Plasma Stability and Dynamical Accessibility

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Overview

- We attempt to provide a unified description of variational methods for establishing stability of plasma equilibria.
- The first approach discussed, which for plasmas was introduced by Bernstein *et al.* [1], is based upon a Lagrangian approach (in the sense of fluid elements). A Lagrangian equilibrium is static.
- If there is a symmetry in the system, one can use the process of reduction to derive a smaller set of equations from the Lagrangian description. Equilibria for the smaller system can have flow (steady, but not static).
- The most important such symmetry is the relabeling symmetry, which leads to an Eulerian description, where knowledge of the position of fluid elements disappears.

Equations of Motion

We consider the equations of motion for an inviscid, ideally conducting fluid:

$$\begin{aligned} \rho\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) &= -\nabla p + \mathbf{J}\times\mathbf{B} \\ \partial_t\rho + \nabla\cdot(\rho\,\mathbf{v}) &= 0 \\ \partial_t s + \mathbf{v}\cdot\nabla s &= 0 \\ \partial_t\mathbf{B} &= \nabla\times(\mathbf{v}\times\mathbf{B})\,. \end{aligned}$$

$\mathbf{v}$ is the fluid velocity, p the pressure, ρ the density, s the entropy, $\mathbf{B}$ the magnetic field, and $\mathbf{J} = \nabla\times\mathbf{B}$ the electric current. The dynamical equations are supplemented by the constraint $\nabla\cdot\mathbf{B} = 0$.

Constants of Motion

The Hamiltonian (total energy) for the system is conserved:

$$H = \int \mathrm{d}^3x \left(\tfrac{1}{2}\,\rho\,|\mathbf{v}|^2 + \tfrac{1}{2}\,|\mathbf{B}|^2 + \rho\,U(\rho, s)\right)$$

U is the internal energy, with $p = \rho^2(\partial U/\partial \rho)_s$.

The system possesses other invariants, such as the helicity and cross-helicity, depending on the initial configuration [15, 19].

Lagrangian (static) Equilibrium

Equilibrium quantities are denoted by a subscript e.

Setting ∂_t and $\mathbf{v}_\mathrm{e}$ to zero, the only condition is

$$\nabla p_\mathrm{e} = (\nabla \times \mathbf{B}_\mathrm{e}) \times \mathbf{B}_\mathrm{e}$$

along with $\nabla \cdot \mathbf{B}_\mathrm{e} = 0$.

To determine a sufficient condition for stability, we consider perturbations about a static equilibrium

$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}(\mathbf{x}_0, t),$$

where $\mathbf{x}$ is the position of a fluid element at time t and $\boldsymbol{\xi}(\mathbf{x}_0, t)$ is the Lagrangian displacement, with $\boldsymbol{\xi}(\mathbf{x}_0, 0) = 0$.

After computing the variations of the various physical quantities and linearizing the equations of motion with respect to $\boldsymbol{\xi}$ (Bernstein *et al.* [1]), we obtain

$$\boxed{\rho_0\,\ddot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi}),}$$

where

$$\mathbf{F}(\boldsymbol{\xi}) := \nabla_0\left[\rho_0\left(\frac{\partial p_0}{\partial \rho_0}\right)_{s_0}\nabla_0\cdot\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla_0)\,p_0\right] + \mathbf{J}_0\times\mathbf{Q} - \mathbf{B}_0\times(\nabla_0\times\mathbf{Q})$$

and

$$\mathbf{Q} := \nabla_0\times(\boldsymbol{\xi}\times\mathbf{B}_0)$$

Linear stability is then guaranteed if

$$\delta^2 W(\boldsymbol{\xi},\boldsymbol{\xi}) := -\tfrac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}^3x \geq 0.$$

This is Lagrange's principle: the potential energy needs to be positive-definite for stability.

Equilibria with Symmetry

If the equations possess a symmetry, we may use the process of reduction (see e.g. Morrison [16]) to decrease the order of the system.

Newcomb [4] finds axially symmetric equilibria, with flow in the toroidal direction. We can move to a reference frame where the equilibrium appears static, even though it has flow along the symmetry direction.

Eulerian Equilibria

An important reduction is the relabeling symmetry, by which we pass from the Lagrangian to the Eulerian picture [16]. The equilibria then represent steady flows. We want to use different methods than before, because we would rather not have to find explicitly the trajectory of fluid elements.

Two approaches:

- "Eulerianized" Lagrangian displacements (Frieman and Rotenberg [3], Newcomb [4]), by which the displacements are re-expressed in terms of Eulerian variables only.

- Dynamically accessible variations [16], a method for generating variations which preserve the Casimir invariants of the system (a generalization of "Arnold's method" [6]).

“Eulerianized” Lagrangian Displacement

The idea here is to express the Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}_0, t)$ in terms of the Eulerian coordinates $\mathbf{x}$:

$$\boldsymbol{\eta}(\mathbf{x}, t) = \boldsymbol{\xi}(\mathbf{x}_0, t)$$

The variations are [4]

$$\begin{aligned}
\delta\mathbf{v} &= \dot{\boldsymbol{\eta}} + \mathbf{v}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\mathbf{v} \\
\delta\rho &= -\nabla\cdot(\rho\,\boldsymbol{\eta}) \\
\delta s &= -\boldsymbol{\eta}\cdot\nabla s \\
\delta\mathbf{B} &= \nabla\times(\boldsymbol{\eta}\times\mathbf{B}).
\end{aligned}$$

Then the energy can be varied with respect to these perturbations, and a sufficient stability criterion is obtained. Note that since the variations are arbitrary, $\boldsymbol{\eta}$ and $\dot{\boldsymbol{\eta}}$ are independent.

Hamiltonian Formulation

The inviscid, ideally conducting fluid has a Hamiltonian formulation in terms of a noncanonical bracket

$$\{F\,,G\} = -\left(\int \mathrm{d}^3x\, F_\rho\, \nabla\cdot G_{\mathbf{v}} + F_{\mathbf{v}}\cdot\left(\frac{(\nabla\times\mathbf{v})}{2\rho}\times G_{\mathbf{v}}\right)\right.$$
$$\left. +\,\rho^{-1}\,\nabla s\,\cdot\,(F_s\; G_{\mathbf{v}}) + \rho^{-1} F_{\mathbf{v}}\cdot(\mathbf{B}\times(\nabla\times G_{\mathbf{B}}))\right) + \Big(F \longleftrightarrow G\Big)\,.$$

F and G are functionals of the dynamical variables $(\mathbf{v},\rho,s,\mathbf{B})$, and subscripts denote functional derivatives. The bracket $\{\ ,\ \}$ is antisymmetric and satisfies the Jacobi identity. The equations of motion of page 3 can be written

$$\boxed{\partial_t(\mathbf{v},\rho,s,\mathbf{B}) = \{(\mathbf{v},\rho,s,\mathbf{B})\,,H\,\}}$$

in terms of the Hamiltonian (page 4).

Dynamical Accessibility

Another method establishing formal stability uses dynamically accessible variations (DAV), defined for the variable ζ as

$$\delta\zeta_{\mathrm{da}} := \{\mathcal{G}\,,\zeta\}\,, \qquad \delta^2\zeta_{\mathrm{da}} := \tfrac{1}{2}\,\{\mathcal{G}\,,\,\{\mathcal{G}\,,\zeta\}\}\,,$$

with $\mathcal{G}$ given in terms of the generating functions χ_μ by

$$\mathcal{G} := \int \zeta^\mu\,\chi_\mu\,\mathrm{d}^3x.$$

DAV are variations that are constrained to remain on the symplectic leaves of the system. They preserve the Casimir invariants to second order (but there is no need to explicitly know the invariants). Stationary solutions ζ_{e} of the Hamiltonian,

$$\delta H_{\mathrm{da}}[\zeta_{\mathrm{e}}] = 0,$$

capture all possible equilibria of the equations of motion.

Energy Associated with DAVs

The energy of the perturbations is

$$\delta^2 H_{\mathrm{da}}[\zeta_{\mathrm{e}}] = \tfrac{1}{2} \int \left(\delta\zeta^\sigma_{\mathrm{da}} \, \frac{\delta^2 H}{\delta\zeta^\sigma \, \delta\zeta^\tau} \, \delta\zeta^\tau_{\mathrm{da}} + \delta^2\zeta^\nu_{\mathrm{da}} \, \frac{\delta H}{\delta\zeta^\nu} \right) \, \mathrm{d}^3 x,$$

with $\zeta = (\mathbf{v}, \rho, s, \mathbf{B})$ and repeated indices are summed.

This is essentially the expression obtained by Isichenko [17], though he "guessed" at the form of $\delta^2\zeta_{\mathrm{da}}$ and so obtained a slightly incorrect result.

Positive-definiteness of $\delta^2 H_{\mathrm{da}}[\zeta_{\mathrm{e}}]$ implies formal stability, which implies linear stability, but not nonlinear stability.

The form of the dynamically accessible perturbations is

$$\rho\,\delta\mathbf{v}_{\mathrm{da}} = (\nabla\times\mathbf{v})\times\boldsymbol{\chi}_{\mathbf{0}} + \rho\,\nabla\chi_1 - \chi_2\,\nabla s + \mathbf{B}\times(\nabla\times\boldsymbol{\chi}_{\mathbf{3}})$$

$$\delta\rho_{\mathrm{da}} = \nabla\cdot\boldsymbol{\chi}_{\mathbf{0}}$$

$$\delta s_{\mathrm{da}} = \rho^{-1}\,\boldsymbol{\chi}_{\mathbf{0}}\cdot\nabla s$$

$$\delta\mathbf{B}_{\mathrm{da}} = \nabla\times\left(\frac{\mathbf{B}\times\boldsymbol{\chi}_{\mathbf{0}}}{\rho}\right)$$

$\boldsymbol{\chi}_{\mathbf{0}}$, χ_1, χ_2, and $\boldsymbol{\chi}_{\mathbf{3}}$ are the arbitrary generating functions of the variations. The variations for ρ, s, and $\mathbf{B}$ are the same as on page 9, with $\boldsymbol{\chi}_{\mathbf{0}} = \rho\,\boldsymbol{\eta}$.

The combination of arbitrary functions in the definition of $\delta\mathbf{v}_{\mathrm{da}}$ makes that perturbation arbitrary, in the same manner as the perturbation $\delta\mathbf{v}$ on page 9.

Remarks

- The two approaches, using Lagrangian perturbations vs dynamical accessibility, lead to essentially the same stability criterion.

- Dynamical accessibility can be used directly at the Hamiltonian level, without any knowledge of underlying Lagrangian dynamics. One needs to know the Poisson bracket and Hamiltonian.

  The magnetofluid system has a semidirect product structure, which implies that it has a simple Lagrangian description. But for some other systems, for example 2D compressible reduced MHD [11], dynamical accessibility is easier to apply [18].

- Dynamical accessibility has also been applied to Vlasov–Maxwell equilibria [12, 13].

- The energy–Casimir method [9, 10], closely related to dynamical accessibility, requires knowledge of the invariants and doesn't quite capture all equilibria. However, it can sometimes be used to yield nonlinear stability criteria.

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