Topological tools for trajectory data analysis

Jean-Luc Thiffeault[1]   Michael Allshouse[2]

[1]Department of Mathematics
University of Wisconsin – Madison
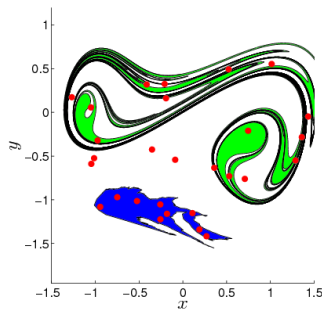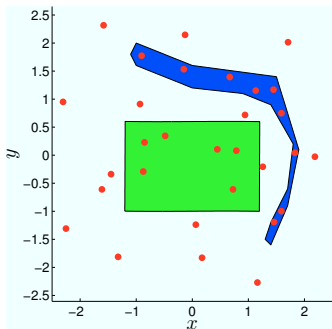
[2]Department of Physics
University of Texas – Austin

SJTU International Forum on Mathematics
10 January 2014

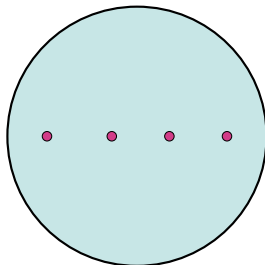## Sparse trajectories and material loops

How do we efficiently detect trajectories that 'bunch' together?

Growth of curves also studied in LCS context by Haller & Beron-Vera (2012).

[movie 1]

Mathematical background: Punctured disks

Low-dimensional topologists have long studied transformations of surfaces such as the punctured disk:

The central object of study is the homeomorphism: a continuous, invertible transformation whose inverse is also continuous.

For instance, this is a model of a two-dimensional vat of viscous fluid with stirring rods.

Punctured disks in experiments

The transformation in this case is given by the solution of a fluid
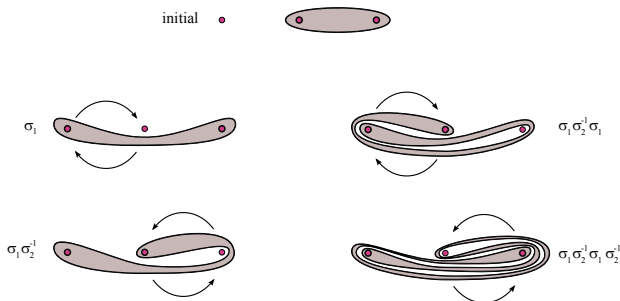equation over one period of rod motion.

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]
[movie 2]   [movie 3]

Growth of curves on a disk

On a disk with 3 punctures (rods), we can also look at the growth of curves:

We use the braid generator notation: σ_i means the clockwise interchange of the ith and $(i + 1)$th rod. (Inverses are counterclockwise.)

The motion above is denoted $\sigma_1 \sigma_2^{-1}$.

Growth of curves on a disk (2)

The rate of growth $h = \log \lambda$ is called the topological entropy.

But how do we find the rate of growth of curves for motions on the disk?

For 3 punctures it's easy: the entropy for $\sigma_1 \sigma_2^{-1}$ is $h = \log \varphi^2$, where φ is the Golden Ratio!

For more punctures, use Moussafir iterative technique (2006).

[Thiffeault, *Phys. Rev. Lett.* (2005); *Chaos* (2010); Gouillart et al., *Phys. Rev. E* (2006) 'ghost rods']

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.
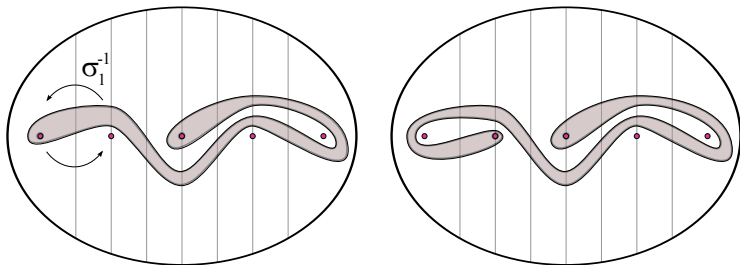
What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:
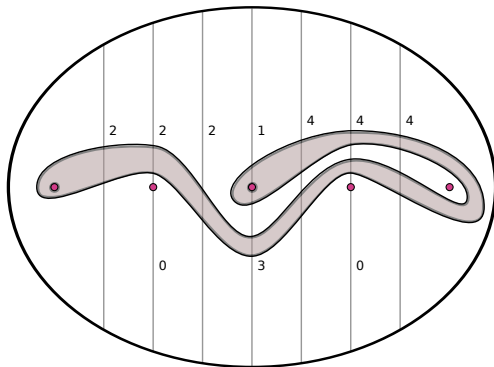
## Solution to problem 2: Action on coordinates

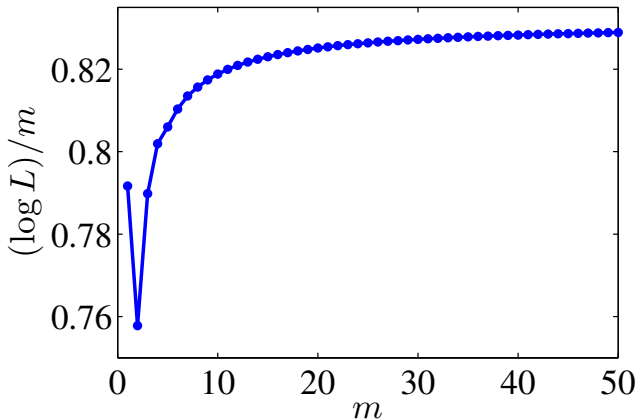Moving the punctures according to a braid generator changes some crossing numbers:

There is an explicit formula for the change in the coordinates!

Easy to code up (see for example Thiffeault (2010), `braidlab`).
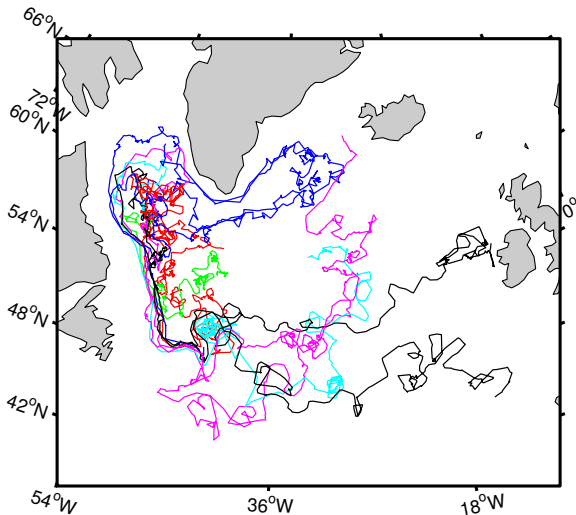
Growth of L

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:
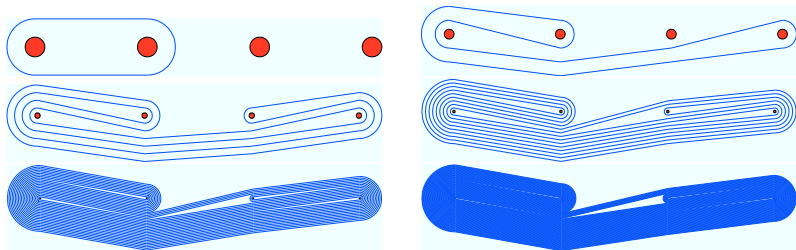
Growth of L (2)

m is the number of times the braid acted on the initial loop.

Oceanic float trajectories

## Oceanic floats: Data analysis

What can we measure?

- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the σ_i for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent).
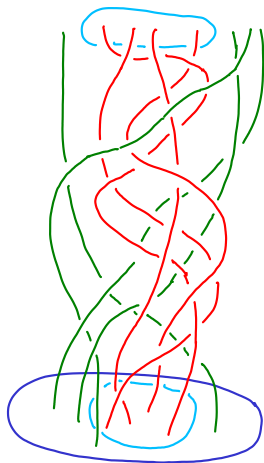
Oceanic floats: Entropy

10 floats from Davis' Labrador sea data:
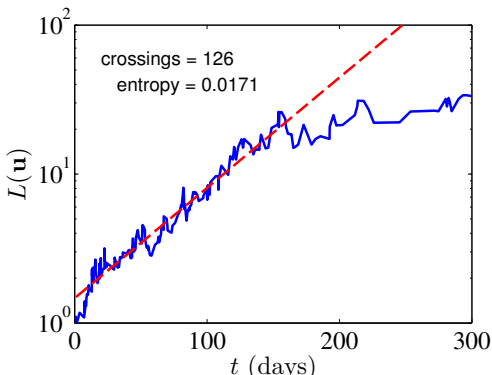
Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)

Lagrangian Coherent Structures

- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- For now: regions are not 'leaky.'
- (See the work of Haller et al.)

Sample system: Modified Duffing oscillator
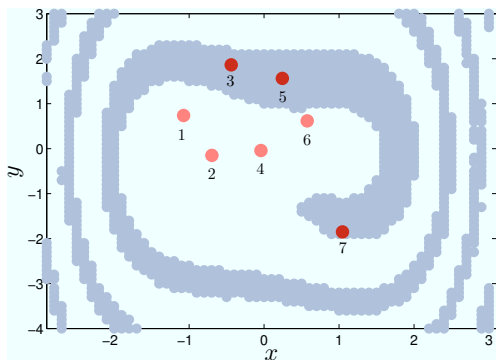
$$\dot{x} = y + \alpha \cos \omega t,$$
$$\dot{y} = x(1 - x^2) + \gamma \cos \omega t - \delta y,$$

+ rotation to further hide two regions. $\alpha = .1$, $\gamma = .14$, $\delta = .08$, $\omega = 1$.

Growth of a vast number of loops
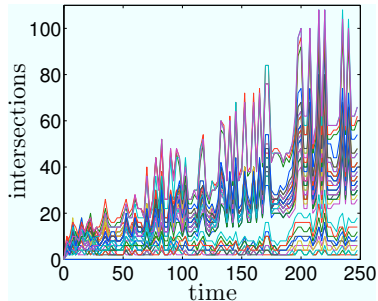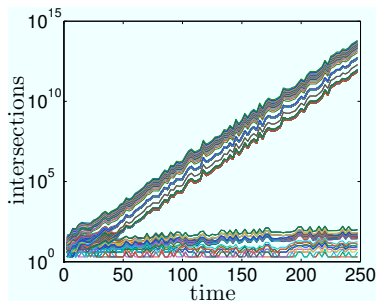
Left: semilog plot; Right: linear plot of slow-growing loops.

Clearly two types of loops!

What do the slowest-growing loops look like?
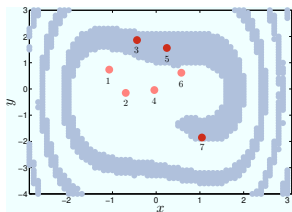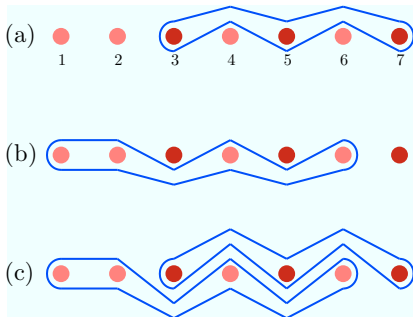
[(c) appears because the coordinates also encode 'multiloops.']

Computational complexity

Here's the bad news:

- There are an infinite number of loops to consider.
- But we don't really expect hyper-convoluted initial loops (nor do we care so much about those).
- Even if we limit ourselves to loops with Dynnikov coordinates between -1 and 1, this is still 3^{2n-4} loops.
- This is too many... can only treat about 10–11 trajectories using this direct method.

An improved method: Pair-loops

The biggest problem is that we only look at whether a loop grows or not. But there is a lot more information to be found in how a loop entangles the punctures as it evolves.
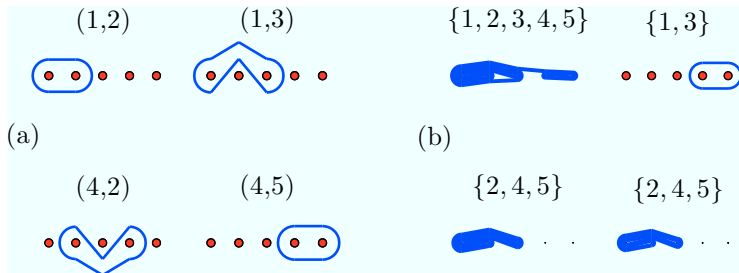
Consider loops that enclose two punctures at once. More involved analysis, but scales *much* better with n.

Improvement

Run times in seconds:

# of trajectories	6	7	8	9	10	11	20
direct method	0.46	0.70	6.0	53	462	3445	N/A
pair-loop method	9.5	11.6	12.3	13	15	20	128

Bottleneck for the pair-loop method is finding the non-growing loops. (Should scale as n^2 for large enough n.)

The downside is that the pair-loop method is much more complicated. But in the end it accomplishes the same thing.

See Allshouse & Thiffeault, *Physica D* **241**, 95–105 (2012).

A benchmark problem: double-gyre

Shadden et al. (2005)

$$\dot{\mathbf{x}} = \pi A \begin{pmatrix} -\sin(\pi f(x,t))\cos(\pi y) \\ \cos(\pi f(x,t))\sin(\pi y)\,\frac{\partial f(x,t)}{\partial x} \end{pmatrix}$$
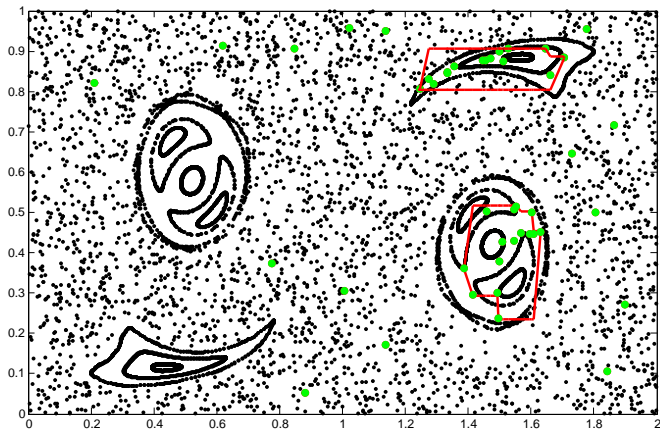
$$f(x,t) = a(t)x^2 + b(t)x$$
$$a(t) = \varepsilon\,\sin(\omega t)$$
$$b(t) = 1 - 2\varepsilon\,\sin(\omega t)$$

$$\varepsilon = 0.1,\ A = 0.1,\ \omega = \pi/5.$$

Double-gyre coherent structures

[movie 4]

Conclusions

- Having rods undergo 'braiding' motion guarantees a minimal amount of entropy (stretching of material lines);

- This idea can also be used on fluid particles to estimate entropy;

- Need a way to compute entropy fast: loop coordinates;

- There is a lot more information in this braid: extract it! (coherent structures);

- However: Difficult to find an appropriate data set.

- We're investigating the limits of the approach (how many trajectories, how long).

- We're developing Matlab tools — `braidlab`.

- Also applicable to granular media Puckett et al. (2012).

- See Thiffeault (2005, 2010) and Allshouse & Thiffeault (2012).

References

Allshouse, M. R. & Thiffeault, J.-L. 2012 Detecting Coherent Structures Using Braids. *Physica D* **241**, 95–105.

Bestvina, M. & Handel, M. 1995 Train-Tracks for Surface Homeomorphisms. *Topology* **34**, 109–140.

Binder, B. J. & Cox, S. M. 2008 A Mixer Design for the Pigtail Braid. *Fluid Dyn. Res.* **49**, 34–44.

Bowen, R. 1978 Entropy and the fundamental group. In *Structure of Attractors*, volume 668 of *Lecture Notes in Math.*, pp. 21–29. New York: Springer.

Boyland, P. L. 1994 Topological methods in surface dynamics. *Topology Appl.* **58**, 223–298.

Boyland, P. L., Aref, H. & Stremler, M. A. 2000 Topological fluid mechanics of stirring. *J. Fluid Mech.* **403**, 277–304.

Boyland, P. L., Stremler, M. A. & Aref, H. 2003 Topological fluid mechanics of point vortex motions. *Physica D* **175**, 69–95.

Dynnikov, I. A. 2002 On a Yang–Baxter map and the Dehornoy ordering. *Russian Math. Surveys* **57**, 592–594.

Finn, M. D. & Thiffeault, J.-L. 2011 Topological optimisation of rod-stirring devices. *SIAM Rev.* **53**, 723–743.

Gouillart, E., Finn, M. D. & Thiffeault, J.-L. 2006 Topological Mixing with Ghost Rods. *Phys. Rev. E* **73**, 036311.

Hall, T. & Yurttaş, S. Ö. 2009 On the Topological Entropy of Families of Braids. *Topology Appl.* **156**, 1554–1564.

Haller, G. & Beron-Vera, F. J.. 2012 Geodesic theory of transport barriers in two-dimensional flows. *Physica D*, submitted

Kolev, B. 1989 Entropie topologique et représentation de Burau. *C. R. Acad. Sci. Sér. I* **309**, 835–838. English translation at arXiv:math.DS/0304105.

Moussafir, J.-O. 2006 On the Entropy of Braids. *Func. Anal. and Other Math.* **1**, 43–54. arXiv:math.DS/0603355.

Puckett, J. G., Lechenault, F., Daniels, K. E. & Thiffeault, J.-L. 2012 Trajectory entanglement in dense granular materials. arXiv:1202.5243.

Thiffeault, J.-L. 2005 Measuring Topological Chaos. *Phys. Rev. Lett.* **94**, 084502.

Thiffeault, J.-L. 2010 Braids of entangled particle trajectories. *Chaos*, **20**, 017516.

Thiffeault, J.-L. & Finn, M. D. 2006 Topology, Braids, and Mixing in Fluids. *Phil. Trans. R. Soc. Lond. A* **364**, 3251–3266.

Thurston, W. P. 1988 On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc.* **19**, 417–431.