Random braids

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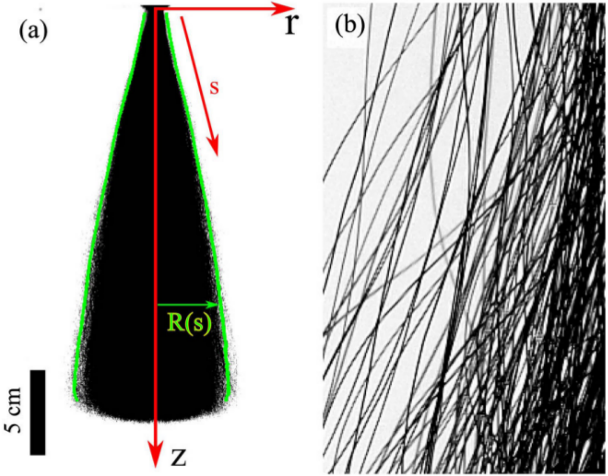
Complex entanglements are everywhere

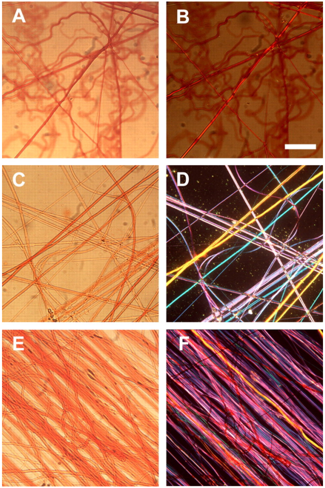
Tangled hair



[Goldstein, R. E., Warren, P. B., & Ball, R. C. (2012). *Phys. Rev. Lett.* **108**, 078101]

Tangled hair in the movies

Tangled hagfish slime

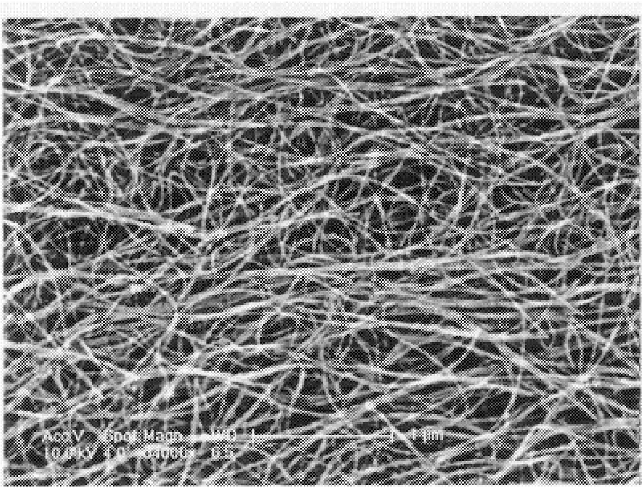


Slime secreted by hagfish is made of microfibers.

The quality of entanglement determines the material properties (rheology) of the slime.

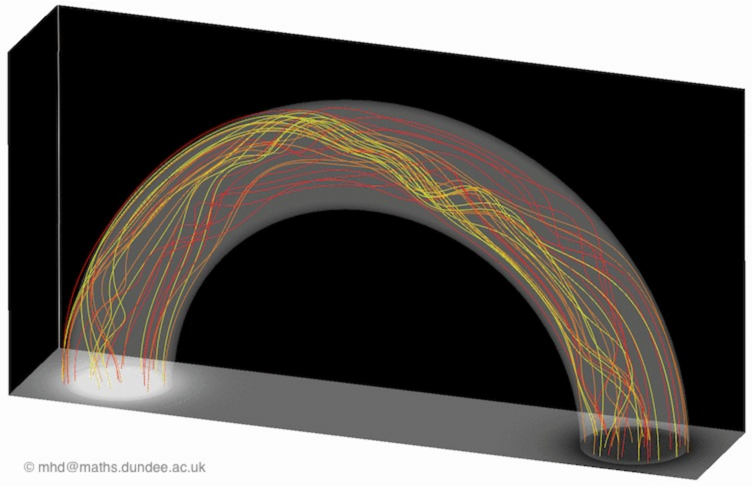
[Fudge, D. S., Levy, N., Chiu, S., & Gosline, J. M. (2005). *J. Exp. Biol.* **208**, 4613–4625]

Tangled carbon nanotubes



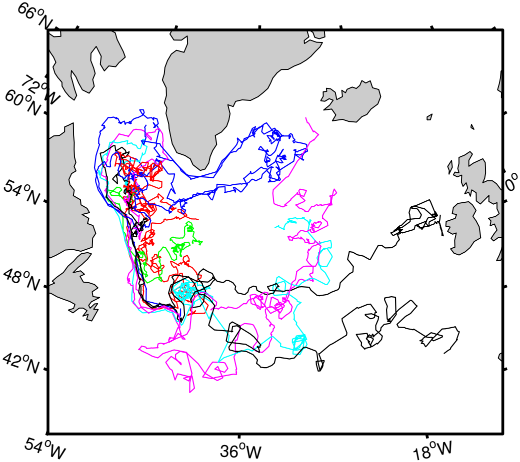
[Source: http://www.ineffableisland.com/2010/04/carbon-nanotubes-used-to-make-smaller.html]

Tangled magnetic fields



[Source: `http://www.maths.dundee.ac.uk/mhd/`]

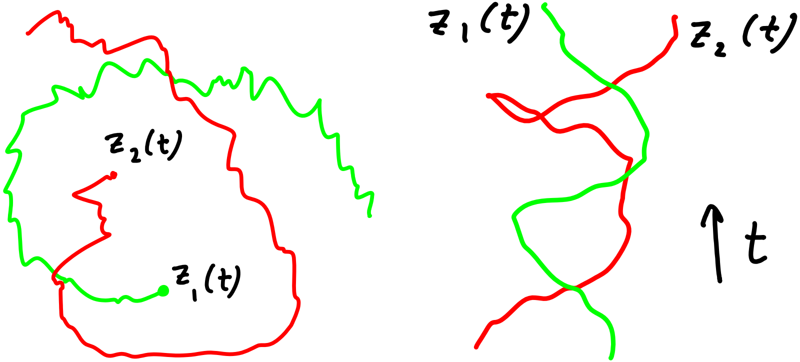
Tangled oceanic float trajectories



[Source: WOCE subsurface float data assembly center, `http://wfdac.whoi.edu`, J-LT (2010). *Chaos*, **20**, 017516]

The simplest tangling problem

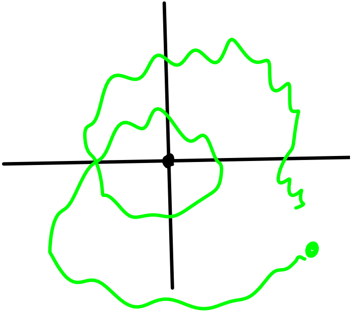
Consider two Brownian motions on the complex plane, each with diffusion constant D:



Viewed as a spacetime plot, these form a ‘braid’ of two strands.

Winding angle

Take the vector $z(t) = z_1(t) - z_2(t)$, which behaves like a Brownian particle of diffusivity $2D$ ($\to D$):



Define $\theta \in (-\infty, \infty)$ to be the total winding angle of $z(t)$ around the origin.

Winding angle distribution

Spitzer (1958) found the time-asymptotic distribution of θ to be Cauchy:

$$P(x) \sim \frac{1}{\pi}\frac{1}{1+x^2}\,, \qquad x := \frac{\theta}{\log(2\sqrt{Dt}/r_0)}\,, \qquad 2\sqrt{Dt}/r_0 \gg 1,$$

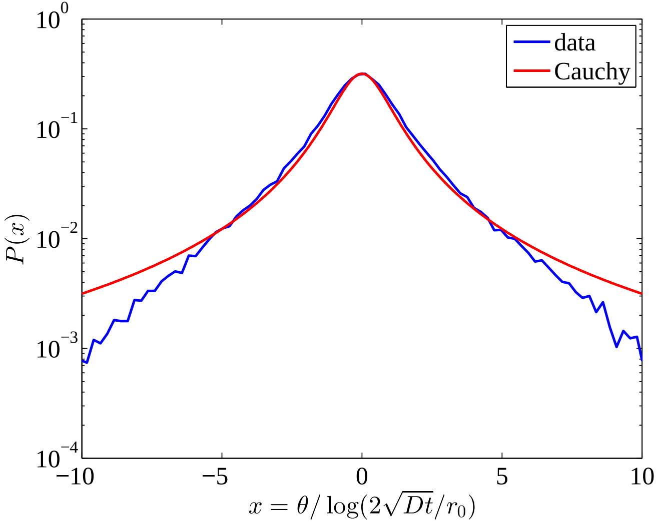
where $r_0 = |z(0)|$.

The scaling variable is $\sim \theta/\log t$.

Note that a Cauchy distribution is a bit strange: the variance is infinite, so large windings are highly probable!

[Spitzer, F. (1958). *Trans. Amer. Math. Soc.* **87**, 187–197]

Winding angle distribution: numerics



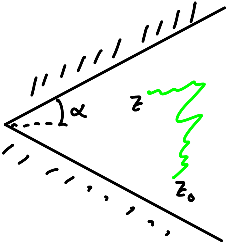
(Well, the tails don't look great: a pathology of Brownian motion.)

Winding angle distribution: derivation

The probability distribution $P(z,t)$ of the Brownian process satisfies the heat equation:

$$\frac{\partial P}{\partial t} = D\Delta P, \qquad P(z,0) = \delta(z - z_0).$$

Consider the solution in a wedge of half-angle α:



(Take either reflecting or absorbing boundary condition at the walls.)

Winding angle distribution: derivation (cont'd)

The solution is standard, but now take the wedge angle α to ∞ (!):

$$P(z,t) = \frac{1}{2\pi Dt}\,\mathrm{e}^{-(r^2+r_0^2)/4Dt}\int_0^\infty \cos\nu(\theta-\theta_0)\,I_\nu\!\left(\frac{r\,r_0}{2Dt}\right)\mathrm{d}\nu$$

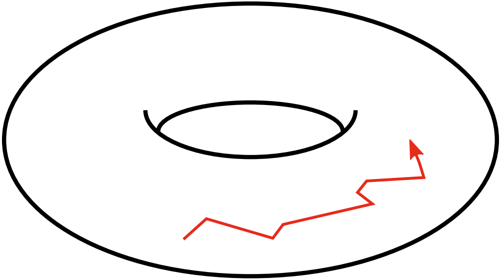
where I_ν is a modified Bessel function of the first kind, and r, θ are the polar coordinates of $z = x + \mathrm{i}y$.

For large t this recovers the Cauchy distribution.

Key point: by allowing the wedge angle to infinity, we are using Riemann sheets to keep track of the winding angle.

Related example: Brownian motion on the torus

A Brownian motion on a torus can wind around the two periodic directions:

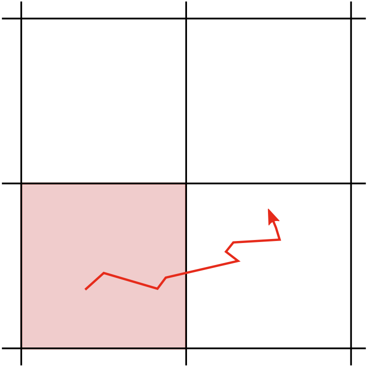


What is the asymptotic distribution of windings?

Mathematically, we are asking what is the homology class of the motion?

Torus: universal cover

We pass to the universal cover of the torus, which is the plane:



The universal cover records the windings as paths on the plane. The original 'copy' is called the fundamental domain.

On the plane the probability distribution is the usual Gaussian heat kernel:

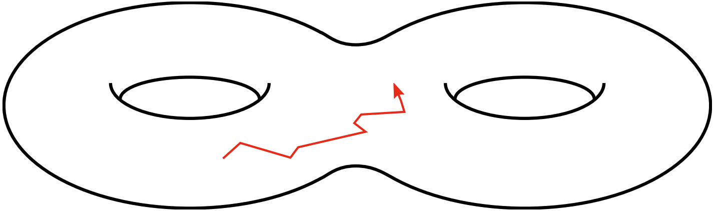
$$P(x, y, t) = \frac{1}{4\pi Dt}\,\mathrm{e}^{-(x^2+y^2)/4Dt}$$

So here $m = \lfloor x \rfloor$ and $n = \lfloor y \rfloor$ will give the homology class: the number of windings of the walk in each direction.

We can think of the motion as entangling with the space itself.

Brownian motion on the double-torus

On a genus two surface (double-torus):

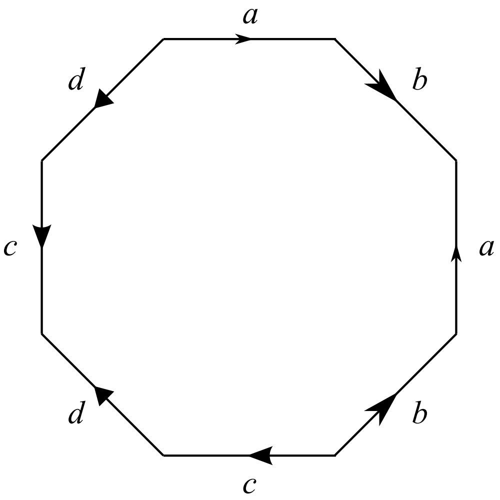


Same question: what is the entanglement of the motion with the space after a long time?

Now homology classes are not enough, since the associated universal cover has a non-Abelian group of deck transformations. In other words, the order of going around the holes matters!

The non-Abelian case involves homotopy classes.

The ‘stop sign’ representation of the double-torus

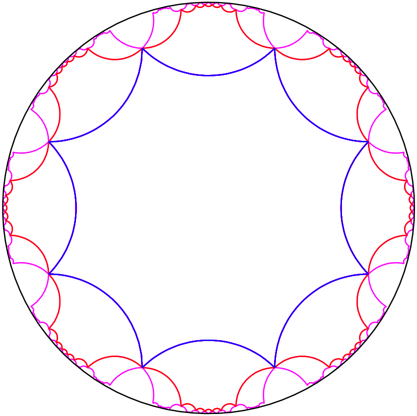


(Identify edges, respecting orientation.)

Problem: can’t tile the plane with this!

Universal cover of the double-torus

Embed the octogon on the Poincaré disk, a space with constant negative curvature:



(These curved lives are actually straight geodesics.)

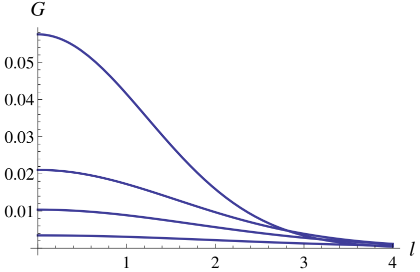
Then we can tile the disk with isometric copies of our octogon (fundamental domain).

Heat kernel on the Poincaré disk

From Chavel (1984), the Green's function for the heat equation $\partial_t \theta = \Delta\theta$ on the Poincaré disk is

$$G(\ell, t) = \frac{\sqrt{2}\,\mathrm{e}^{-t/4}}{(4\pi t)^{3/2}} \int_\ell^\infty \frac{\beta\,\mathrm{e}^{-\beta^2/4t}}{\sqrt{\cosh\beta - \cosh\ell}}\,\mathrm{d}\beta,$$

where ℓ is the hyperbolic distance between the source and target points.

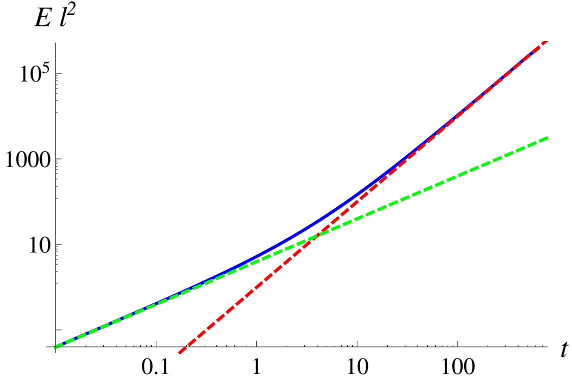


$t = 1, 2, 3, 5$

A priori, this behaves a lot like the planar heat kernel.

Squared-displacement on the Poincaré disk

Expected value of ℓ^2 as a function of time:



The green dashed line is $4t$ (diffusive), the red dashed line is t^2 (ballistic).

Surprising result: not diffusive for large time! Why?

Recurrence on the Poincaré disk

The probability of recurrence (coming back to the origin) from a distance ℓ is

$$\int_0^\infty G(\ell, t)\,\mathrm{d}t = \frac{1}{2\pi}\log\coth(\ell/2) \sim \frac{1}{\pi}\,\mathrm{e}^{-\ell}, \quad \ell \gg 1.$$

Hence, even though it is two-dimensional, a Brownian motion on the hyperbolic plane is transient.

Put another way, if the particle wanders too far from the origin, then it will almost certainly not return. It is hopelessly entangled.

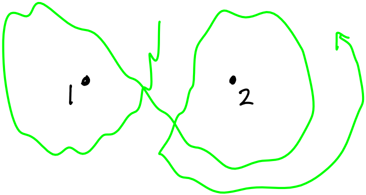
This spontaneous entanglement property arises because of the natural hyperbolicity of the surface, i.e., its universal cover is the Poincaré disk.

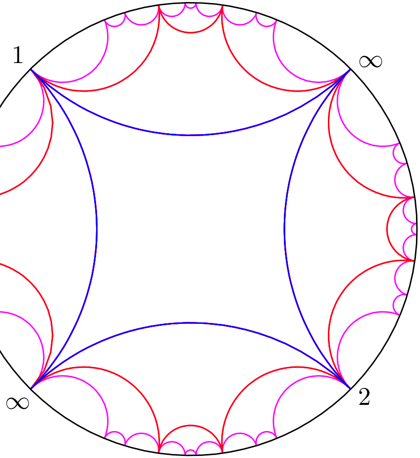
[Nechaev, S. K. (1996). *Statistics of Knots and Entangled Random Walks*. Singapore; London: World Scientific]

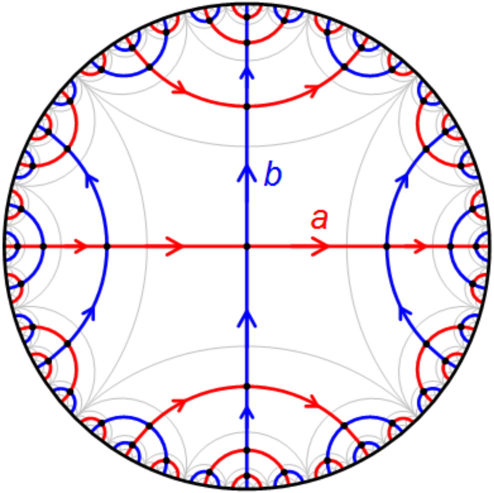
Universal cover of twice-punctured plane

Consider now winding around two points in the complex plane.

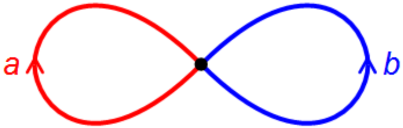
Topologically, this space is like the sphere with 3 punctures, where the third puncture is the point at infinity.







We really only care about which 'copy' of the fundamental domain we're in. Can use a tree to record this.



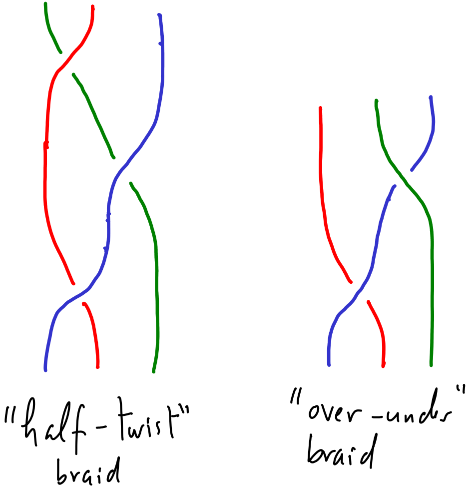
The history of a path is encoded in a 'word' in the letters a, b, a^{-1}, b^{-1}.

(Free group with two generators.)

[Source: Wikipedia]

Quality of entanglement

Compare these two braids:



Repeating these increases distance in the universal cover. . .

But clearly the pigtail is more “entangled”

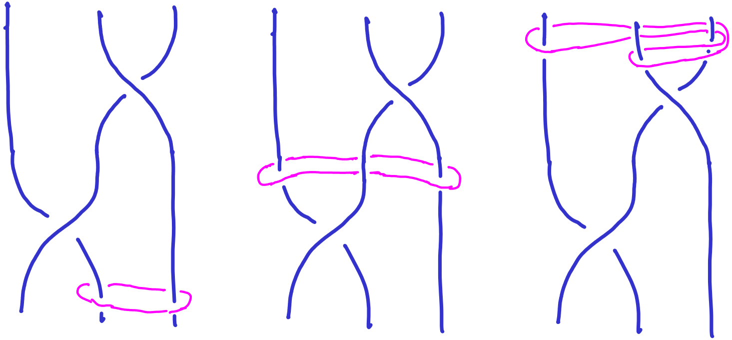
Over-under (pigtail) is very robust, unlike simply twisting. How do we capture this difference?

[http://www.lovethispic.com/image/24844/pigtail-braid]

Topological entropy

Inspired by dynamical systems. (Related to: braiding factor, braid complexity.)

Cartoon: compute the growth rate of a loop slid along the rigid braid.



This is relatively easy to compute using braid groups and loop coordinates.

[See Dynnikov (2002); J-LT (2005); J-LT & Finn (2006); Moussafir (2006); Dynnikov & Wiest (2007); J-LT (2010)]

Topological entropy: bounds

In Finn & J-LT (2011) we proved that

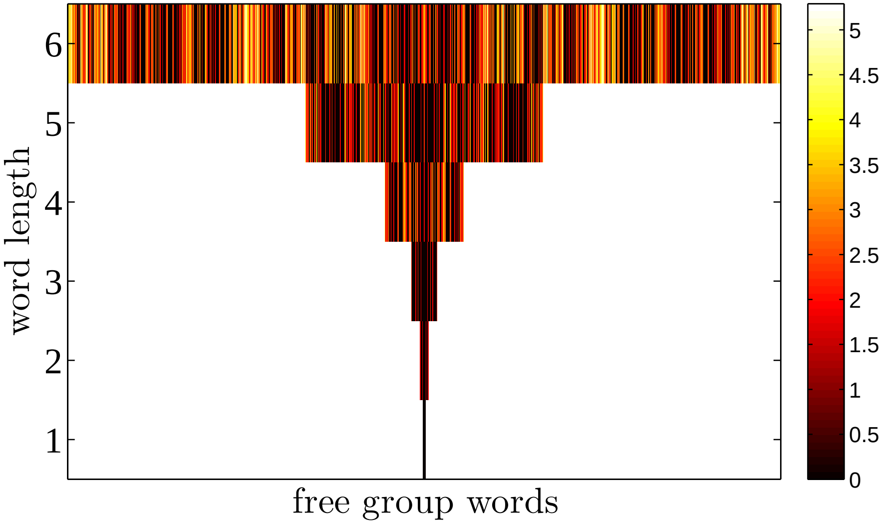
$$\boxed{\frac{\text{topological entropy}}{\text{braid length}} \le \log(\text{Golden ratio})}$$

This maximum entropy is exactly realized by the pigtail braid, reinforcing the intuition that it is somehow the most 'sturdy' braid.

[Finn, M. D. & J-LT (2011). *SIAM Rev.* **53** (4), 723–743]

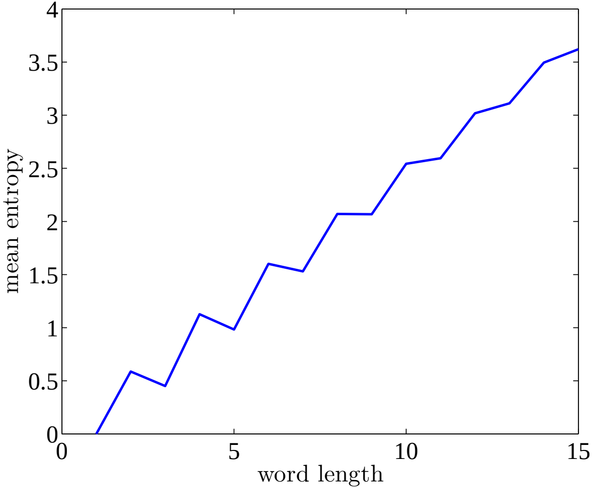
Word length vs topological entropy

For the plane with two punctures, we can relate entropy to word length.



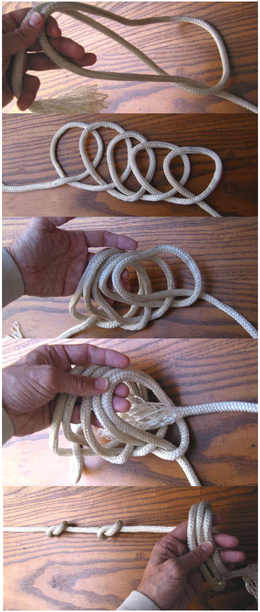
Mean topological entropy

Averaged over all words of a given length, entropy grows linearly:



(This assumes all words are equally probable, which is not necessarily true.)

Another viewpoint: how hard is detangling?



Buck & Scharein (2014) take another approach: the 'rope trick' on the left shows how to create a sequence of simple knots with a single final 'pull.'

They show that creating the knots takes work proportional to the length, but undoing the knots is quadratic in the length, because the knots must be loosened one-by-one.

This asymmetry suggests why it's easy to tangle things, but hard to disentangle.

[Buck, G. & Scharein, R. (2014). preprint]

Conclusions & outlook

- Entanglement at confluence of dynamics, probability, topology, and combinatorics.
- Instead of Brownian motion, can use orbits from a dynamical system. This yields dynamical information.
- More generally, study random processes on configuration spaces of sets of points (also finite size objects).
- Other applications: Crowd dynamics (Ali, 2013), granular media (Puckett *et al.*, 2012).
- With Michael Allshouse: develop tools for analyzing orbit data from this topological viewpoint (Allshouse & J-LT, 2012).
- With Tom Peacock and Margaux Filippi: apply to orbits in a fluid dynamics experiments.

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