl(\mathcal{G}^{(i)}\bigr)_{\boldsymbol{k}\boldsymbol{\ell}} \hat{\theta}_{\boldsymbol{\ell}}^{(i-1)} \,,$$

where i denotes the ith iterate of the map, and

$$\mathcal{G}_{\boldsymbol{k}\boldsymbol{\ell}}^{(i)} = -4\pi^2 T \int_{\mathbb{T}^2} (\boldsymbol{k} \cdot \mathbb{D}^{(i)} \cdot \boldsymbol{\ell})\, \mathrm{e}^{-2\pi\mathrm{i}(\boldsymbol{k}-\boldsymbol{\ell})\cdot\boldsymbol{X}}\, \mathrm{d}^2 X \,.$$

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This is an exact result, but the great difficulty lies in calculating the exponential of $\mathcal{G}^{(i)}$. We shall accomplish this perturbatively.

Back to the Beginning

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$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \boldsymbol{\phi}(\boldsymbol{x}) = \frac{\varepsilon}{2\pi}\begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

The eigenvalues of $\mathbb{M}$ are

$$\Lambda_{\mathrm{u}} = \Lambda = \tfrac{1}{2}(3+\sqrt{5}) = \cot^2\theta, \quad \Lambda_{\mathrm{s}} = \Lambda^{-1} = \tfrac{1}{2}(3-\sqrt{5}) = \tan^2\theta$$

and the corresponding eigenvectors,

$$(\hat{\mathbf{u}} \;\; \hat{\mathbf{s}}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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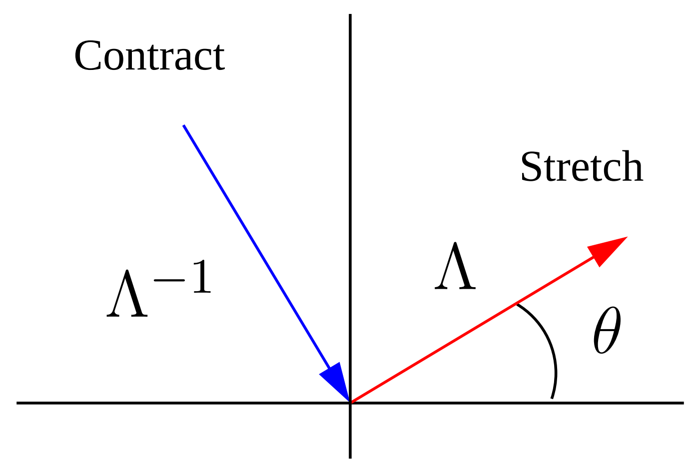

Coefficients of Expansion: Perturbation Theory

The coefficients of expansion and characteristic directions for the linear cat map are uniform in space. Perturb off this.

To leading order in ε, the coefficient of expansion is written as

$$\Lambda_\varepsilon^{(i)} = \Lambda^i \, (1 + \varepsilon \, \eta^{(i)})$$

where Λ is the coefficient of expansion for the unperturbed cat map; the perturbed eigenvectors are similarly written

$$\hat{\mathbf{u}}_\varepsilon^{(i)} = \hat{\mathbf{u}} + \varepsilon \, \zeta^{(i)} \, \hat{\mathbf{s}} \,, \qquad \hat{\mathbf{s}}_\varepsilon^{(i)} = \hat{\mathbf{s}} - \varepsilon \, \zeta^{(i)} \, \hat{\mathbf{u}} \,.$$

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Simple application of matrix perturbation theory to Jacobian matrix of the map. The symmetrised Jacobian is the metric:

$$g_\varepsilon^{(i)} = [\Lambda_\varepsilon^{(i)}]^2\, \hat{\mathbf{u}}_\varepsilon^{(i)} \hat{\mathbf{u}}_\varepsilon^{(i)} + [\Lambda_\varepsilon^{(i)}]^{-2}\, \hat{\mathbf{s}}_\varepsilon^{(i)} \hat{\mathbf{s}}_\varepsilon^{(i)}\,.$$

Perturbation Results

skip

$$\Lambda_\varepsilon^{(i)} = \Lambda^i\,(1+\varepsilon\,\eta^{(i)}), \qquad \hat{\mathbf{u}}_\varepsilon^{(i)} = \hat{\mathbf{u}} + \varepsilon\,\zeta^{(i)}\,\hat{\mathbf{s}}\,,$$

$$\eta^{(i)} = \tfrac{1}{2}\sin 2\theta \sum_{j=0}^{i-1} \cos\left(2\pi(\mathbb{M}^j \cdot \boldsymbol{X})_1\right);$$

$$\zeta^{(i)} = \frac{1}{\Lambda^{2i} - \Lambda^{-2i}}\,(\zeta_+^{(i)} + \zeta_-^{(i)}),$$

$$\zeta_\pm^{(i)} = \tfrac{1}{2}(\cos 2\theta \mp 1) \sum_{j=0}^{i-1} \Lambda^{\pm 2(i-j)} \cos\left(2\pi(\mathbb{M}^j \cdot \boldsymbol{X})_1\right).$$

Observe that the perturbation to the eigenvectors converges exponentially, as required.

Perturbed Metric Tensor

skip

$$\mathbb{D}^{(i)} = \kappa\,[g_\varepsilon^{(i)}]^{-1}; \qquad [g_\varepsilon^{(i)}]^{-1} = [\Lambda_\varepsilon^{(i)}]^2 \hat{\mathbf{s}}_\varepsilon^{(i)} \hat{\mathbf{s}}_\varepsilon^{(i)} + [\Lambda_\varepsilon^{(i)}]^{-2} \hat{\mathbf{u}}_\varepsilon^{(i)} \hat{\mathbf{u}}_\varepsilon^{(i)}\,.$$

To leading order in ε, we have

$$\begin{aligned}[g_\varepsilon^{(i)}]^{-1} = \Lambda^{2i}\,\hat{\mathbf{s}}\,\hat{\mathbf{s}} + \Lambda^{-2i}\,\hat{\mathbf{u}}\,\hat{\mathbf{u}} + 2\varepsilon\,\eta^{(i)}(\Lambda^{2i}\,\hat{\mathbf{s}}\,\hat{\mathbf{s}} - \Lambda^{-2i}\,\hat{\mathbf{u}}\,\hat{\mathbf{u}}) \\ - \varepsilon\,\zeta^{(i)}\left(\Lambda^{2i} - \Lambda^{-2i}\right)(\hat{\mathbf{u}}\,\hat{\mathbf{s}} + \hat{\mathbf{s}}\,\hat{\mathbf{u}}),\end{aligned}$$

where the only functions of $\boldsymbol{X}$ are $\eta^{(i)}$ and $\zeta^{(i)}$.

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where the only functions of $\boldsymbol{X}$ are $\eta^{(i)}$ and $\zeta^{(i)}$.

Recall the solution to the A–D equation:

$$\hat{\theta}_{\boldsymbol{k}}^{(i)} = \sum_{\boldsymbol{\ell}} \exp\big(\mathcal{G}^{(i)}\big)_{\boldsymbol{k}\boldsymbol{\ell}}\hat{\theta}_{\boldsymbol{\ell}}^{(i-1)}\,.$$

The Exponent $\mathcal{G}^{(i)}$ skip

$$\begin{aligned}\mathcal{G}^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} &= -4\pi^2 T \int_{\mathbb{T}^2} (\boldsymbol{k}\cdot\mathbb{D}^{(i)}\cdot\boldsymbol{\ell})\,\mathrm{e}^{-2\pi\mathrm{i}(\boldsymbol{k}-\boldsymbol{\ell})\cdot\boldsymbol{X}}\,\mathrm{d}^2X \\ &= A^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} + \varepsilon B^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}}\end{aligned}$$

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$$= A^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} + \varepsilon B^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}}$$

where

$$A^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} = -\kappa \left(\Lambda^{2i}\, k_{\mathrm{s}}^2 + \Lambda^{-2i}\, k_{\mathrm{u}}^2\right) \delta_{\boldsymbol{k}\boldsymbol{\ell}}, \qquad \kappa := 4\pi^2 \widetilde{\kappa}\, T$$

$$B^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} = -\kappa \Big(2\left(\Lambda^{2i}\, k_{\mathrm{s}}\, \ell_{\mathrm{s}} - \Lambda^{-2i}\, k_{\mathrm{u}}\, \ell_{\mathrm{u}}\right) \eta^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} - (k_{\mathrm{u}}\, \ell_{\mathrm{s}} + k_{\mathrm{s}}\, \ell_{\mathrm{u}}) \left({\zeta_+^{(i)}}_{\boldsymbol{k}\boldsymbol{\ell}} + {\zeta_-^{(i)}}_{\boldsymbol{k}\boldsymbol{\ell}}\right) \Big).$$

with $k_{\mathrm{u}} := (\boldsymbol{k} \cdot \hat{\mathbf{u}})$, $k_{\mathrm{s}} := (\boldsymbol{k} \cdot \hat{\mathbf{s}})$.

The Exponent $\mathcal{G}^{(i)} = A^{(i)} + \varepsilon B^{(i)}$ (cont'd)

skip

The diagonal part, $A^{(i)}$, inexorably leads to superexponential decay of variance, because it grows exponentially.

Upon making use of the Fourier-transformed $\zeta^{(i)}$ and $\eta^{(i)}$, we find

$$B^{(i)}_{\boldsymbol{k}\boldsymbol{\ell}} = -\tfrac{1}{2}\kappa \sum_{j=0}^{i-1} \mathcal{B}^{ij}_{\boldsymbol{k}\boldsymbol{\ell}} \left(\delta_{\boldsymbol{k},\boldsymbol{\ell}+\hat{\mathbf{e}}_1\cdot\mathbb{M}^j} + \delta_{\boldsymbol{k},\boldsymbol{\ell}-\hat{\mathbf{e}}_1\cdot\mathbb{M}^j}\right)$$

$$\begin{aligned}\mathcal{B}^{ij}_{\boldsymbol{k}\boldsymbol{\ell}} = \sin 2\theta \left(\Lambda^{2i}\, k_{\mathrm{s}}\, \ell_{\mathrm{s}} - \Lambda^{-2i}\, k_{\mathrm{u}}\, \ell_{\mathrm{u}}\right) \\ + \left(k_{\mathrm{u}}\, \ell_{\mathrm{s}} + k_{\mathrm{s}}\, \ell_{\mathrm{u}}\right)\left(\Lambda^{2(i-j)} \sin^2\theta - \Lambda^{-2(i-j)} \cos^2\theta\right).\end{aligned}$$

So $B^{(i)}$ is not diagonal (it couples different modes to each other).

$\Longrightarrow$ Dispersive in Fourier space.

But can we Compute the Exponential, $\exp(\mathcal{G}^{(i)})$?

To leading order in ε, for A diagonal, we have $\mathcal{G}^{(i)} = A^{(i)} + \varepsilon B^{(i)}$,

$$[\exp(A^{(i)}+\varepsilon B^{(i)})]_{\boldsymbol{k\ell}} = \mathrm{e}^{A^{(i)}_{\boldsymbol{kk}}}\,\delta_{\boldsymbol{k\ell}}+\varepsilon E^{(i)}_{\boldsymbol{k\ell}}; \qquad E^{(i)}_{\boldsymbol{k\ell}} = B^{(i)}_{\boldsymbol{k\ell}}\,\frac{\mathrm{e}^{A^{(i)}_{\boldsymbol{kk}}} - \mathrm{e}^{A^{(i)}_{\boldsymbol{\ell\ell}}}}{A^{(i)}_{\boldsymbol{kk}} - A^{(i)}_{\boldsymbol{\ell\ell}}}\,.$$

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- For $\varepsilon = 0$, this is indeed what happens. But for a finite value of ε, the E term breaks the diagonality of $\mathcal{G}$, so that given some initial set of wavevectors, the variance contained in those modes can be transferred elsewhere.

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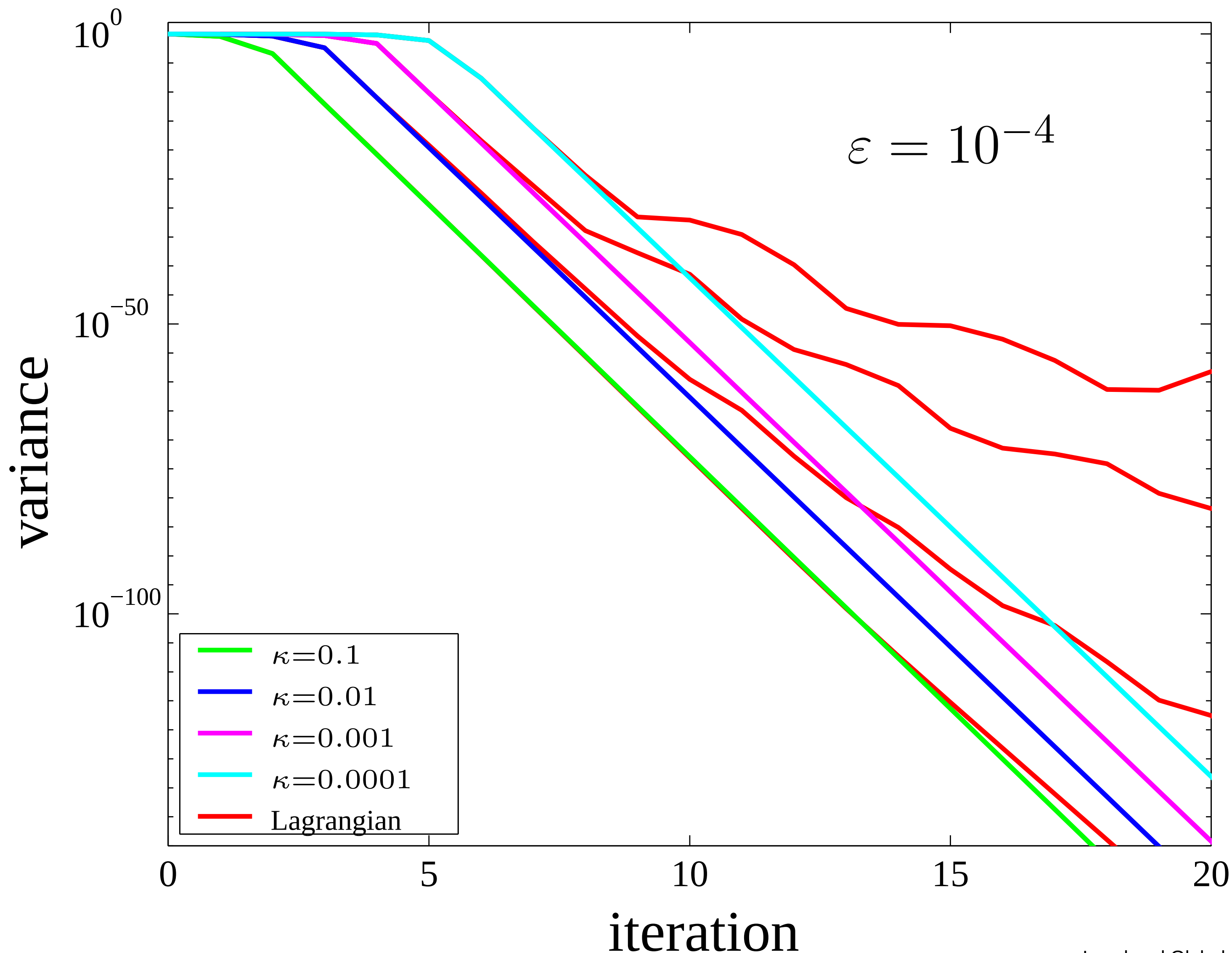
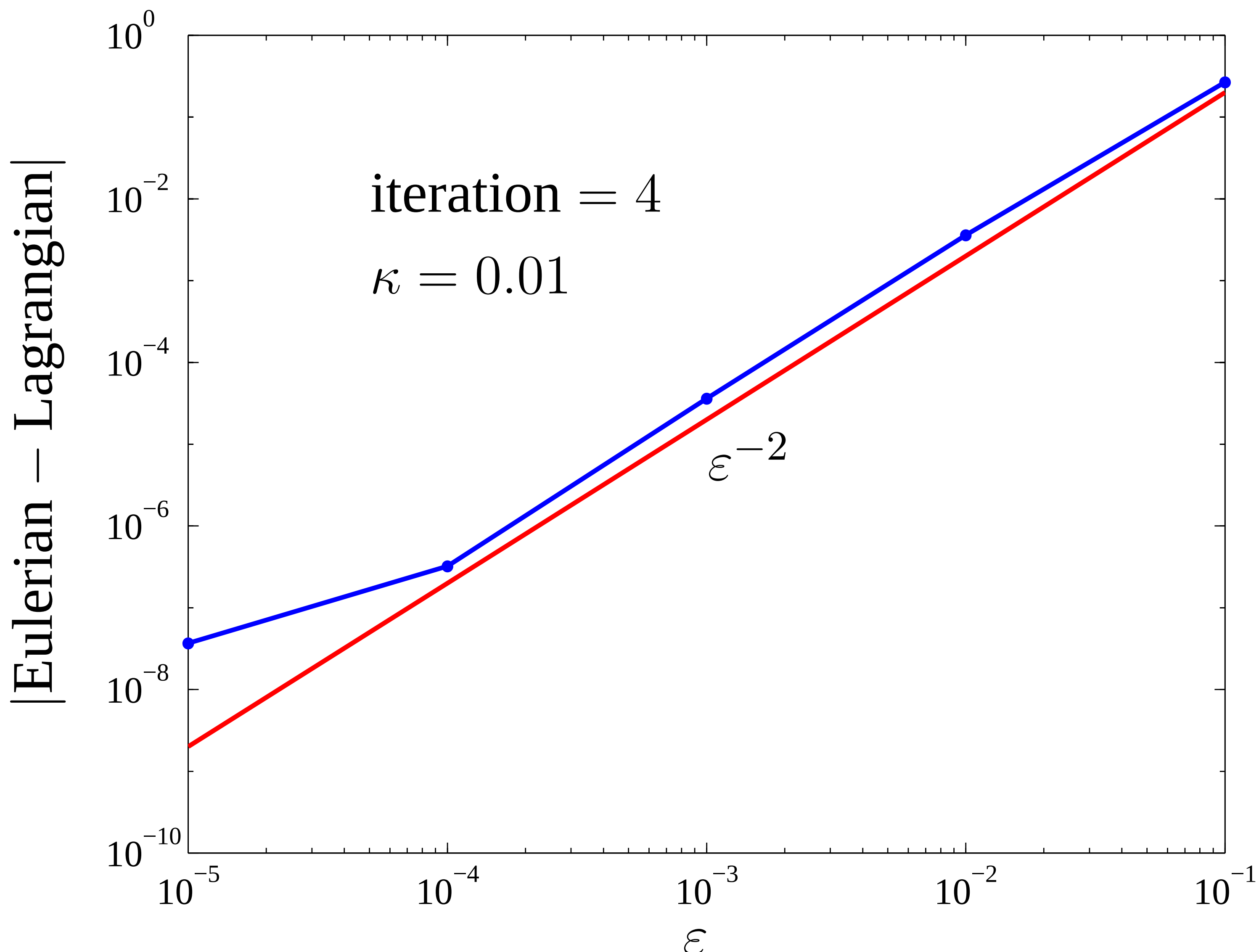
- Just let computer take care of pruning via underflow!
- The surviving modes need to become more and more aligned with $\hat{\mathbf{u}}$ as time goes on.

Comparison: Eulerian and Lagrangian Views

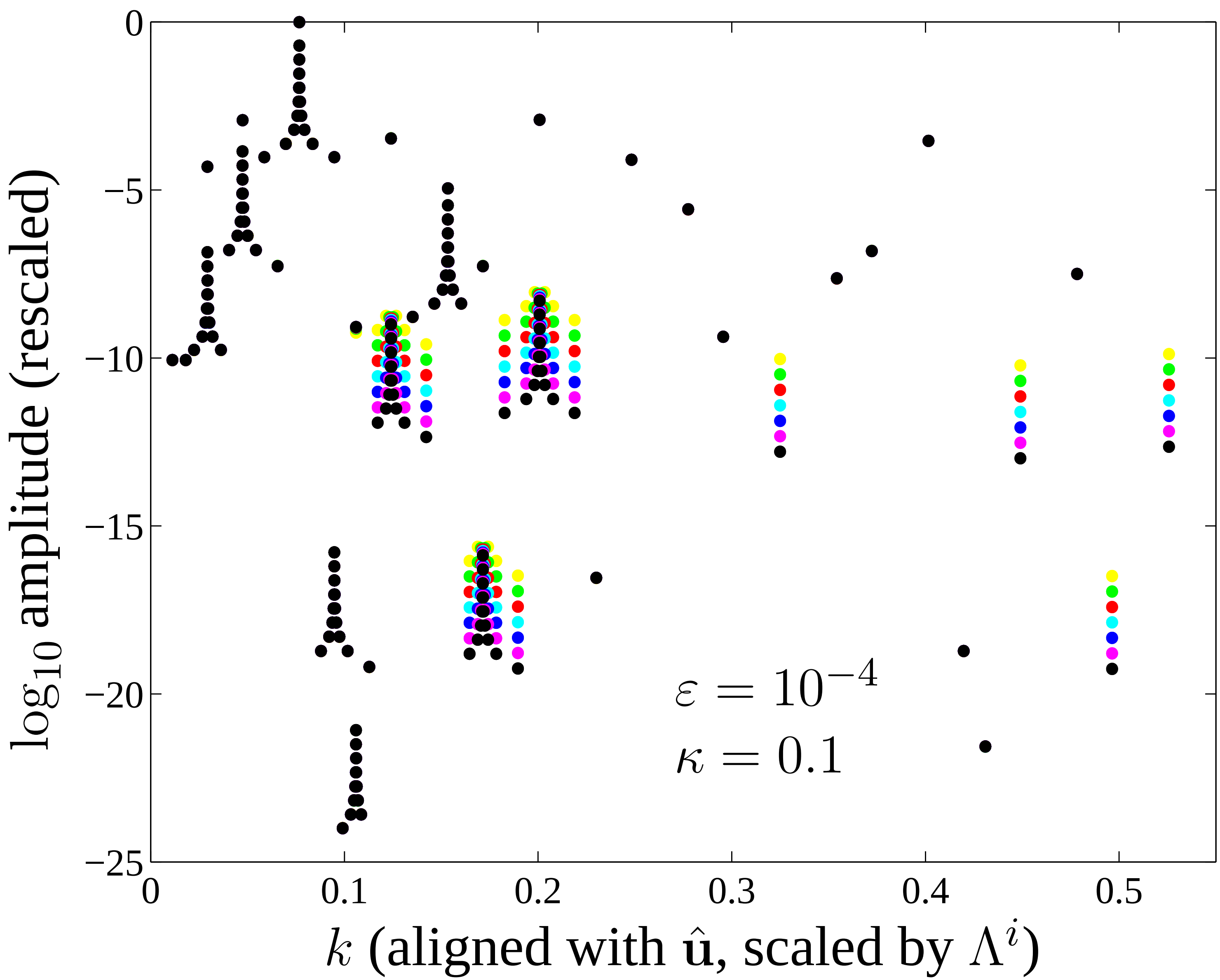


Convergence

skip



Rescaled Pattern for $i = 6, \ldots, 12$



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