STIRRING UP TROUBLE

Multi-scale mixing measures for steady scalar sources

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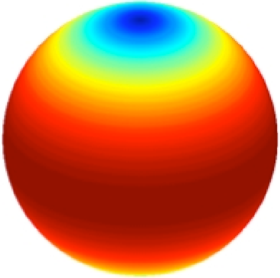
Sample motivation

- The earth is differentially heated: there is a constant input of 'fresh' temperature fluctuations.



- Stirring and mixing suppress fluctuations.
- How does the steady-state level of fluctuations depend on the stirring and the source distribution?

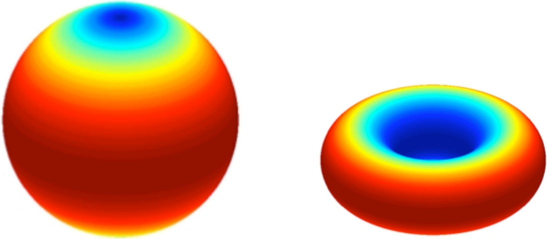
Outline

- Advection & diffusion with steady sources & sinks.
- Quantify mixing efficiency on different length scales.
- Estimating the mixing efficiencies via bounds.
- Source dependence of the efficiencies.
- Optimal flows that 'saturate' the bounds.
- Scalings for shear flows.
- Summary & discussion.

Advection, diffusion and all that

- Advection-diffusion of a passive scalar $\theta(\mathbf{x}, t)$ maintained by a steady and spatially mean zero body source:

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta + s(\mathbf{x})$$



- Stirring field $\mathbf{u}(\mathbf{x}, t)$ is given, divergence-free ($\nabla \cdot \mathbf{u} = 0$) with finite time-averaged kinetic energy (L^2 norm).
- Statistically stationary, homogeneous, isotropic flows.

Multiscale variances

- Variance is natural measure of the well-mixedness of the scalar.
- Weight on different length scales to produces family of space-time averaged variances $\langle|\nabla^p\theta|^2\rangle$:

small scales for $p = 1$: $$\langle|\nabla\theta|^2\rangle = \left\langle \sum_{\mathbf{k}} k^2 |\hat{\theta}(\mathbf{k})|^2 \right\rangle$$

intermediate scales for $p = 0$: $$\langle\theta^2\rangle = \left\langle \sum_{\mathbf{k}} |\hat{\theta}(\mathbf{k})|^2 \right\rangle$$

large scales for $p = -1$: $$\langle|\nabla^{-1}\theta|^2\rangle = \left\langle \sum_{\mathbf{k}} \frac{|\hat{\theta}(\mathbf{k})|^2}{k^2} \right\rangle$$

- Better mixing $\Longleftrightarrow$ reduced variance.

Mixing efficiencies

- Dimensionless mixing efficiencies compare variances with and without stirring

$$M_p := \left(\frac{\langle |\nabla^p \theta_0|^2 \rangle}{\langle |\nabla^p \theta|^2 \rangle} \right)^{1/2}$$

where the purely-diffusive solution is

$$\theta_0(\mathbf{x}) = \kappa^{-1} (-\Delta)^{-1} s.$$

- Péclet number is the dimensionless gauge of the stirring strength

$$Pe := \frac{UL}{\kappa}$$

where $U^2 = \langle |\mathbf{u}|^2 \rangle$.

- Challenge: estimate $M_p(Pe)$ ... what might be expected?

Equivalent diffusivity & eddy diffusivity

- Equivalent diffusivity is the molecular diffusion necessary to achieve the same variance in the absense of stirring:

$$\kappa_{\mathrm{eq},p} = \kappa \, M_p$$

- The case of large-scale source and small-scale stirring is the setting for homogenization theory to estimate κ_{eq}.
- Classical mixing length theory $\Rightarrow$ Eddy diffusivity $\kappa_{\mathrm{eddy}} = U\ell$ where ℓ is a characteristic length scale of the flow.
- Guess:
  $\kappa_{\mathrm{eq},p} \approx \kappa_{\mathrm{eddy}} = U\ell = \kappa \, \frac{UL}{\kappa} \, \frac{\ell}{L} = \kappa \, Pe \, \frac{\ell}{L} \quad \Rightarrow \quad M_p \lesssim Pe.$
- $M_p \sim Pe$ is "classical" scaling; any other is "anomalous".

Estimating the mixing efficiencies

- Multiply ADE by smooth, time-independent function $\varphi(\mathbf{x})$, then space-time average & integrate by parts:

$$\langle (\mathbf{u}\varphi + \kappa\nabla\varphi) \cdot \nabla\theta \rangle = \langle \varphi s \rangle$$

- $\Longrightarrow$ Variational problem for lower-bounds for each p:

$$\langle |\nabla^p\theta|^2 \rangle \geq \max_{\varphi} \min_{\tilde{\theta}} \ \{ \langle |\nabla^p\tilde{\theta}|^2 \rangle \mid \langle (\mathbf{u}\varphi + \kappa\nabla\varphi) \cdot \nabla\tilde{\theta} \rangle = \langle \varphi s \rangle \}$$

- TD&G (JFM 2004) derived simple estimates for $p = 0$ for general $\mathbf{u}$, $s \in L^2$, without variation over φ : $M_p \leq c_\varphi Pe$.

Bounds for SHIF

- Statistically Stationary, Homogeneous, Isotropic Flows (SHIF):

$$\overline{u_i(\mathbf{x},\cdot)} = 0, \qquad \overline{u_i(\mathbf{x},\cdot)u_j(\mathbf{x},\cdot)} = \frac{U^2}{d}\delta_{ij}$$

$$\overline{u_i(\mathbf{x},\cdot)\frac{\partial u_j(\mathbf{x},\cdot)}{\partial x_k}} = 0, \qquad \overline{\frac{\partial u_i(\mathbf{x},\cdot)}{\partial x_k}\frac{\partial u_j(\mathbf{x},\cdot)}{\partial x_k}} = \frac{\Gamma^2}{d}\delta_{ij}$$

- $\lambda = U/\Gamma \ \sim$ Taylor microscale of turbulence theory.
- Variational upper bounds on $M_p(Pe)$ can be evaluated exactly for SHIFs!

- Gradient variance

$$1 \;\le M_1^2 \;\le\; \frac{\langle s(-\Delta^{-1})s\rangle}{\langle s\{-\Delta + U^2/\kappa^2\}^{-1}s\rangle} = \frac{\left(\sum_{\mathbf{k}} \frac{|\hat{s}(\mathbf{k})|^2}{k^2L^2}\right)}{\left(\sum_{\mathbf{k}} \frac{|\hat{s}(\mathbf{k})|^2}{k^2L^2+Pe^2}\right)}$$

- Variance

$$c_0\{s\} \;\le\; M_0^2 \;\le\; \frac{\langle s(\Delta^{-2})s\rangle}{\langle s\{\Delta^2 - \frac{U^2}{\kappa^2 d}\Delta\}^{-1}s\rangle} = \frac{\left(\sum_{\mathbf{k}} \frac{|\hat{s}(\mathbf{k})|^2}{k^4L^4}\right)}{\left(\sum_{\mathbf{k}} \frac{|\hat{s}(\mathbf{k})|^2}{k^4L^4+\frac{Pe^2k^2L^2}{d}}\right)}$$

- Inverse-gradient variance

$$c_{-1}\{s\} \;\le\; M_{-1}^2 \;\le\; \left(\sum_{\mathbf{k}} \frac{|\hat{s}(\mathbf{k})|^2}{k^6L^6}\right) \Big/ \left(\sum_{\mathbf{k}} \frac{|\hat{s}(\mathbf{k})|^2}{k^6L^6 + \frac{Pe^2k^4L^4}{d} + \frac{Pe^2k^2L^4}{\lambda^2 d}}\right)$$

Source dependence

- For "smooth" sources, i.e., $s(\mathbf{x}) \in L^2$, as $Pe \to \infty$,

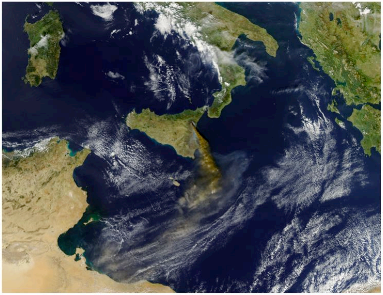
$$M_1 \;\le\; Pe\sqrt{\frac{\sum_{\mathbf{k}} |\hat{s}(\mathbf{k})|^2/k^2L^2}{\sum_{\mathbf{k}} |\hat{s}(\mathbf{k})|^2}} \;=\; Pe\,\frac{\ell_1}{L} \;=\; \frac{U\ell_1}{\kappa},$$

$$M_0 \;\le\; Pe\sqrt{\frac{\sum_{\mathbf{k}} |\hat{s}(\mathbf{k})|^2/k^4L^2}{d\sum_{\mathbf{k}} |\hat{s}(\mathbf{k})|^2/k^2}} \;=\; Pe\,\frac{\ell_0}{L} \;=\; \frac{U\ell_0}{\kappa},$$

$$M_{-1} \;\le\; Pe\sqrt{\frac{\sum_{\mathbf{k}} |\hat{s}(\mathbf{k})|^2/k^6L^2}{d\sum_{\mathbf{k}} |\hat{s}(\mathbf{k})|^2/(k^4+\frac{k^2}{\lambda^2})}} \;=\; Pe\,\frac{\ell_{-1}}{L} \;=\; \frac{U\ell_{-1}}{\kappa}.$$

- Classical mixing length scaling may hold but...
- Mixing lengths ℓ_{p} defining the equivalent diffusivities generally depend on scales in source rather than just those in the flow.

- High-*Pe* scalings for "rough" sources with many small scales, i.e., for $s(\mathbf{x}) \notin L^2$, are necessarily anomalous.



Mt. Etna

- Measure-valued (e.g., delta-function) sources or sinks $\Rightarrow$

$$d = 2: \quad M_1 = 1 \qquad M_0 \lesssim Pe/(\log Pe)^{1/2} \qquad M_{-1} \lesssim Pe$$

$$d = 3: \quad M_1 = 1 \qquad M_0 \lesssim Pe^{1/2} \qquad M_{-1} \lesssim Pe$$

Source dependence of scalings

- D&T (PRL submitted 2006) summarize the scalings of the mixing efficiency bounds for general "rough" sources when spectrum $|\hat{s}(\mathbf{k})|$ decays $\sim k^{-\gamma}$, $\gamma \geq 0$ ($\gamma > d/2 \Rightarrow$ "smooth").

$d = 2$	$p = 1$	$p = 0$	$p = -1$
$\gamma = 0$	1	$Pe/(\log Pe)^{1/2}$	Pe
$0 < \gamma < 1$	Pe^{γ}	Pe	Pe
$\gamma = 1$	$Pe/(\log Pe)^{1/2}$	Pe	Pe
$\gamma > 1$	Pe	Pe	Pe

$d = 3$	$p = 1$	$p = 0$	$p = -1$
$\gamma = 0$	1	$Pe^{1/2}$	Pe
$0 \leq \gamma < 1/2$	1	$Pe^{\gamma+1/2}$	Pe
$\gamma = 1/2$	1	$Pe/(\log Pe)^{1/2}$	Pe
$1/2 < \gamma < 3/2$	$Pe^{\gamma-1/2}$	Pe	Pe
$\gamma = 3/2$	$Pe/(\log Pe)^{1/2}$	Pe	Pe
$\gamma > 3/2$	Pe	Pe	Pe

Monochromatic bounds

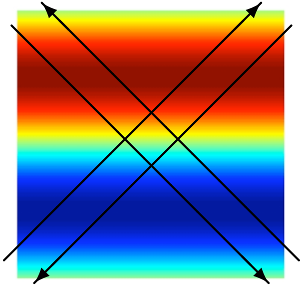
- For monochromatic sources depending on a single wave number k_s:

$$
\begin{aligned}
1 \;&\leq\; M_1 \leq \sqrt{1 + \frac{Pe^2}{k_s^2 L^2}}, \\
1 \;&\leq\; M_0 \leq \sqrt{1 + \frac{Pe^2}{d k_s^2 L^2}}, \\
1 \;&\leq M_{-1} \leq \sqrt{1 + \frac{Pe^2}{d k_s^2 L^2} + \frac{Pe^2}{d \lambda^2 k_s^4 L^2}}
\end{aligned}
$$

- Questions:
  1. can these bounds be achieved by any SHIF?
  2. are they generally achieved for “typical” SHIFs?

Saturating the bounds

- These bounds *can* be saturated!
- Optimal SHIF for monochromatic sources are “sweeping flows” transporting source onto sink and vice versa (W.R. Young).

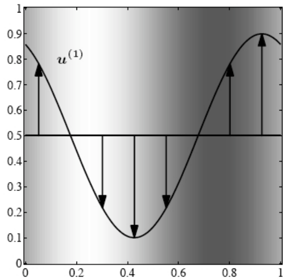


steady wind switching direction slowly to achieve SHIF

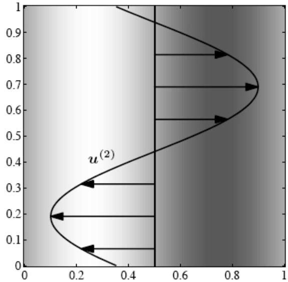
- Sweeping flows *not* the optimal for measure valued sources.
- Note: uniform “sweeps” do not *exist* on the sphere!

Zeldovich sine flow

- Alternating horizontal and vertical sine shear flows



first half-period

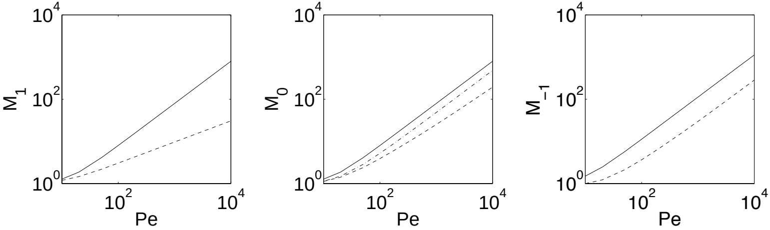


second half-period

- Phase selected randomly for each cycle.
- Frequently used to study fundamental mixing characteristics.
- MOVIES

Efficiencies for monochromatic source

- Bounds and DNS results for Zeldovich sine flow:

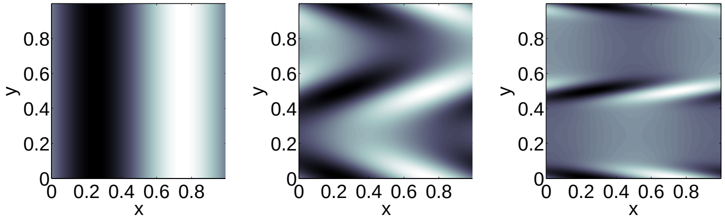


solid: Bound dot-dashed: PY bound dashed: DNS data

- Data scale anomalously for M_1 and M_0.

Steady shear flows

- Explore multiscale mixing efficiency scalings for sine flow via quasi-steady shearing perpendicular to the source.
- Consider steady shear flow $\mathbf{u} = \hat{\mathbf{i}} U \cos k_u y$, source $s = S \sin k_s x$.
- Non-dimensional number indicating relative shear $r = k_u / k_s$.
- Interesting limits include $Pe \to \infty$ at fixed r, etc.
- For $r = 1$ and Pe increasing:



- Steady solution of ADE is of the form

$$\theta(x,y) = f(y)\sin kx + g(y)\cos kx.$$

where

$$-\sqrt{2}Uk_s\sin(k_u y)g = \kappa\left[-k_s^2 f + f''\right] + \sqrt{2}S$$
$$\sqrt{2}Uk_s\sin(k_u y)f = \kappa\left[-k_s^2 g + g''\right].$$

- Internal layer analysis:

$$\hat{f} = \sum_{n=-1}^{\infty}\epsilon^n \hat{f}_n, \quad \hat{g} = \sum_{n=-1}^{\infty}\epsilon^n \hat{g}_n$$

where $\hat{f} = fUk_s/S$, $\hat{g} = gUk_s/S$, and $\epsilon^{-3} = \sqrt{2}U/\kappa k_s$.

- Leading order the composite solutions are

$$f(y) \sim \frac{S}{Uk_s}\frac{1}{k_u\delta}F\left(\frac{y}{\delta}\right), \quad g(y) \sim \frac{S}{Uk_s}\frac{1}{k_u\delta}G\left(\frac{y}{\delta}\right)\frac{k_u y}{\sin(k_u y)}$$

where $F = r^{2/3}\hat{f}_{-1}$, and $G = r^{2/3}\hat{g}_{-1}$, and $\delta = \epsilon/r^{1/3}k_s$.

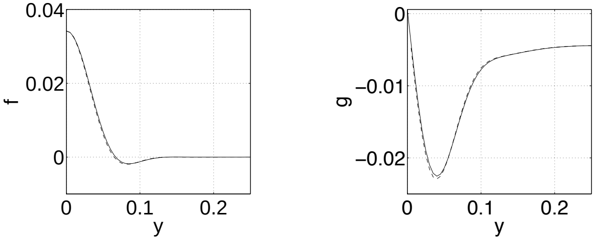
- Just solve "internal layer" equations

$$F'' + \xi G + 1 = 0 \quad \text{and} \quad G'' - \xi F = 0$$

with

$$F'(0) = 0 = G(0), \quad F \sim -2\xi^{-4}, \quad G \sim -\xi^{-1} \ \text{ as } \ \xi \to \infty.$$

- For $Pe = 1000$ (ϵ=0.2)

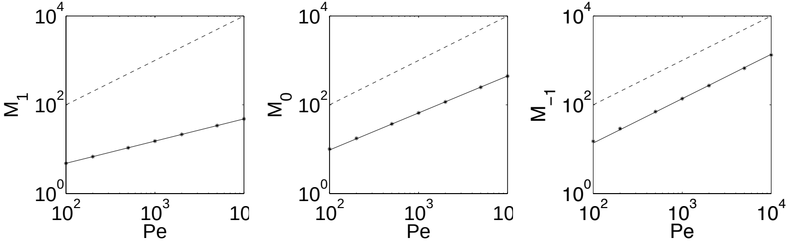


solid: exact numerical solution dashed: composite asymptotic approximation

- Leading terms capture asymptotic behavior!

- Scaling of mixing efficiencies as $Pe \to \infty$

$$M_1 \sim Pe^{1/2}, \quad M_0 \sim r^{1/3} Pe^{5/6}, \quad M_{-1} \sim Pe + r^{1/3} Pe^{5/6}$$



dots: exact numerical solution   solid: asymptotic theory
dashed: $\sim Pe$ bound scaling

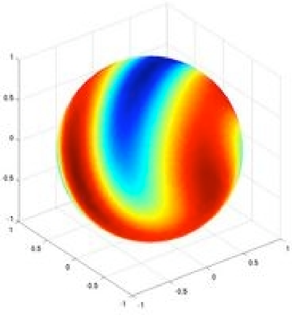
- Exponents match the DNS data scalings!

Conclusions

- Multiscale mixing efficiencies susceptible to rigorous analysis.
- Upper estimates on multiscale efficiencies *may* be saturated.
- Steady state mixing *very* different from transient mixing!
- Source structure is crucial to Pe-scaling of efficiencies.
- Can understand some flows via quasi-static analysis.
- More info than just U, Γ generally needed.
- What about more complicated sources?
- What about "real" SHI-turbulence?

Future Work

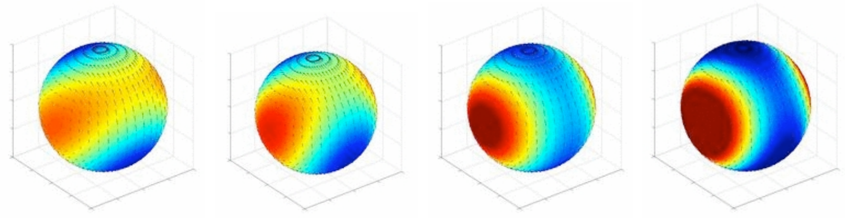
- Search for optimal stirring strategies (not necessarily SHIF) for given source distributions by maximizing the bounds.
- Consider the problem with scalar decay.
- Extend the analysis to the sphere.



Zeldovich sine flow with wave number 2

Mixing on a Sphere

- Constant rotation perpendicular to equatorial heating.



- 8 sector cellular flow.