Long-wave Instability in Anisotropic Double-Diffusion

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Takens–Bogdanov

TB bifurcation occurs when two modes go unstable at the same parameter values.

Equations for the reduced dynamics near this bifurcation point capture more of the diverse behaviour of the system than simple steady or Hopf bifurcation.

For thermohaline convection in long-wave theory such a bifurcation is present.

Unfortunately, one problematic feature of the Takens–Bogdanov bifurcation is that the correct reduced equations contain terms of differing order in the standard asymptotic expansion parameter (Coullet 1983). The trouble is that the asymptotic theory fails to collect a dissipative nonlinear term; the amplitude equations is conservative to leading order (Childress and Spiegel, 1981).

Possible solutions

- Normal form theory: not available for extended systems.
- Reconstitution: difficult to judge validity. Clearly flawed in some cases (Clune, Depassier, and Knobloch, 1994).
- Alternative route: if more parameters were available, could tune out resonant terms (augmenting the codimension of the bifurcation).

To introduce needed extra parameters, we choose *anisotropic* double-diffusion as our system. (ocean, astrophysics, tokamak plasmas)

Model Equations

The equations for anisotropic double-diffusion are

$$
\begin{aligned}
\sigma^{-1}\frac{d}{dt}\nabla^2\psi &= R_T\,\partial_x\Theta - R_S\,\partial_x\Sigma + (D^2 + \gamma_\psi\,\partial_x^2)\nabla^2\psi,\\
\frac{d}{dt}\Theta &= \partial_x\psi + (D^2 + \gamma_\Theta\,\partial_x^2)\Theta,\\
\mathrm{Le}\,\frac{d}{dt}\Sigma &= \mathrm{Le}\,\partial_x\psi + (D^2 + \gamma_\Sigma\,\partial_x^2)\Sigma;
\end{aligned}
$$

with stress-free, fixed-flux boundary conditions

$$\psi = \nabla^2\,\psi = 0, \quad D\,\Theta = D\,\Sigma = 0, \; z = 0 \text{ and } 1$$

Fixed flux favours convection cells that are as large as the system will permit. Use this to define small parameter ϵ.

Scaling:

$$\partial_x = \epsilon\,\partial_X, \quad \partial_t = \epsilon^4\,\partial_T, \quad \psi = \epsilon\,\phi_X$$

Order ϵ^0

$$\begin{aligned} D^4\phi_{0X} &= -R_{T0}\,\Theta_{0X} + R_{S0}\,\Sigma_{0X} \\ D^2\Theta_0 &= 0 \\ D^2\Sigma_0 &= 0 \end{aligned}$$

The fixed flux boundary conditions give $\Theta_0 = \text{const.}$, $\Sigma_0 = \text{const.}$ in z. Applying stress-free boundary conditions on ϕ, we have

$$\phi_{0X} = \frac{1}{24}\, z(z-1)(z^2-z-1)\left(R_{S0}\,\Sigma_{0X} - R_{T0}\,\Theta_{0X}\right).$$

Order ϵ^2

$$
\begin{aligned}
D^4\phi_{2X} &= -R_{T0}\,\Theta_{2X} + R_{S0}\,\Sigma_{2X} - R_{T2}\,\Theta_{0X} + R_{S2}\,\Sigma_{0X} \\
&\quad - (1+\gamma_{\psi_0})D^2\phi_{0XXX} \\
&\quad + \sigma^{-1}\left(\phi_{0XX}D^3\phi_{0X} - D\phi_{0X}D^2\phi_{0XX}\right) \\
D^2\Theta_2 &= -\gamma_{\Theta 0}\Theta_{0XX} - \phi_{0XX} - D\phi_{0X}\Theta_{0X} \\
D^2\Sigma_2 &= -\gamma_{\Sigma 0}\Sigma_{0XX} - \mathrm{Le}_0\,\phi_{0XX} - \mathrm{Le}_0\,D\phi_{0X}\Sigma_{0X}
\end{aligned}
$$

Solvability condition (linear at this order):

$$
\begin{pmatrix} a_0 - \gamma_{\Theta 0} & -b_0 \\ \mathrm{Le}_0\,a_0 & -\mathrm{Le}_0\,b_0 - \gamma_{\Sigma 0} \end{pmatrix}
\begin{pmatrix} \Theta_{0XX} \\ \Sigma_{0XX} \end{pmatrix} = 0.
$$

where $a_0 = R_{T0}/120$, $b_0 = R_{S0}/120$.

(120 $\longrightarrow$ 720 for no-slip)

Codimension two point

The requirement that the matrix M have zero eigenvalues means that its trace and determinant must vanish.

This is obtained by letting

$$R_{T0} = 120 \frac{\gamma_{\Theta_0}^2}{\gamma_{\Theta_0} - \gamma_{\Sigma_0}}, \quad R_{S0} = \frac{120}{\mathrm{Le}_0} \frac{\gamma_{\Sigma_0}^2}{\gamma_{\Theta_0} - \gamma_{\Sigma_0}},$$

so that we now have

$$M = \begin{pmatrix} \gamma_{\Theta_0} \gamma_{\Sigma_0} & -\mathrm{Le}_0^{-1} \gamma_{\Sigma_0}^2 \\ \mathrm{Le}_0 \gamma_{\Theta_0}^2 & -\gamma_{\Theta_0} \gamma_{\Sigma_0} \end{pmatrix}.$$

The eigenvector for the matrix M is parametrized by $\Sigma_0 = \gamma_{\Theta_0} \mathrm{Le}_0 \Theta_0 / \gamma_{\Sigma_0}$ (it only has one).

Order ϵ^4

Expressions getting ugly fast. . . Get two solvability conditions again,

$$\begin{aligned} \Theta_{0T} &= \cdots \\ \Sigma_{0T} &= \gamma_{\Theta 0}\,\mathrm{Le}_0\,\Theta_0/\gamma_{\Sigma 0} = \cdots \end{aligned}$$

Must be compatible since Θ_0 and Σ_{0T} are related. This is not satisfied automatically; this is why we now make use of the extra parameters. By letting

$$\begin{aligned} \mathrm{Le}_0 &= 1 \\ 31(\gamma_{\Theta 0} + \gamma_{\Sigma 0}) &= 561(1 + \gamma_{\psi_0}) \\ R_{T2} - \frac{\gamma_{\Theta 0}}{\gamma_{\Sigma 0}} R_{S2} &= \frac{120\gamma_{\Theta 0}(\gamma_{\Theta 2} - \gamma_{\Sigma 2} + \mathrm{Le}_2\,\gamma_{\Sigma 0})}{\gamma_{\Theta 0} - \gamma_{\Sigma 0}} \end{aligned}$$

the two become compatible. This increases the codimension by three.

We can now write

$$f_T = \bar{\alpha}\, g_{XX} + \bar{\mu}\, f_{XX} + \bar{\nu}\, f_{XXXX} + \bar{p}\left(f_X^3\right)_X$$

with

$$\begin{aligned}
f(X,T) &:= \Theta_0(X,T) \\
g(X,T) &:= \Sigma_{2,0}(X,T) - \frac{\gamma_{\Theta_0}}{\gamma_{\Sigma_0}}\,\mathrm{Le}_0\,\Theta_{2,0}(X,T) \\
\bar{\alpha} &:= \frac{\gamma_{\Sigma_0}^2}{\gamma_{\Theta_0} - \gamma_{\Sigma_0}} \\
\bar{\mu} &:= \frac{\gamma_{\Theta_0}\,\gamma_{\Sigma_2} - \gamma_{\Theta_2}\,\gamma_{\Sigma_0} - \mathrm{Le}_2\,\gamma_{\Theta_0}\,\gamma_{\Sigma_0}}{\gamma_{\Theta_0} - \gamma_{\Sigma_0}} \\
\bar{\nu} &:= \frac{\gamma_{\Theta_0}\,\gamma_{\Sigma_0}}{56} \\
\bar{p} &:= \frac{155}{126}\,\gamma_{\Theta_0}^2
\end{aligned}$$

Order ϵ^6

We get a solvability condition involving only g_T at this order. After rescaling to eliminate some parameters we have the coupled system

$$
\begin{aligned}
f_T &= g_{XX} + \mu\, f_{XX} + f_{XXXX} + \left(f_X^3\right)_X \\
g_T &= \lambda\, f_{XX} + \kappa\, f_{XXXX} - \gamma\, f_{XXXXXX} + \beta\, g_{XX} \\
&\quad - \rho\, g_{XXXX} + \xi\left(f_X^3\right)_X + \left(f_X^2\, g_X\right)_X \\
&\quad + \eta\left(f_X\, f_{XX}^2\right)_X - \zeta\left(f_X^3\right)_{XXX}
\end{aligned}
$$

We fixed Le_0, γ_{ψ_0}, and γ_{Θ_2}. However, we are left with enough parameters to vary independently all the coefficients except η and ζ.

In detail...

$$\mu = \frac{7}{15}\,\frac{120\,\gamma_{\Theta 0}\,(\gamma_{\Sigma 2} - \mathrm{Le}_2\,\gamma_{\Sigma 0}) - R_{T2}\,\gamma_{\Sigma 0} + R_{S2}\,\gamma_{\Theta 0}}{\gamma_{\Sigma 0}\,{\gamma_{\Theta 0}}^2}$$

$$\lambda = \frac{392}{15}\,\frac{1}{\gamma_{\Theta_0^2}\,\gamma_{\Sigma_0^2}(\gamma_{\Theta 0} - \gamma_{\Sigma 0})}\Big((R_{T4}\,\gamma_{\Sigma 0} - R_{S4}\,\gamma_{\Theta 0})(\gamma_{\Theta 0} - \gamma_{\Sigma 0})$$

$$+ 120\gamma_{\Theta 0}\,\gamma_{\Sigma 0}(\gamma_{\Sigma 4} - \gamma_{\Theta 4} - \gamma_{\Sigma 0}\,\mathrm{Le}_4 + \gamma_{\Sigma 0}\,\mathrm{Le}_2^2 - \gamma_{\Sigma 2}\,\mathrm{Le}_2)\Big)$$

$$\kappa = \frac{7}{1485}\frac{1}{\sigma\gamma_{\Theta_0^2}\,\gamma_{\Sigma_0^2}}\Big(67320\gamma_{\Theta 0}\,\gamma_{\Sigma 0}(\sigma\,\gamma_{\psi 2} - \gamma_{\Sigma 2} + \gamma_{\Sigma 0}\,\mathrm{Le}_2)$$

$$+ 561\gamma_{\Sigma 0}(R_{T2}\,\gamma_{\Sigma 0} - R_{S2}\,\gamma_{\Theta 0}) + 11880\sigma\,\gamma_{\Theta 0}\,\gamma_{\Sigma 0}\,\gamma_{\Sigma 2}$$

$$- 31\sigma\,\gamma_{\Theta 0}\,\gamma_{\Sigma 0}\,R_{T2} - \sigma\,\gamma_{\Theta 0}\,R_{S2}(99\gamma_{\Theta 0} - 130\gamma_{\Sigma 0})\Big)$$

$$+ \left(\frac{130}{99}\,{\gamma_{\Theta 0}}^{-1} - 1\right)\xi$$

$$\gamma = \rho + \frac{1}{552335355\,\gamma_{\Theta 0}\,\gamma_{\Sigma 0}}\Big(241025959\,({\gamma_{\Theta 0}}^2 + {\gamma_{\Sigma 0}}^2)$$

$$+ 981200220\,(\gamma_{\Theta 0} + \gamma_{\Sigma 0}) + 238887029\,\gamma_{\Sigma 0}\,\gamma_{\Theta 0} - 17756558820\Big)$$

$$\beta = \frac{7}{15}\frac{120\,\gamma_{\Sigma 0}\,(\gamma_{\Sigma 2} - \mathrm{Le}_2\,\gamma_{\Sigma 0}) - R_{S2}\,(\gamma_{\Theta 0} - \gamma_{\Sigma 0})}{{\gamma_{\Sigma 0}}^2\,\gamma_{\Theta 0}}$$

$$\rho = \frac{17}{3\,\sigma}$$

$$\xi = 56\,\frac{\mathrm{Le}_2}{\gamma_{\Theta 0} - \gamma_{\Sigma 0}}$$

$$\eta = \frac{1}{66495}\,\frac{569621\,\sigma^2 + 89877\,\sigma + 4565}{\sigma^2}$$

$$\zeta = \frac{2}{199485}\,\frac{360069\,\sigma^2 + 516149\,\sigma - 2385}{\sigma^2}$$

Small-amplitude Normal Form

Near criticality, if the system is in a box so that only discrete modes are excited, we use a single spatial mode, i.e. $f, g \sim \exp(iKX)$. Then we can use normal form theory. After a nonlinear transformation the system can be written as the (unfolded) normal form

$$
\begin{aligned}
\dot{u} &= w \\
\dot{w} &= \mu_1 u + \mu_2 w + \left[A\,|u|^2 - (u^* w + u w^*)\right] u - |u|^2 w,
\end{aligned}
$$

with

$$
\begin{aligned}
\mu_1 &:= -K^4\left((\mu\beta - \lambda) + (\mu\rho - \beta + \kappa)K^2 + (\gamma - \rho)K^4\right) \\
\mu_2 &:= K^2\left(\mu + \beta + (\rho - 1)\,K^2\right)\ , \\
A &:= \frac{3}{4}K^2\left((\mu + \beta + \xi) - (\frac{1}{3}\eta - \zeta - \rho + 1)K^2\right).
\end{aligned}
$$

A can have either sign, so criticality can change. Systems of this type are classified in Dangelmayr and Knobloch, 1987.

Slaving and Limits

We can slave g to f by assuming that its decay rate is large. The variable g will then quickly reach its equilibrium value ($\dot{g} = 0$). There are two such limits we wish to consider: $\beta \gg 1$ and $\rho \gg 1$.

Large β

$$f_T = \left(\mu - \frac{\lambda}{\beta}\right) f_{XX} + \left(1 - \frac{\kappa}{\beta}\right) f_{XXXX} + \frac{\gamma}{\beta} f_{XXXXXX} + \left(1 - \frac{\xi}{\beta}\right) \left(f_X^3\right)_X - \frac{\eta}{\beta} \left(f_X \, f_{XX}^2\right)_X + \frac{\zeta}{\beta} \left(f_X^3\right)_{XXX}$$

In the limit $\gamma = \eta = \zeta = 0$ (or $\beta \to \infty$) this equation reduces to the one derived by Chapman and Proctor (1980) for Rayleigh–Bénard convection with Boussinesq symmetry.

Consider the functional

$$\mathcal{V}[f] = \left\langle \frac{1}{2}\left(\mu - \frac{\lambda}{\beta}\right) f_X^2 - \frac{1}{2}\left(1 - \frac{\kappa}{\beta}\right) f_{XX}^2 + \frac{1}{2}\frac{\gamma}{\beta} f_{XXX}^2 + \frac{1}{4}\left(1 - \frac{\xi}{\beta}\right) f_X^4 - \frac{1}{2}\frac{\eta}{\beta} f_X^2 f_{XX}^2 \right\rangle,$$

where $\langle\,\rangle$ denotes an integration over the domain of f. For appropriate boundary conditions on f this system is variational

$$f_T = -\frac{\delta \mathcal{V}}{\delta f}$$

only if $\zeta = \frac{1}{3}\eta$. Otherwise the equation is not variational.

For appropriate signs of the coefficients $\mathcal{V}$ is bounded from below and so is a Lyapunov functional. Hence all solutions tend to steady states for long times.

Large ρ

Letting $F := f_X$, we then have

$$\begin{aligned} F_T &= \frac{\lambda}{\rho} F + \left(\mu + \frac{\kappa}{\rho}\right) F_{XX} + \left(1 - \frac{\gamma}{\rho}\right) F_{XXXX} \\ &\quad + \frac{\xi}{\rho} F^3 + \frac{\eta}{\rho} F F_X^2 + \left(1 - \frac{\zeta}{\rho}\right) \left(F^3\right)_{XX} . \end{aligned}$$

In the limit where $\gamma = \rho,\ \eta = 0,\ \zeta = \rho$ we recover a real Ginzburg–Landau equation. The functional

$$\begin{aligned} \mathcal{V}[f] &= \left\langle -\frac{1}{2}\frac{\lambda}{\rho} F^2 + \frac{1}{2}\left(\mu + \frac{\kappa}{\rho}\right) F_X^2 - \frac{1}{2}\left(1 - \frac{\gamma}{\rho}\right) F_{XX}^2 \right. \\ &\quad \left. - \frac{1}{4}\frac{\xi}{\rho} F^4 - \frac{1}{2}\frac{\eta}{\rho} F^2 F_X^2 \right\rangle \end{aligned}$$

will generate F_T if $\zeta - \rho = \frac{1}{3}\eta$. It is not variational otherwise.

Hamiltonian Limit

If we go back to the unscaled equations there are values of the parameters for which the system is Hamiltonian with

$$H[f,g] = \frac{1}{2}\left\langle -\bar{\alpha}\, g_X^2 + \bar{\lambda}\, f_X^2 - \bar{\kappa}\, f_{XX}^2 - \bar{\gamma}\, f_{XXX}^2 - 2\bar{\mu}\, f_X\, g_X \right.$$
$$\left. +2\bar{\nu}\, f_{XX}\, g_{XX} - \bar{p}\, \left(f_X^3\right)\, g_X + \frac{1}{2}\bar{\xi}\, f_X^4 - \bar{\eta}\, f_X^2\, f_{XX}^2 \right\rangle$$

In particular, when $\bar{\mu} = \bar{\nu} = \bar{p} = \bar{\eta} = 0$, we have

$$\begin{aligned} \bar{\alpha}^{-1} f_{TT} \quad = \quad & \bar{\lambda}\, f_{XXXX} + \bar{\kappa}\, f_{XXXXXX} \\ & - \bar{\gamma}\, f_{XXXXXXXX} + \bar{\xi}\left(f_X^3\right)_{XXX}. \end{aligned}$$

For $\gamma = 0$ this is the same equation that was derived by Childress and Spiegel 1981.

Steady Solutions

If we let $F = f_X$ and $G = g_X$, their time evolution is given by

$$\begin{aligned}
F_T &= G_{XX} + \mu\, F_{XX} + F_{XXXX} + \left(F^3\right)_{XX} \\
G_T &= \lambda\, F_{XX} + \kappa\, F_{XXXX} - \gamma\, F_{XXXXXX} + \beta\, G_{XX} \\
&\quad - \rho\, G_{XXXX} + \xi \left(F^3\right)_{XX} + \left(F^2\, G\right)_{XX} \\
&\quad + \eta \left(F\, F_X^2\right)_{XX} - \zeta \left(F^3\right)_{XXXX}.
\end{aligned}$$

For steady solutions we have just one equation

$$\begin{aligned}
0 &= (\lambda - \mu\beta)F + (\kappa - \beta + \mu\rho)F_{XX} - (\gamma - \rho)F_{XXXX} \\
&\quad - (\mu + \beta - \xi)F^3 + \left(\rho - \zeta - \frac{1}{3}\right)\left(F^3\right)_{XX} \\
&\quad + (\eta + 2)F F_X^2 - F^5
\end{aligned}$$

Conclusions

- For anisotropic double-diffusion in long-wave theory, we have shown that an extended system equation can be asymptotically derived.

- The equation contains several known equations as limits.

- We've begun exploring steady nonlinear solutions. Their stability still remains to be determined.

- Make connection with physics.