pseudo-Anosovs with small dilatation

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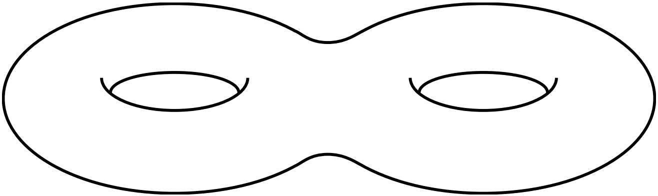
Spring Topology and Dynamics Conference,
Mississippi State University, 19 March 2010

Surface homeomorphisms

homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a compact orientable surface without boundary, such as 2-torus:



φ and ψ are isotopic if ψ can be continuously 'reached' from φ. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

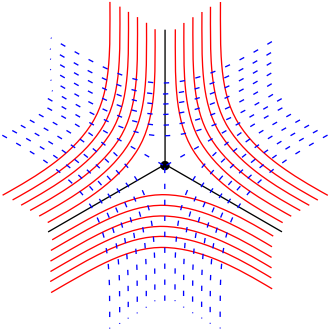
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

Focus on number 3 (most interesting case).

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.



3-pronged singularity

The Minimizer problem (‘Systole’)

- On a given surface $\mathcal{S}$, which pA has the least λ?
- Known to exist (Thurston);
- Punctured discs: Known for $n = 3$ to 7 [Song et al. (2002); Ham & Song (2007); Lanneau & Thiffeault (2009a,b, 2010)];
- Minimizer is simple for n odd [Hironaka & Kin (2006)], though not proved in general;
- Surfaces: known for genus 2 [Zhirov (1995); Cho & Ham (2008); Lanneau & Thiffeault (2010)].

Orientable minimizer

- No punctures: surface of genus g;
- If the foliation is orientable, then things are much simpler;
- Action of the pA on first homology captures dilatation λ;
- Polynomials of degree $2g$;
- Procedure:
  - We have a guess for the minimizer;
  - Find all integer-coefficient, reciprocal polynomials that have largest root smaller than λ;
  - Show that they can't correspond to pAs;
  - For the smallest one that can, construct pA.
- To appear in *Ann. Inst. Fourier* (2010). See also article in Dynamical Systems Magazine.

Newton's formulas

We need an efficient way to bound the number of polynomials with largest root smaller than λ. Given a reciprocal polynomial

$$P(x) = x^{2g} + a_1\, x^{2g-1} + \ldots + a_2\, x^2 + a_1\, x + 1$$

we have Newton's formulas for the traces,

$$\mathrm{Tr}(\phi_*^k) = -\sum_{m=1}^{k-1} a_m \mathrm{Tr}(\phi_*^{k-m}) - k a_k,$$

where

- ϕ is a (hypothetical) pA associated with $P(x)$;
- ϕ_* is the matrix giving the action of the pA ϕ on first homology;
- $\mathrm{Tr}(\phi_*)$ is its trace.

Bounding the traces

The trace satisfies

$$|\mathrm{Tr}(\phi_*^k)| = \left|\sum_{m=1}^{g} (\lambda_m^k + \lambda_m^{-k})\right| \leq g(r^k + r^{-k})$$

where λ_m are the roots of ϕ_*, and $r = \max_m(|\lambda_m|)$.

- Bound $\mathrm{Tr}(\phi_*^k)$ with $r < \lambda$, $k = 1, \ldots, g$;
- Use these g traces and Newton's formulas to construct candidate $P(x)$;
- Overwhelming majority have fractional coeffs $\rightarrow$ discard!
- Carefully check the remaining polynomials:
  - Is their largest root real?
  - Is it strictly greater than all the other roots?
  - Is it really less than λ?
- Largest tractable case: $g = 8$ (10^{12} polynomials).

Lefschetz's fixed point theorem

This procedure still leaves a fair number of polynomials — though not enormous (10's to 100's, even for $g = 8$.)
The next step involves using Lefschetz's fixed point theorem to eliminate more polynomials:

$$L(\phi) = 2 - \mathrm{Tr}(\phi_*) = \sum_{p \in \mathrm{Fix}(\phi)} \mathrm{Ind}(\phi, p)$$

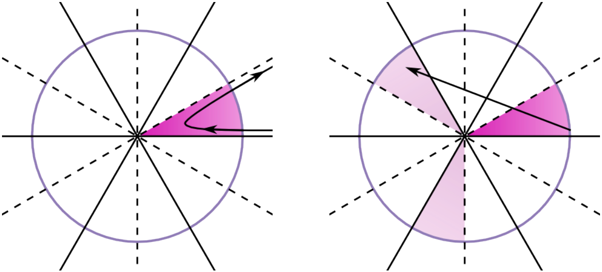
where

- $L(\phi)$ is the Lefschetz number;
- $\mathrm{Fix}(\phi)$ is set of fixed points of ϕ;
- $\mathrm{Ind}(\phi, p)$ is index of ϕ at p.

We can easily compute $L(\phi^k)$ for every iterate using Newton's formula.

Topological index at a fixed point

The index is defined as the number of turns of a vector joining x to $\phi(x)$ as x travels counterclockwise around a small circle.



For this case, each sector can map to itself (left, index $1 - 6 = -5$) or to one of two other sectors (right, index $+1$).

Eliminating polynomials

Outline of procedure: for a surface of genus g,

- Use the Euler–Poincaré formula to list possible singularity data for the foliations;
- For each singularity data, compute possible contributions to the index (depending on how the singularities and their separatrices are permuted);
- Check if index is consistent with Lefschetz's theorem.

With this, we can reduce the number of polynomials to one or two!

Minimizers for orientable foliations

Our approach gives lower bound. Construct the pA explicitly to prove minimal.

g	polynomial	minimizer
2	$X^4 - X^3 - X^2 - X + 1$	1.72208 [1]
3	$X^6 - X^4 - X^3 - X^2 + 1$	1.40127 [2]
4	$X^8 - X^5 - X^4 - X^3 + 1$	1.28064 [2]
5	$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$	1.17628 [3]
6	$X^{12} - X^7 - X^6 - X^5 + 1$	$\gtrsim$ 1.17628 [4]
7	$X^{14} + X^{13} - X^9 - X^8 - X^7 - X^6 - X^5 + X + 1$	1.11548 [5]
8	$X^{16} - X^9 - X^8 - X^7 + 1$	1.12876 [6]

[1] Zhirov (1995)'s result; also for nonorientable [Lanneau–T];

[2] Constructed by Lanneau–T;

[3] Lehmer's number; constructed by Leininger (2004)'s pA;

[4] Genus 6 is the first nondecreasing case. No explicitly construction of pA;

[5] Constructed by Aaber & Dunfield (2010);

[6] Constructed by Hironaka (2009).

Question

Examining the cases with even g leads to a natural question:

> *Is the minimum value of the dilatation of pseudo-Anosov homeomorphisms on a genus g surface, for g even, with orientable invariant foliations, equal to the largest root of the polynomial $X^{2g} - X^{g+1} - X^{g} - X^{g-1} + 1$?*

This would imply that the minimum dilatation asymptotes to $(\text{Golden ratio})^{2/g}$ for $g \gg 1$.

This appears to be the 'sparsest' reciprocal polynomial that also satisfies the Lefschetz formula. Don't know the pA in general, however.

This work was supported by the Division of Mathematical Sciences of the US National Science Foundation, under grant DMS-0806821.

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