

Random entanglements

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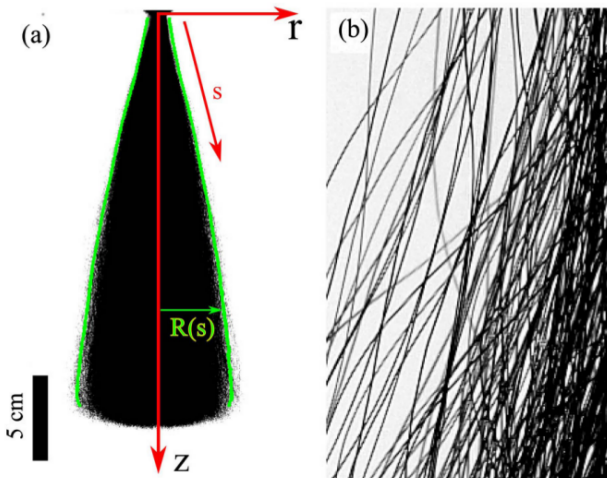


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A dense, chaotic tangle of various colored wires and cables, illustrating complex entanglements. The wires are of many colors, including black, white, grey, blue, red, and yellow, and are intertwined in a complex, overlapping pattern. The background is a dark, textured surface, possibly a wall or a large piece of equipment. The overall appearance is one of extreme complexity and disorder.

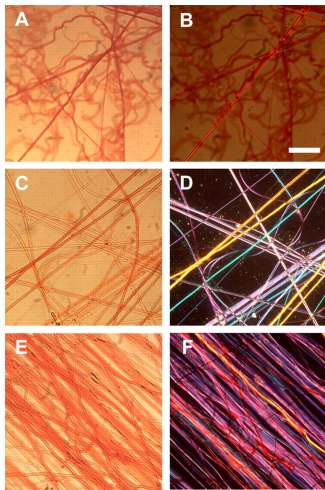
Complex entanglements are everywhere

Tangled hair



[Goldstein, R. E., Warren, P. B., & Ball, R. C. (2012). *Phys. Rev. Lett.* **108**, 078101]

Tangled hagfish slime

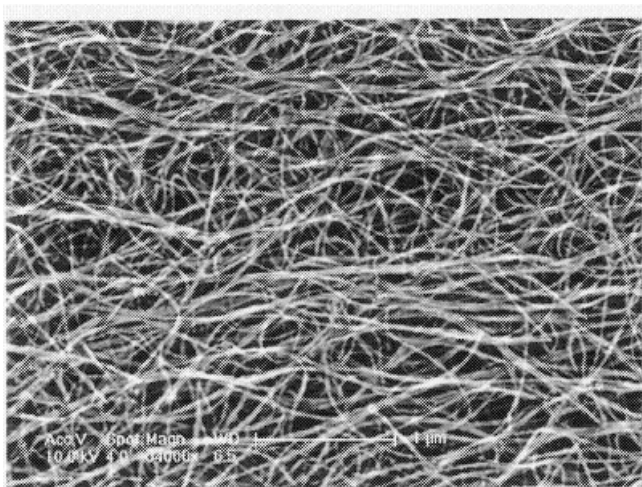


Slime secreted by **hagfish** is made of microfibers.

The quality of entanglement determines the material properties (**rheology**) of the slime.

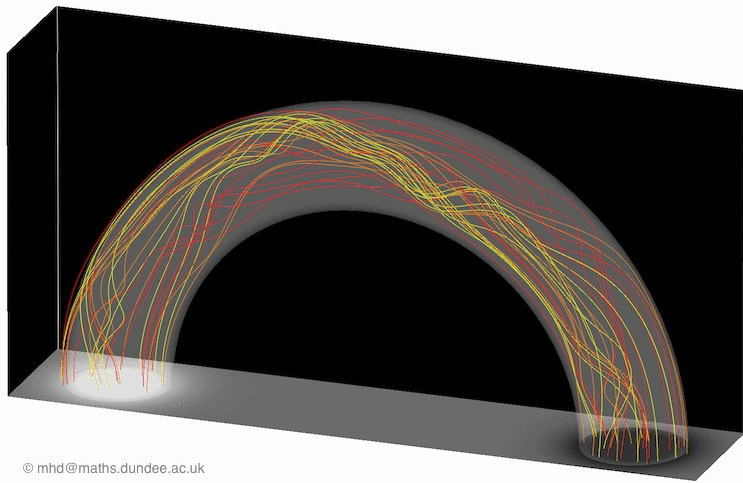
[Fudge, D. S., Levy, N., Chiu, S., & Gosline, J. M. (2005). *J. Exp. Biol.* **208**, 4613–4625]

Tangled carbon nanotubes



[Source: <http://www.ineffableisland.com/2010/04/carbon-nanotubes-used-to-make-smaller.html>]

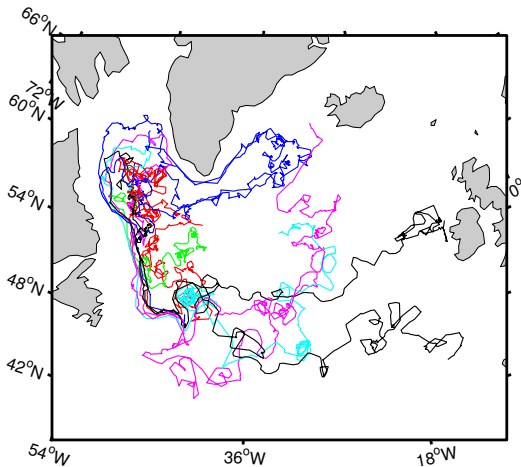
Tangled magnetic fields



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[Source: <http://www.maths.dundee.ac.uk/mhd/>]

Tangled oceanic float trajectories

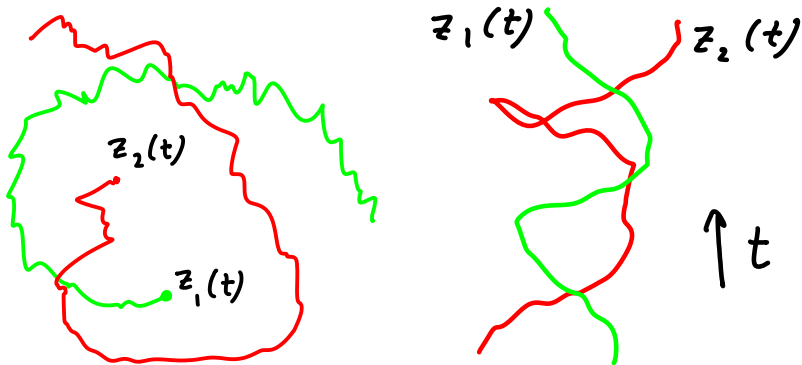


[Source: WOCE subsurface float data assembly center, <http://wfdac.whoi.edu>, J-LT (2010). *Chaos*, **20**, 017516]

The simplest tangling problem

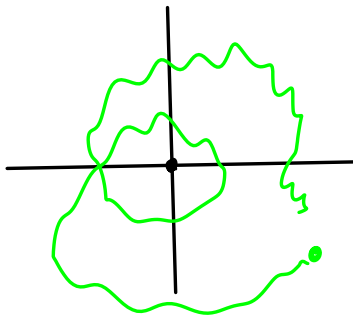


Consider two Brownian motions on the complex plane, each with diffusion constant D :



Viewed as a spacetime plot, these form a 'braid' of two strands.

Take the vector $z(t) = z_1(t) - z_2(t)$, which behaves like a Brownian particle of diffusivity $2D$ ($\rightarrow D$):



Define $\theta \in (-\infty, \infty)$ to be the **total winding angle** of $z(t)$ around the origin.

Spitzer (1958) found the time-asymptotic distribution of θ to be **Cauchy**:

$$P(x) \sim \frac{1}{\pi} \frac{1}{1+x^2}, \quad x := \frac{\theta}{\log(2\sqrt{Dt}/r_0)}, \quad 2\sqrt{Dt}/r_0 \gg 1,$$

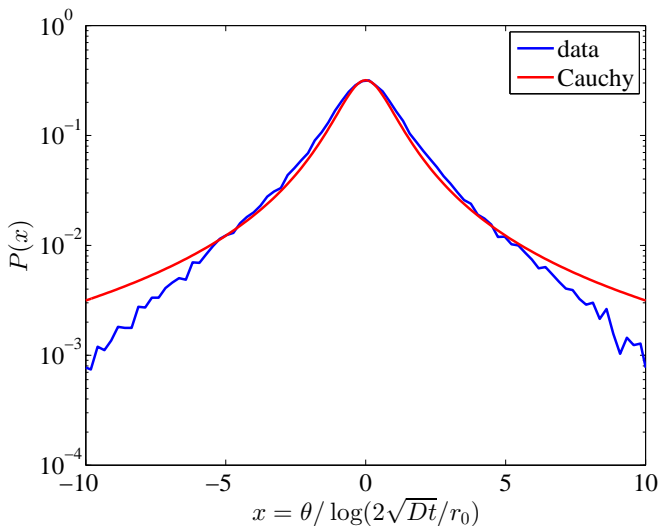
where $r_0 = |z(0)|$.

The scaling variable is $\sim \theta / \log t$.

Note that a Cauchy distribution is a bit strange: the variance is infinite, so **large windings are highly probable!**

[Spitzer, F. (1958). *Trans. Amer. Math. Soc.* **87**, 187–197]

Winding angle distribution: numerics



(Well, the tails don't look great: a pathology of Brownian motion.)

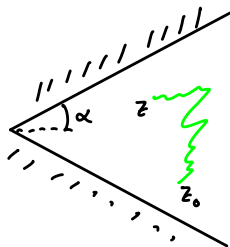
Winding angle distribution: derivation



The probability distribution $P(z, t)$ of the Brownian process satisfies the heat equation:

$$\frac{\partial P}{\partial t} = D\Delta P, \quad P(z, 0) = \delta(z - z_0).$$

Consider the solution in a **wedge** of half-angle α :



(Take either reflecting or absorbing boundary condition at the walls.)



The solution is standard, but now take the wedge angle α to ∞ (!):

$$P(z, t) = \frac{1}{2\pi Dt} e^{-(r^2+r_0^2)/4Dt} \int_0^\infty \cos \nu(\theta - \theta_0) I_\nu\left(\frac{r r_0}{2Dt}\right) d\nu$$

where I_ν is a modified Bessel function of the first kind, and r, θ are the polar coordinates of $z = x + iy$.

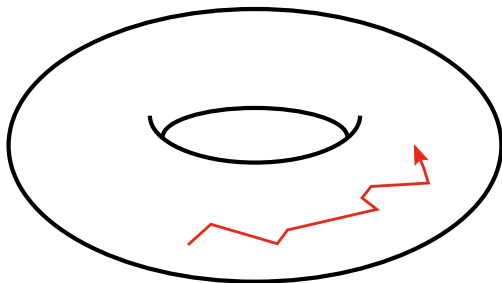
For large t this recovers the Cauchy distribution.

Key point: by allowing the wedge angle to infinity, we are using Riemann sheets to keep track of the winding angle.

Related example: Brownian motion on the torus



A Brownian motion on a torus can wind around the **two periodic directions**:



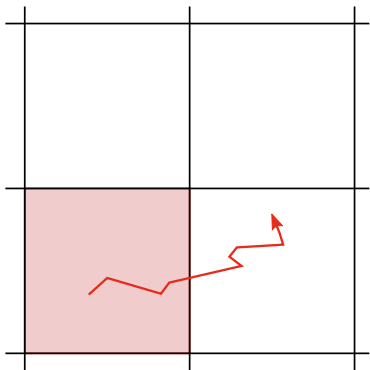
What is the **asymptotic distribution of windings**?

Mathematically, we are asking what is the **homology class** of the motion?

Torus: universal cover



We pass to the **universal cover** of the torus, which is the plane:



The universal cover records the windings as paths on the plane. The original 'copy' is called the **fundamental domain**.

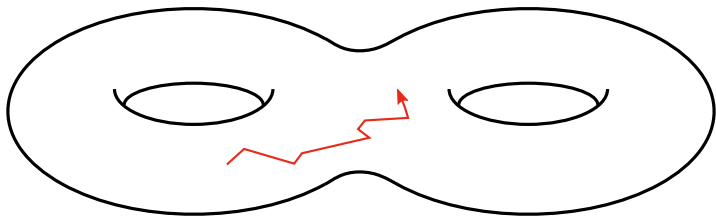
On the plane the probability distribution is the usual **Gaussian heat kernel**:

$$P(x, y, t) = \frac{1}{4\pi Dt} e^{-(x^2+y^2)/4Dt}$$

So here $m = \lfloor x \rfloor$ and $n = \lfloor y \rfloor$ will give the **homology class**: the number of windings of the walk in each direction.

We can think of the motion as **entangling with the space itself**.

On a **genus two surface** (double-torus):

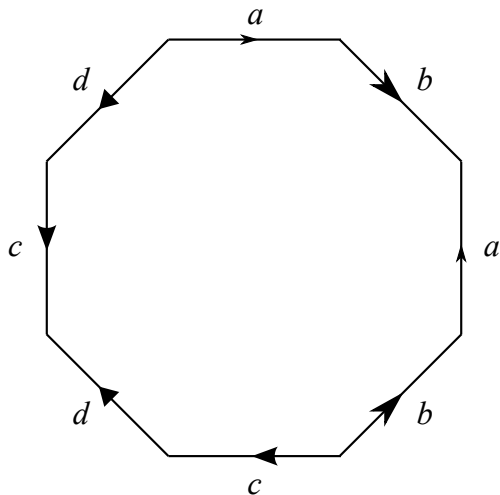


Same question: **what is the entanglement of the motion with the space after a long time?**

Now **homology classes** are not enough, since the associated universal cover has a **non-Abelian** group of deck transformations. In other words, the **order** of going around the holes matters!

The non-Abelian case involves **homotopy classes**.

The 'stop sign' representation of the double-torus



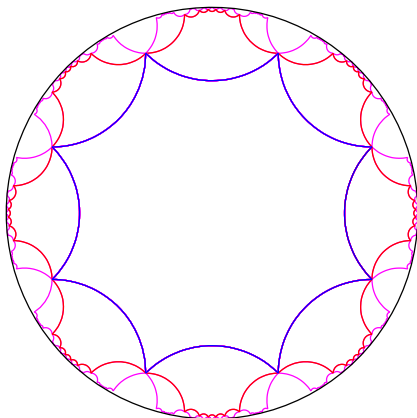
(Identify edges, respecting orientation.)

Problem: can't tile the plane with this!

Universal cover of the double-torus



Embed the octagon on the **Poincaré disk**, a space with constant negative curvature:



(These curved lines are actually **straight geodesics**.)

Then we can tile the disk with **isometric copies** of our octagon (fundamental domain).

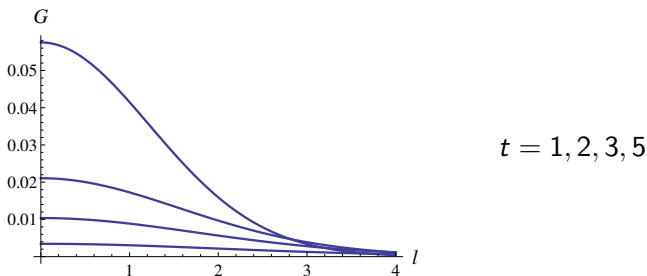
Heat kernel on the Poincaré disk



From Chavel (1984), the Green's function for the heat equation $\partial_t \theta = \Delta \theta$ on the Poincaré disk is

$$G(\ell, t) = \frac{\sqrt{2} e^{-t/4}}{(4\pi t)^{3/2}} \int_{\ell}^{\infty} \frac{\beta e^{-\beta^2/4t}}{\sqrt{\cosh \beta - \cosh \ell}} d\beta,$$

where ℓ is the hyperbolic distance between the source and target points.

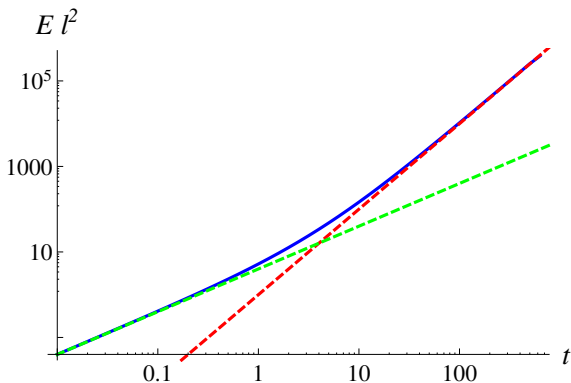


A priori, this behaves a lot like the planar heat kernel.

Squared-displacement on the Poincaré disk



Expected value of ℓ^2 as a function of time:



The **green dashed line** is $4t$ (diffusive), the **red dashed line** is t^2 (ballistic).

Surprising result: **not diffusive for large time!** Why?

The probability of **recurrence** (coming back to the origin) from a distance ℓ is [Lyons & McKean (1984)]

$$\int_0^\infty G(\ell, t) dt = \frac{1}{2\pi} \log \coth(\ell/2) \sim \frac{1}{\pi} e^{-\ell}, \quad \ell \gg 1.$$

Hence, even though it is two-dimensional, a Brownian motion on the hyperbolic plane is **transient**.

Put another way, if the particle wanders too far from the origin, then it will **almost certainly not return**. It is hopelessly entangled.

This **spontaneous entanglement** property arises because of the natural hyperbolicity of the surface, i.e., its universal cover is the Poincaré disk.

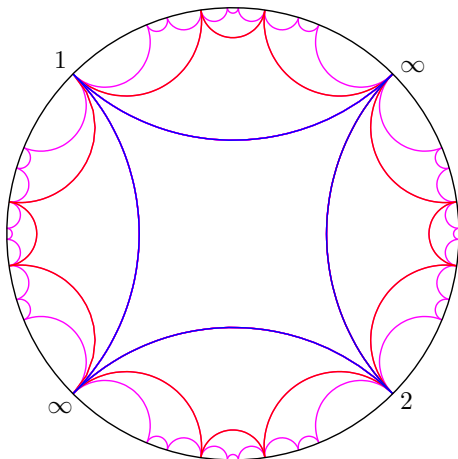
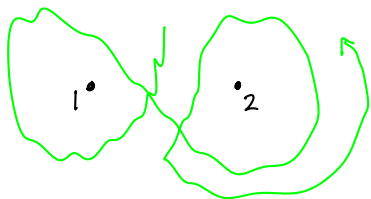
[Nechaev, S. K. (1996). *Statistics of Knots and Entangled Random Walks*. Singapore; London: World Scientific]

Universal cover of twice-punctured plane

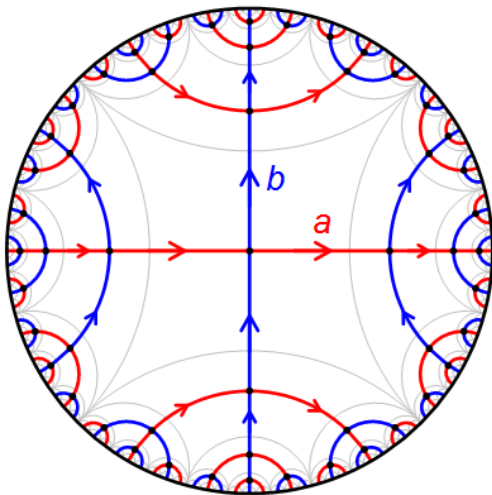


Consider now winding around **two points** in the complex plane.

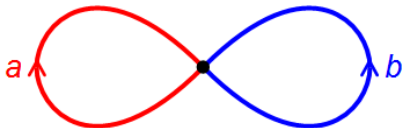
Topologically, this space is like the **sphere with 3 punctures**, where the third puncture is the point at infinity.



Cayley graph of free group



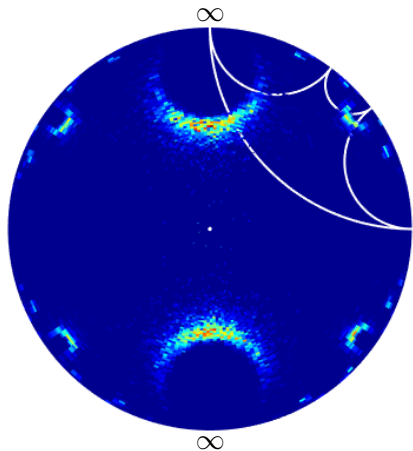
We really only care about which 'copy' of the fundamental domain we're in. Can use a **tree** to record this.



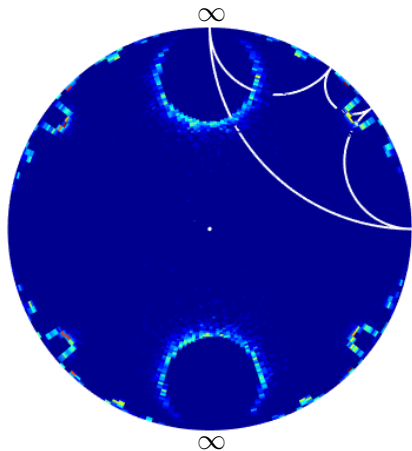
The history of a path is encoded in a 'word' in the letters a , b , a^{-1} , b^{-1} .

(Free group with two generators.)

[Source: Wikipedia]



unbounded domain



bounded domain

Note the secondary peaks: the windings occur mostly at the start in the unbounded case.



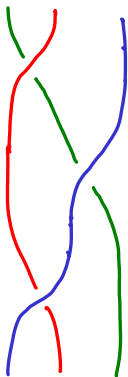
Many people have worked on aspects of this problem:

- [Spitzer (1958); Durrett (1982); Messulam & Yor (1982); Berger (1987); Shi (1998)] [winding of Brownian motion around a point in \$\mathbb{R}^2\$.](#)
- [Berger & Roberts (1988); Bélisle (1989); Bélisle & Faraway (1991); Rudnick & Hu (1987)] [winding of random walk around a point.](#)
- [Drossel & Kardar (1996); Grosberg & Frisch (2003)] [finite obstacle, closed domain.](#)
- [Itô & McKean (1974); McKean (1969); Lyons & McKean (1984)] [doubly-punctured plane.](#)
- [McKean & Sullivan (1984)] [three-punctured sphere.](#)
- [Pitman & Yor (1986, 1989)] [more points.](#)
- [Watanabe (2000)] [Riemann surfaces.](#)
- [Nechaev (1988)] [lattice of obstacles.](#)
- [Nechaev (1996); Revuz & Yor (1999)] [comprehensive books.](#)

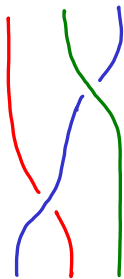
Quality of entanglement



Compare these two braids:



"half-twist"
braid



"over-unders"
braid

Repeating these increases distance in the universal cover...

But clearly the pigtail is more “entangled”



Over-under (pigtail) is very robust, unlike simply twisting. How do we capture this difference?

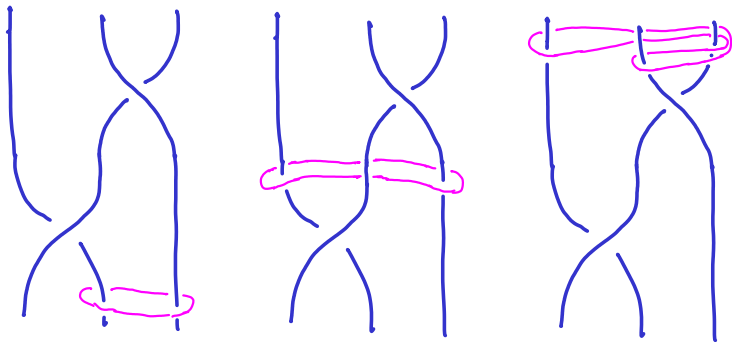
[<http://www.lovethispic.com/image/24844/pigtail-braid>]

Topological entropy



Inspired by dynamical systems. (Related to: braiding factor, braid complexity.)

Cartoon: compute the **growth rate** of a loop slid along the rigid braid.



This is relatively easy to compute using **braid groups** and **loop coordinates**.

[See Dynnikov (2002); J-LT (2005); J-LT & Finn (2006); Moussafir (2006); Dynnikov & Wiest (2007); J-LT (2010)]

In Finn & J-LT (2011) we proved that

$$\frac{\text{topological entropy}}{\text{braid length}} \leq \log(\text{Golden ratio})$$

This maximum entropy is exactly realized by the pigtail braid, reinforcing the intuition that it is somehow the most 'sturdy' braid.

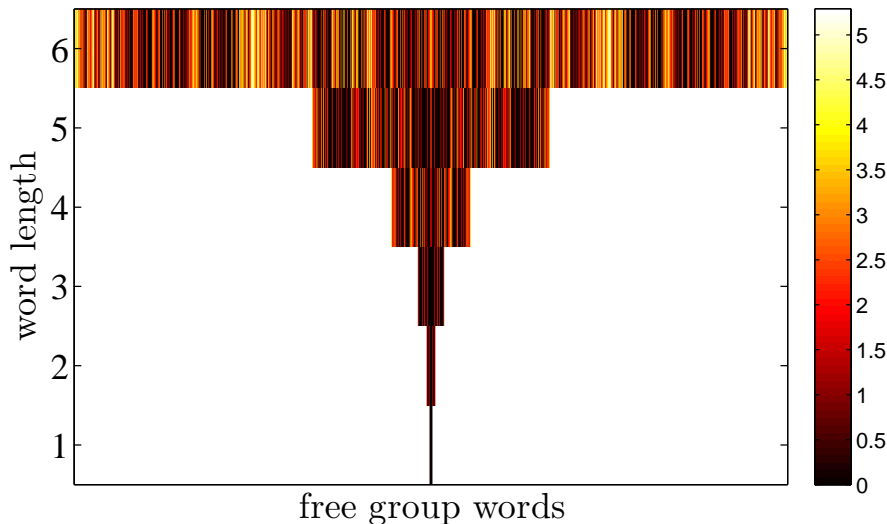
[Finn, M. D. & J-LT (2011). *SIAM Rev.* **53** (4), 723–743]



Word length vs topological entropy



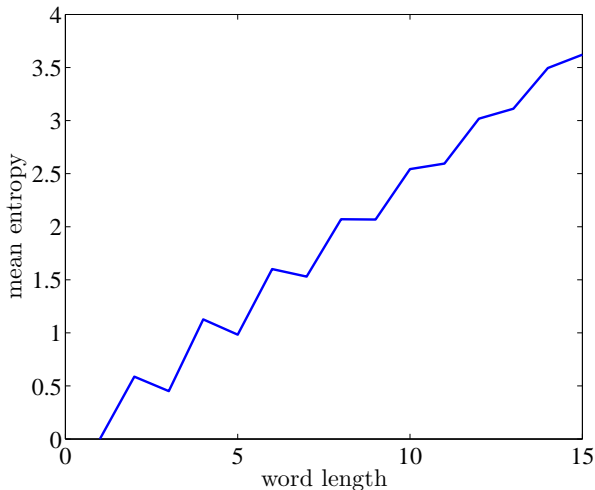
For the plane with two punctures, we can relate entropy to word length.



Mean topological entropy



Averaged over all words of a given length, entropy grows linearly:



(This assumes all words are equally probable, which is not necessarily true.)

Another viewpoint: how hard is detangling?



Buck & Scharein (2014) take another approach: the 'rope trick' on the left shows how to create a **sequence of simple knots** with a single final 'pull.'

They show that creating the knots takes work proportional to the length, but undoing the knots is **quadratic in the length**, because the knots must be loosened one-by-one.

This asymmetry suggests why it's easy to tangle things, but hard to disentangle.

[Buck, G. & Scharein, R. (2014). preprint]



- Entanglement at confluence of **dynamics**, **probability**, **topology**, and **combinatorics**.
- Instead of Brownian motion, can use orbits from a **dynamical system**. This yields dynamical information.
- More generally, study random processes on **configuration spaces** of sets of points (also finite size objects).
- Other applications: **Crowd dynamics** (Ali, 2013), **granular media** (Puckett *et al.*, 2012).
- With **Michael Allshouse**: develop tools for analyzing orbit data from this topological viewpoint (Allshouse & J-LT, 2012).
- With **Tom Peacock** and **Margaux Filippi**: apply to orbits in a fluid dynamics experiments.

- Ali, S. (2013). In: *IEEE International Conference on Computer Vision (ICCV)* pp. 1097–1104, .:
- Allshouse, M. R. & J-LT (2012). *Physica D*, **241** (2), 95–105.
- Bélisle, C. (1989). *Ann. Prob.* **17** (4), 1377–1402.
- Bélisle, C. & Faraway, J. (1991). *J. Appl. Prob.* **28** (4), 717–726.
- Berger, M. A. (1987). *J. Phys. A*, **20**, 5949–5960.
- Berger, M. A. & Roberts, P. H. (1988). *Adv. Appl. Prob.* **20** (2), 261–274.
- Buck, G. & Scharein, R. (2014). preprint.
- Chavel, I. (1984). *Eigenvalues in Riemannian Geometry*. Orlando: Academic Press.
- Drossel, B. & Kardar, M. (1996). *Phys. Rev. E*, **53** (6), 5861–5871.
- Durrett, R. (1982). *Ann. Prob.* **10** (1), 244–246.
- Dynnikov, I. A. (2002). *Russian Math. Surveys*, **57** (3), 592–594.
- Dynnikov, I. A. & Wiest, B. (2007). *Journal of the European Mathematical Society*, **9** (4), 801–840.
- Finn, M. D. & J-LT (2011). *SIAM Rev.* **53** (4), 723–743.
- Fudge, D. S., Levy, N., Chiu, S., & Gosline, J. M. (2005). *J. Exp. Biol.* **208**, 4613–4625.
- Goldstein, R. E., Warren, P. B., & Ball, R. C. (2012). *Phys. Rev. Lett.* **108**, 078101.
- Grosberg, A. & Frisch, H. (2003). *J. Phys. A*, **36** (34), 8955–8981.

- Itô, K. & McKean, H. P. (1974). *Diffusion processes and their sample paths*. Berlin: Springer.
- Lyons, T. J. & McKean, H. P. (1984). *Adv. Math.* **51**, 212–225.
- McKean, H. P. (1969). *Stochastic Integrals*. New York: Academic Press.
- McKean, H. P. & Sullivan, D. (1984). *Adv. Math.* **51**, 203–211.
- Messulam, P. & Yor, M. (1982). *J. London Math. Soc.* (2), **26**, 348–364.
- Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* **1** (1), 37–46.
- Nechaev, S. K. (1988). *J. Phys. A*, **21**, 3659–3671.
- Nechaev, S. K. (1996). *Statistics of Knots and Entangled Random Walks*. Singapore; London: World Scientific.
- Pitman, J. & Yor, M. (1986). *Ann. Prob.* **14** (3), 733–779.
- Pitman, J. & Yor, M. (1989). *Ann. Prob.* **17** (3), 965–1011.
- Puckett, J. G., Lechenault, F., Daniels, K. E., & J-LT (2012). *Journal of Statistical Mechanics: Theory and Experiment*, **2012** (6), P06008.
- Revuz, D. & Yor, M. (1999). *Continuous Martingales and Brownian motion*. Berlin: Springer, third edition.
- Rudnick, J. & Hu, Y. (1987). *J. Phys. A*, **20**, 4421–4438.
- Shi, Z. (1998). *Ann. Prob.* **26** (1), 112–131.
- Spitzer, F. (1958). *Trans. Amer. Math. Soc.* **87**, 187–197.



J-LT (2005). *Phys. Rev. Lett.* **94** (8), 084502.

J-LT (2010). *Chaos*, **20**, 017516.

J-LT & Finn, M. D. (2006). *Phil. Trans. R. Soc. Lond. A*, **364**, 3251–3266.

Watanabe, S. (2000). *Acta Applicandae Mathematicae*, **63**, 441–464.