Braids and Dynamics

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

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part 1
periodic motion

Taffy is a type of candy.

Needs to be pulled: this aerates it and makes it lighter and chewier.

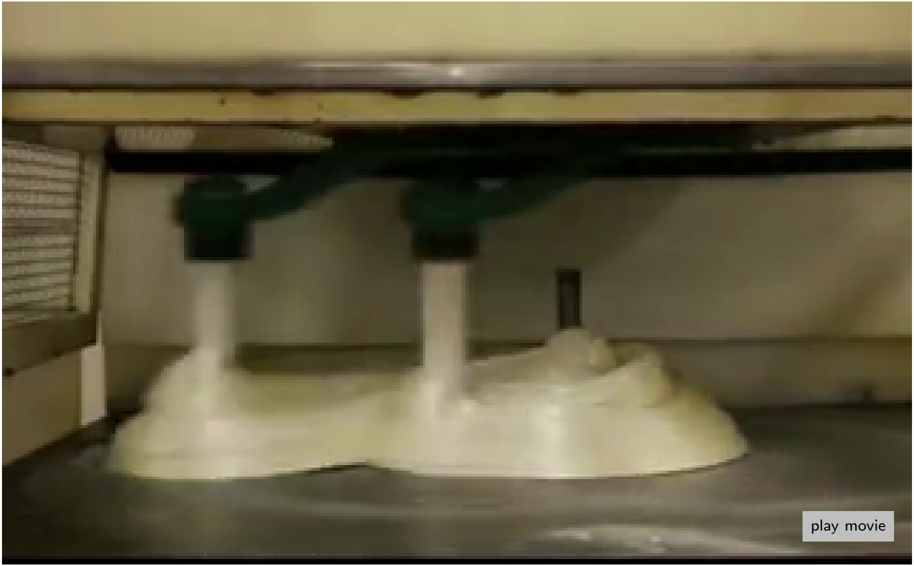
We can assign a growth: length multiplier per period.

(Here $(1+\sqrt{2})^2\ldots$ more later.)

[movie by M. D. Finn]

play movie

making candy cane



[*Wired:* This Is How You Craft 16,000 Candy Canes in a Day]

four-pronged taffy puller

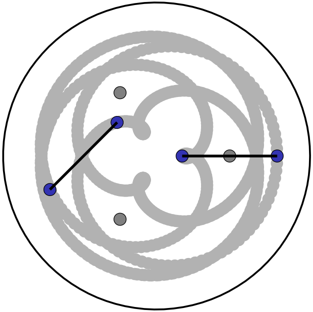
play movie

http://www.youtube.com/watch?v=Y7tlHDsquVM

[MacKay (2001); Halbert & Yorke (2014)]

the mixograph

Experimental device for kneading bread dough:



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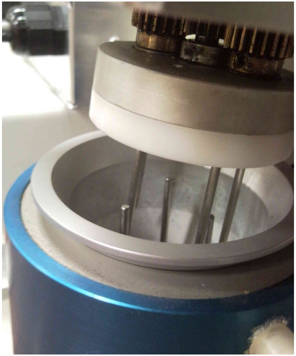
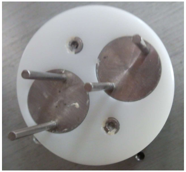
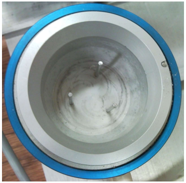






[Department of Food Science, University of Wisconsin. Photos by J-LT.]

For a *lot* more, see ‘The Mathematics of Taffy Pullers,’ Thiffeault, J.-L. (2018). *Math. Intelligencer,* **40** (1), 26–35. arXiv:1608.00152.

Encode the topological information as a sequence of generators of the Artin braid group B_n.

Equivalent to the 7-braid

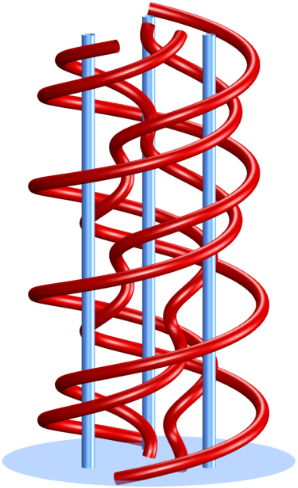
$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

The growth is the largest root of

$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1$$

$\simeq 4.186$

Compare to taffy pullers: 5.828



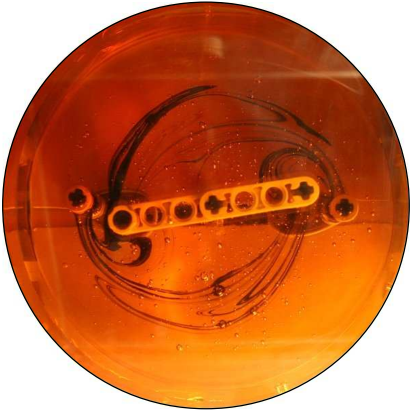
braids and rod-stirring

play movie

play movie

[Boyland, P. L., Aref, H., & Stremler, M. A. (2000). *J. Fluid Mech.* **403**, 277–304; Simulations by M. D. Finn, S. E. Tumasz, and J-LT.]

$4 + 1$ rods



play movie

[Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743]

mathematical description

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained by solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods.

Goal: Topological characterization of φ.

[The theory extends to handlebodies, but not as relevant for applications. . .]

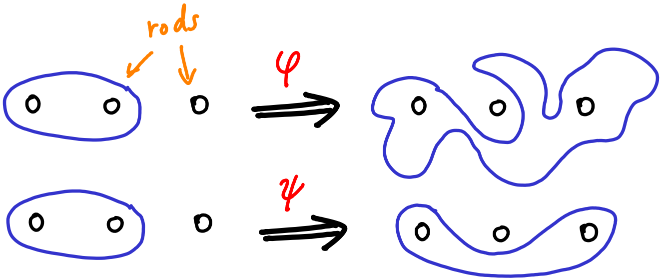
three main ingredients

1. The Thurston–Nielsen classification theorem (idealized φ);
2. Handel's isotopy stability theorem (link to real φ);
3. Topological entropy (quantitative measure of mixing).

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

(Defines isotopy classes.)

Convenient to think of isotopy in terms of material loops. Isotopic maps act the same way on loops (up to continuous deformation).



(Loops will always mean essential loops.)

Thurston–Nielsen classification theorem

Theorem

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

- *finite-order* — *for some integer $k > 0$, $\psi^k \simeq$ identity;*
- *reducible* — *ψ leaves invariant a disjoint union of essential simple closed curves, called reducing curves;*
- *pseudo-Anosov* — *ψ leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\psi(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\psi(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda > 1$.*

The three categories characterize the isotopy class of φ.

We want pseudo-Anosov for good mixing.

Handel's isotopy stability theorem

The TN classification tells us about a simpler map ψ, the TN representative. What about the original map φ?

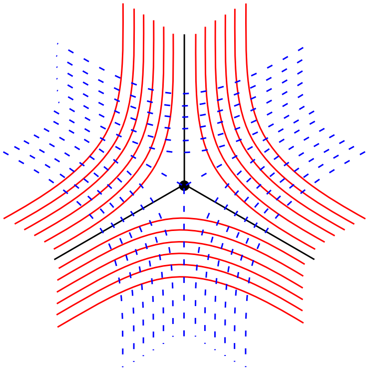
Theorem (Handel, 1985)

If ψ is pseudo-Anosov and isotopic to $\varphi : \mathcal{S} \to \mathcal{S}$, then there is a compact, φ-invariant set, $\mathcal{Y} \subset \mathcal{S}$, and a continuous, onto mapping $\alpha : \mathcal{Y} \to \mathcal{S}$, so that $\alpha\varphi = \psi\alpha$.

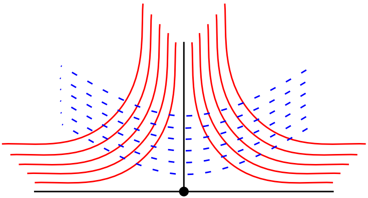
This is called a semiconjugacy (α not generally invertible).

Succinctly: the dynamics of the pseudo-Anosov map 'survive' isotopy, and so φ is at least as complicated as ψ. (In particular, it has at least as much topological entropy.)

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

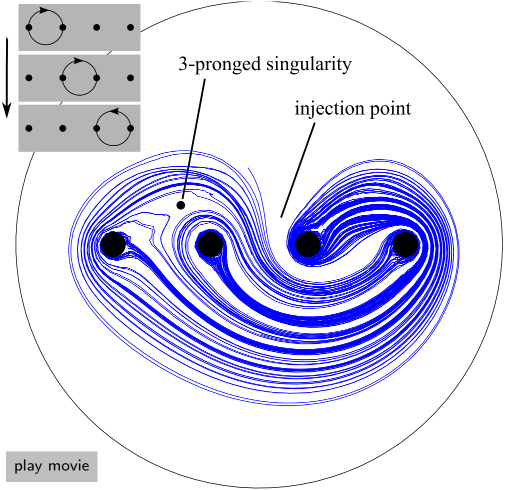


3-pronged singularity



Boundary singularity

visualizing a singular foliation



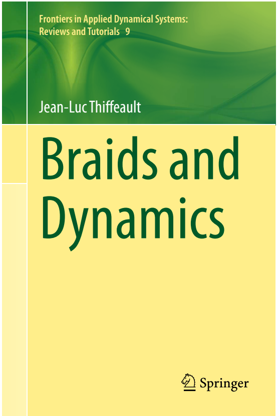
play movie

- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

[Thiffeault, J.-L., Finn, M. D., Gouillart, E., & Hall, T. (2008). *Chaos*, **18**, 033123]

the topological program

- Consider a motion of stirring elements, such as rods.
- Determine if the motion is isotopic to a pseudo-Anosov mapping.
- Compute topological quantities, such as foliation, entropy, etc.
- Analyze and optimize.



[shameless plug for new book]

insight: do we need the rods?

[Gouillart, E., Kuncio, N., Dauchot, O., Dubrulle, B., Roux, S., & Thiffeault, J.-L. (2007). *Phys. Rev. Lett.* **99**, 114501]

play movie

ghost rods (‘tiges fantômes’)

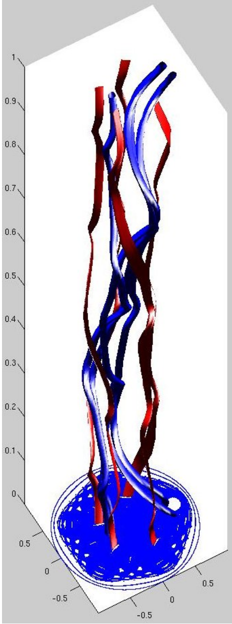
Topological analysis can be done on other objects than rods – for instance, islands or unstable periodic orbits.

We simply follow the islands and examine the braid they form, which gives us bounds on topological entropy.

In this framework we call the islands ghost rods.

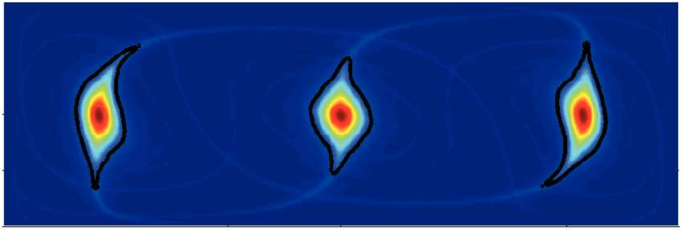
[Gouillart, E., Finn, M. D., & Thiffeault, J.-L. (2006). *Phys. Rev. E,* **73**, 036311]

[implemented by Stremler & Chen (2007); Thiffeault *et al.* (2009); Binder (2010); Stremler *et al.* (2011)]



ghost rods (cont'd)

One of the best examples of ghost rods is from Stremler *et al.* (2011):



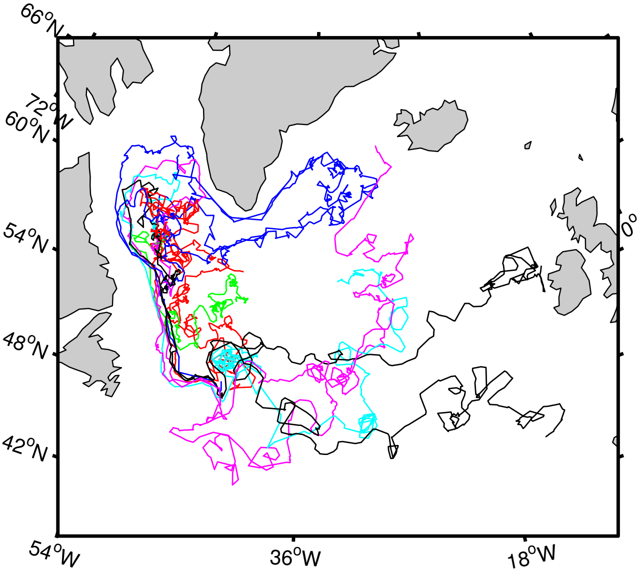
The islands are made to follow the $\sigma_2\sigma_1^{-1}$ stirring protocol by clever wall motions! (viscous Stokes flow)

[Stremler, M. A., Ross, S. D., Grover, P., & Kumar, P. (2011). *Phys. Rev. Lett.* **106**, 114101]

play movie

part 2
non-periodic motion

oceanic float trajectories



oceanic floats: data analysis

What can we measure?

- single-particle dispersion (not a good use of all data)
- correlation functions (useful)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the braid group generators σ_i for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent, or to the 'growth' of taffy pullers).

iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

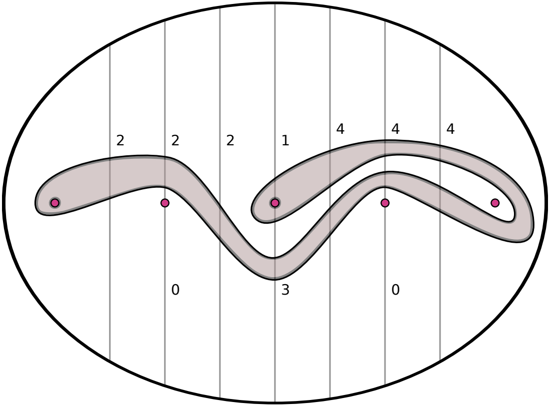
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

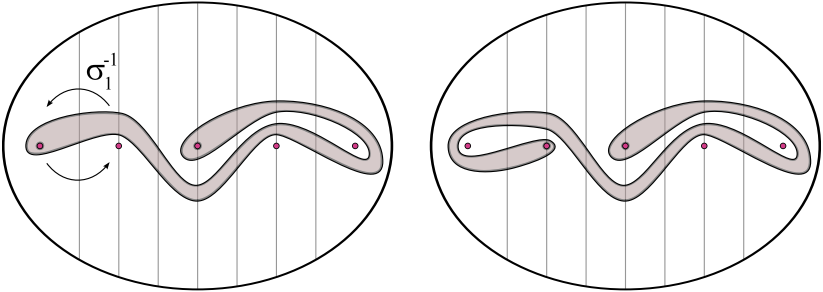
solution to problem 1: loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



solution to problem 2: action on coordinates

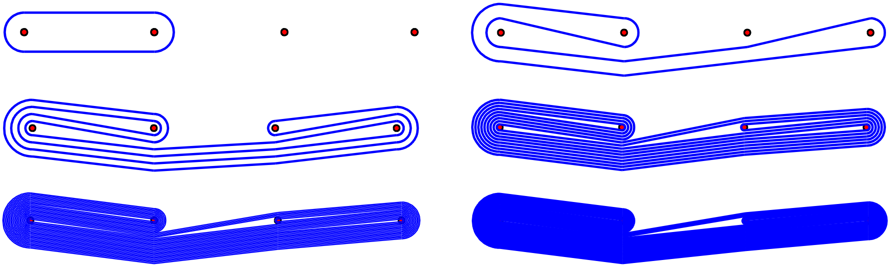
Moving the punctures according to a braid generator changes some crossing numbers:



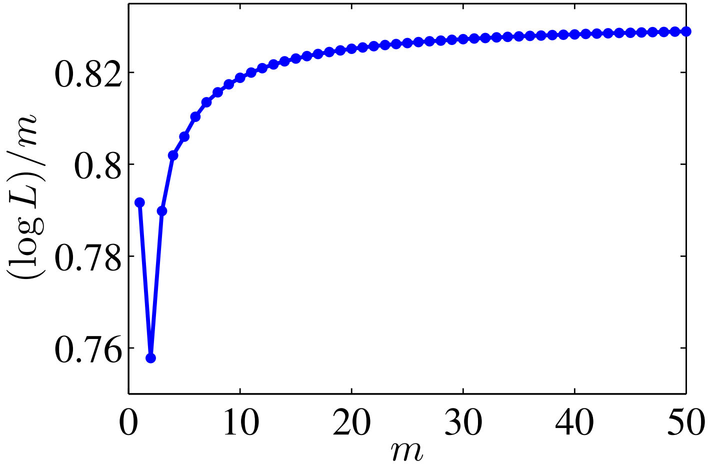
There is an explicit formula for the change in the coordinates! [Dynnikov (2002); Moussafir (2006); Hall & Yurttaş (2009); Thiffeault (2010)]

growth of L

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:



growth of L (2)

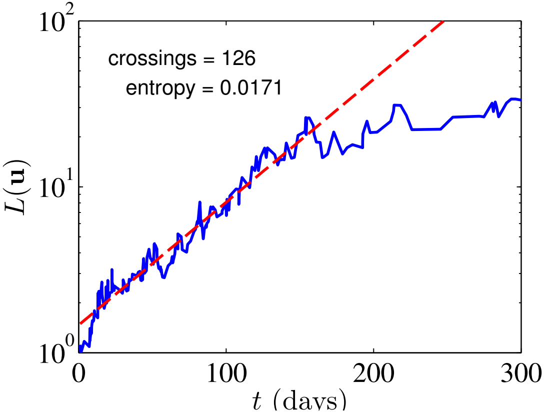


m is the number of times the braid acted on the initial loop.

[Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* **1** (1), 37–46]

oceanic floats: entropy

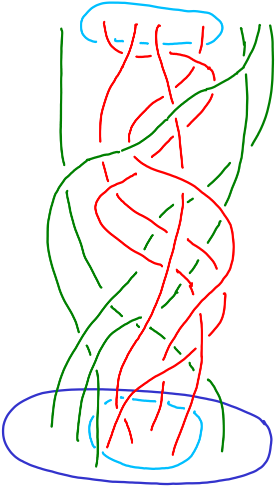
10 floats from Davis' Labrador sea data:



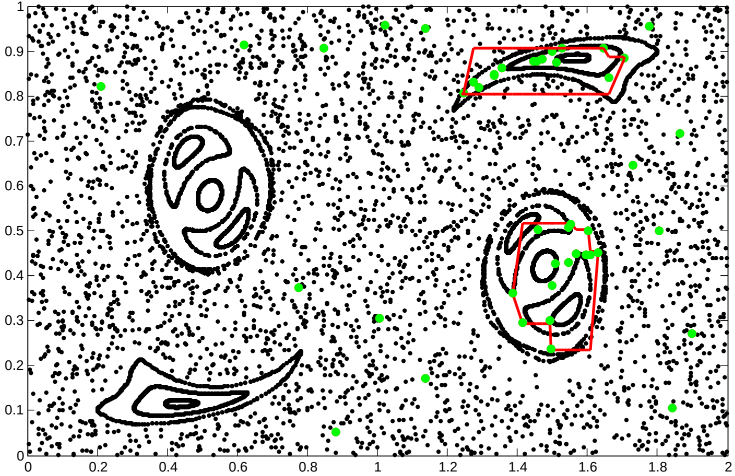
Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)

Lagrangian Coherent Structures

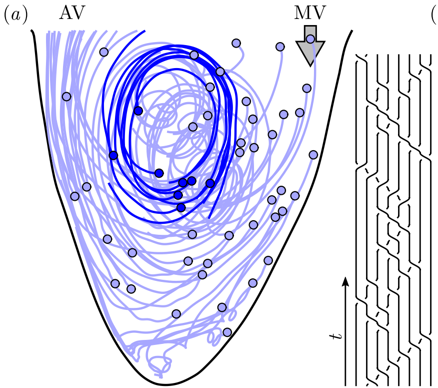


- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- [see Haller, G. & Beron-Vera, F. J. (2012). *Physica D*, **241** (20), 1680–1702.]
- Topological approach: [Allshouse & Thiffeault (2012); Filippi *et al.* (2020); Yeung *et al.* (2020)].

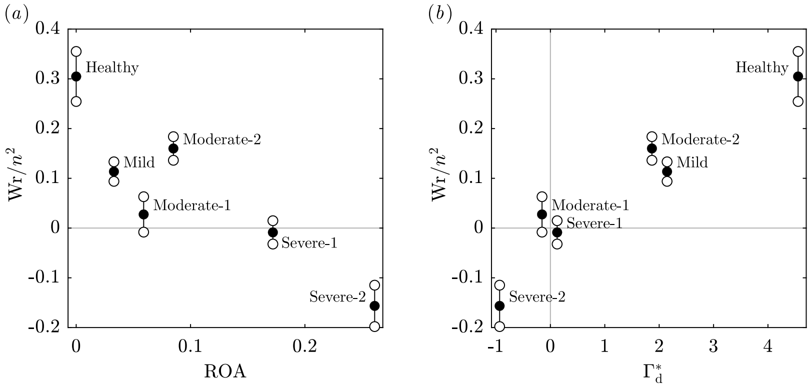


play movie [Allshouse, M. R. & Thiffeault, J.-L. (2012). *Physica D,* **241** (2), 95–105]

braids in the heart



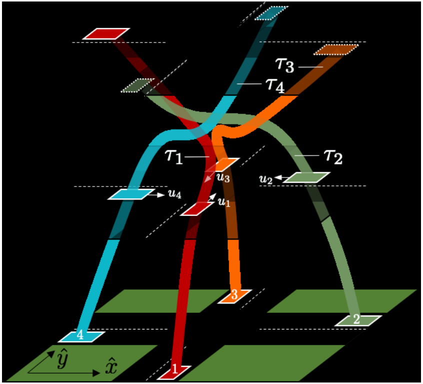
[Di Labbio, G., Thiffeault, J.-L., & Kadem, L. (2022). *Flow,* **2**, E12]



Wr	writhe of braid
ROA	Regurgitant Orifice Area
Γ_d^*	circulation

[Di Labbio, G., Thiffeault, J.-L., & Kadem, L. (2022). *Flow*, **2**, E12]

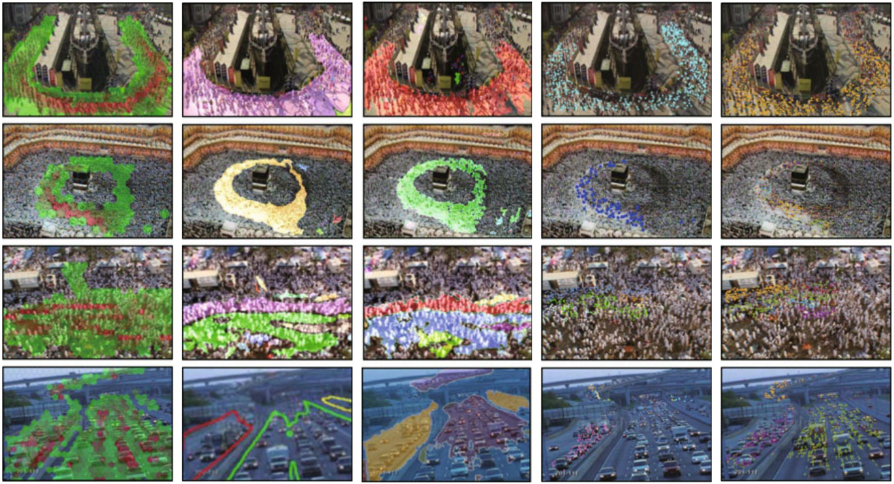
multiagent modeling: cars at an intersection



[Mavrogiannis, C., DeCastro, J., & Srinivasa, S. (2022). preprint]

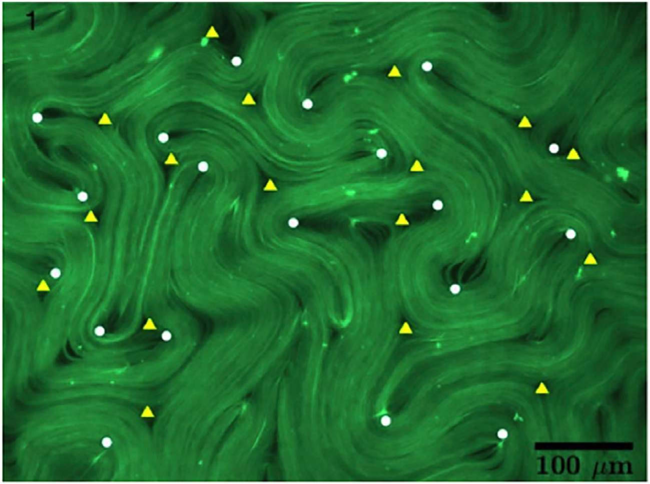
This view of agent coordination is closely linked to Rob Ghrist's work on configuration space (U. Penn).

complexity of crowd movement



[Akpulat, M. & Ekinci, M. (2019). *Frontiers of Information Technology & Electronic Engineering*, **20** (6), 849–861]

braiding in active nematics



[Smith, S. A. & Gong, R. (2022). *Frontiers in Physics,* **10**]

some research directions

- We don't have solid theory for aperodic or open braids.
- Computational methods for isotopy class (random entanglements of trajectories – LCS method, see Allshouse & Thiffeault (2012); Filippi *et al.* (2020); Yeung *et al.* (2020).
- 'Designing' for topological chaos (see Stremler & Chen (2007)).
- Combine with other measures, e.g., mix-norms (Mathew *et al.*, 2005; Lin *et al.*, 2011; Thiffeault, 2012).
- Matlab toolbox — `https://github.com/jeanluct/braidlab`.
- 3D?! (lots of missing theory; E-Tec approach shows promise [Roberts, E., Sindi, S., Smith, S. A., & Mitchell, K. A. (2019). *Chaos,* **29** (1), 013124]).

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