Invariants and Labels in Lie–Poisson Systems

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Overview

We explore the meaning of the Casimir invariants of some Lie–Poisson brackets.

Lie–Poisson brackets are a type of noncanonical bracket and are ubiquitous in the reduction of canonical Hamiltonian systems with symmetry. Examples include the heavy top, the moment reduction of the Kida vortex, the 2–D ideal fluid, reduced MHD, and the 1–D Vlasov equation.

Casimir invariants are conserved for all Hamiltonians; they only occur for brackets with degeneracy.

We review the derivation of Lie–Poisson brackets by the method of reduction, and interpret their Casimir invariants.

We then turn to building Lie–Poisson brackets directly from Lie algebras by the method of extension, specifically the *semidirect product* of algebras. Equations for the heavy top and low-beta reduced MHD are obtained in this manner.

The Casimir invariants of the bracket determine the manifold on which the system evolves. It is thus important to understand what sort of constraints they impose on a system. We show that, for the semidirect product, the Casimir invariants yield information about the configuration of the system, which was lost in the reduction.

Finally, we investigate a case where the extension is not of the semidirect type, namely compressible reduced MHD. The Casimir invariants in that case lend only partial information about the configuration of the system.

Reduction for the Free Rigid Body

Hamiltonian for the free rigid body in terms of Euler angles:

$$
\begin{aligned}
&H(p_\phi, p_\psi, p_\theta, \phi, \psi, \theta) = \\
&\quad \frac{1}{2}\left\{ \frac{\left[(p_\phi - p_\psi \cos\theta)\sin\psi + p_\theta \sin\theta\cos\psi\right]^2}{I_1 \, \sin^2\theta} \right. \\
&\quad \left. + \frac{\left[(p_\theta - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta\sin\psi\right]^2}{I_2 \, \sin^2\theta} + \frac{p_\psi^2}{I_3} \right\}
\end{aligned}
$$

Equations of motion are generated using the canonical bracket:

$$
\{f\,, g\} = \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial p_\phi} + \frac{\partial f}{\partial \psi}\frac{\partial g}{\partial p_\psi} + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial p_\theta} - (f \longleftrightarrow g)
$$

Here we have 3 degrees of freedom (6 coordinates). The configuration space is the rotation group $SO(3)$, the phase space is $T^*SO(3)$.

A reduction is possible for this system. In terms of angular momenta,

$$H(p_\phi, p_\psi, p_\theta, \phi, \psi, \theta) \longrightarrow H(\ell_1, \ell_2, \ell_3) = \sum_{i=1}^{3} \frac{\ell_i^2}{2I_i}$$

Under this *noncanonical* mapping, the bracket becomes of the *Lie–Poisson* form

$$\{f\,, g\} = -\ell \cdot \frac{\partial f}{\partial \ell} \times \frac{\partial g}{\partial \ell}$$

The equations of motion generated by the bracket from H are permutations of

$$\dot{\ell}_1 = \frac{I_2 - I_3}{I_2\, I_3}\, \ell_2\, \ell_3$$

These are Euler's equations for the rigid body. The Hamiltonian is conserved, and so is the quantity

$$C = \sum_{i=1}^{3} \ell_i^2$$

which commutes with any $f(\ell)$. We call C a *Casimir invariant*.

Casimirs are conserved quantities for any Hamiltonian, so they tell us about the topology of the manifold on which the motion takes place. For the simple case of the rigid body, the motion takes place on the two-sphere, S^2

The symmetry that permits the reduction is the invariance of the equations of motion for $(\phi, \psi, \theta, p_\phi, p_\psi, p_\theta)$ under rotations. This symmetry amounts to the freedom of choosing axes from which the Euler angles are measured. In that sense it is a $relabeling$ symmetry, since the choice of axes amounts to making "marks", or labels, on the rigid body.

At any time the exact configuration of the system is known ($Lagrangian$ system), whereas the reduced system is $Eulerian$ because only the angular momentum of the body is known.

Reduction for the 2–D Ideal Fluid

Hamiltonian functional for a 2–D incompressible, dissipationless fluid:

$$H[q;\pi] = \int_D \left(\frac{\pi^2}{2\rho_0} - p(a,t)\left(\left| \frac{\partial q}{\partial a} \right| - 1 \right) \right) \, d^2 a,$$

$\pi(a,t)$ is the momentum and $q(a,t)$ is the position of a fluid elements labeled by a.

This H together with the canonical bracket

$$\{F\,,G\}_{\rm can} = \int_D \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q}\frac{\delta F}{\delta \pi} \right) \, d^2 a$$

generates the equations of motion for a Lagrangian fluid. The information about the position of every fluid element at any time is contained in the model. There is a *relabeling* symmetry of the initial condition labels, a.

We introduce the streamfunction ϕ

$$v(\mathbf{x}, t) = (-\partial_y \phi, \partial_x \phi)$$

so that $\nabla \cdot v = 0$ is automatically satisfied, and the vorticity

$$\omega(\mathbf{x}, t) = \hat{z} \cdot \nabla \times v \, .$$

The noncanonical transformation from Lagrangian to Eulerian variables is

$$\omega(\mathbf{x}, t) = \int_D \frac{\pi(a, t)}{\rho_0} \; \times \nabla \delta(\mathbf{x} - q(a, t)) \, d^2 a \, .$$

Then, after some manipulation involving integration by parts we get the bracket

$$\{F \, , G\} = \int_D \omega \, \left[\frac{\delta F}{\delta \omega} \, , \frac{\delta G}{\delta \omega} \right] \, d^2 x$$

where

$$[f \, , g] := \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \, .$$

The equation of motion generated by the bracket and the transformed Hamiltonian

$$H[\omega] = -\frac{1}{2}\int_D \phi\,\omega\; d^2x = \frac{1}{2}\int_D |\nabla\phi|^2\; d^2x$$

is just Euler's equation for an the ideal fluid

$$\dot{\omega}(\mathbf{x}) = -[\phi\,,\omega]\,.$$

This has a Casimir given by

$$C[\omega] = \int_D f\,(\omega(\mathbf{x}))\; d^2x,$$

where f is arbitrary. This invariant implies the preservation of contours of ω, so that the value ω_0 on a contour labels that contour for all times. By choosing $f(\omega) = \theta(\omega(\mathbf{x}) - \omega_0)$, a heavyside function, it follows that the area inside of any ω-contour is conserved.

Extensions and the Semidirect Product

The simplest extension is the *direct product* of Lie algebras. Let ξ, ξ' be elements of a Lie algebra g and η, η' be elements of a vector space v (which is an Abelian Lie algebra under addition). The direct product of these two algebras is an algebra h of 2-tuples, (ξ, η) with bracket

$$[\,(\xi\,,\eta)\,,(\xi'\,,\eta')\,] := ([\,\xi\,,\xi'\,]\;,\;[\,\eta\,,\eta'\,])\,.$$

A less trivial extension is the *semidirect product* with an operation defined by

$$[\,(\xi\,,\eta)\,,(\xi'\,,\eta')\,] := ([\,\xi\,,\xi'\,]\;,\;[\,\xi\,,\eta'\,]+[\,\eta\,,\xi'\,])\,.$$

An example of a semidirect product structure is when g is the Lie algebra $so(3)$ associated with the rotation group $SO(3)$ and v is $\mathbb{R}^3$. Their semidirect product is the algebra of the 6-parameter Galilean group of rotations and translations.

We can build Lie–Poisson brackets from these algebras by extension by defining

$$\{F\,,G\} := \pm\left\langle \mu\,,\ \left[\frac{\delta F}{\delta \mu}\,,\frac{\delta G}{\delta \mu}\right]\right\rangle,$$

where $\mu \in h^*$, the dual of h under the pairing $\langle\,,\rangle : h^* \times h \to R$. The dynamical variables of the system are the elements of the n-tuple $\mu = \mu(t)$. These elements may be fields or variables, so the Lie–Poisson bracket derived from an algebra by extension generates the dynamics for a system involving several dynamical quantities.

Using this procedure to make a Lie–Poisson bracket from a direct product of algebras leads to a sum of n independent brackets.

We illustrate the process of building a Lie–Poisson bracket from a semidirect product of algebras by two examples, which are extensions of the rigid body and ideal fluid examples.

The Heavy Top

The Lie–Poisson bracket for the semidirect product of the rotation group and R^3 is

$$\{f\,,g\} = -\ell\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \ell}\right) - \alpha\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \alpha}+\frac{\partial f}{\partial \alpha}\times\frac{\partial g}{\partial \ell}\right)$$

where α denote a 3-vector. The Casimirs for this bracket are

$$C_1 = \alpha^2\,, \quad C_2 = \ell\cdot\alpha\ .$$

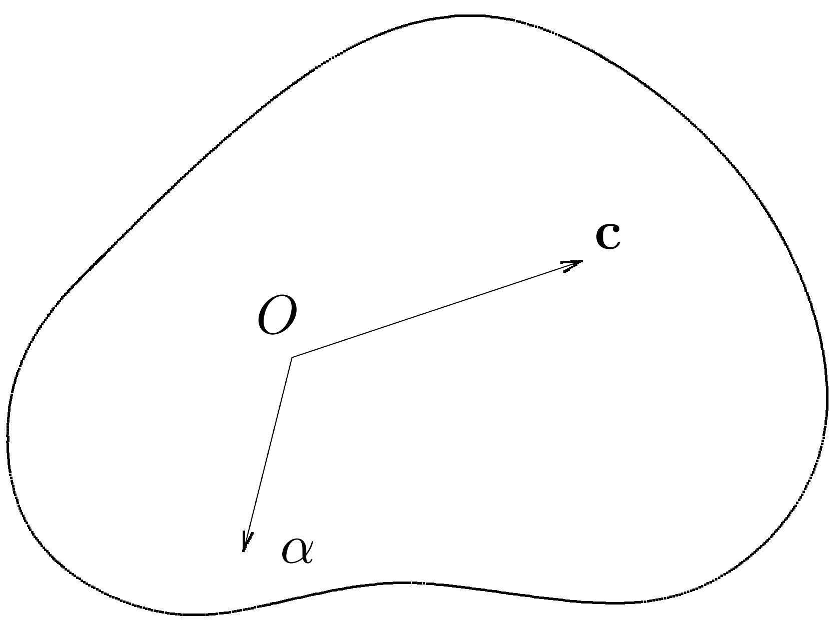
For a Hamiltonian quadratic in ℓ the vector α rotates rigidly with the body. Knowing α does not lead to a determination of the orientation of the rigid body: there is still a symmetry of rotation about α. Taking the semidirect product has led to the recovery of some of the Lagrangian (configuration) information.

By using

$$H(\ell, \alpha) = \sum_{i=1}^{3} \frac{\ell_i^2}{2 I_i} + \alpha \cdot \mathbf{c}$$

we get the prototypical example of a semidirect product system, the heavy rigid body (in the body frame):

$$\begin{aligned} \dot{\ell}_1 &= \frac{I_2 - I_3}{I_2 \, I_3} \ell_2 \, \ell_3 + \alpha_2 \, c_3 - \alpha_3 \, c_2 \\ \dot{\alpha}_1 &= \frac{\ell_3 \, \alpha_2}{I_3} - \frac{\ell_2 \, \alpha_3}{I_2} \end{aligned}$$



Low–β Reduced MHD

The semidirect product bracket for two fields is

$$\begin{aligned}\{F\,,G\} \;&=\; \int_D \left(\omega \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right]\right. \\ &\quad \left. + \psi \left(\left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \psi}\right] + \left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \omega}\right]\right)\right) d^2x\end{aligned}$$

If $\omega = \nabla^2\phi$, where ϕ is the electric potential, ψ is the magnetic flux, and $J = \nabla^2\psi$ is the current, then the Hamiltonian

$$H[\omega;\psi] = \frac{1}{2}\int_D \left(|\nabla\phi|^2 + |\nabla\psi|^2\right)\, d^2x$$

with the above bracket gives us

$$\begin{aligned}\dot{\omega} \;&=\; [\psi, J] + [\omega, \phi]\ , \\ \dot{\psi} \;&=\; [\psi, \phi]\ ,\end{aligned}$$

a model for low-β reduced MHD derived by Morrison and Hazeltine.

The bracket has Casimir invariants

$$C_1[\psi] = \int_D f(\psi)\, d^2x, \qquad C_2[\omega;\psi] = \int_D \omega\, g(\psi)\, d^2x.$$

The first has the same form as the one for 2–D Euler and has the same interpretation. To make sense of the second one let $g(\psi) = \theta(\psi - \psi_0)$.

$$C_2[\omega;\psi] = \oint_{\Psi_0} \omega\, d^2x.$$

where Ψ_0 represents the (not necessarily connected) region of D enclosed by the contour $\psi = \psi_0$, and $\partial\Psi_0$ is its boundary. The contour $\partial\Psi_0$ moves with the fluid, so this just expresses Kelvin's circulation theorem: the circulation around a closed material loop is conserved.

Putting Labels on a Rigid-body

Remember that taking a semidirect product restricted the symmetry group of the body to rotations about α. If we take another semidirect product to get

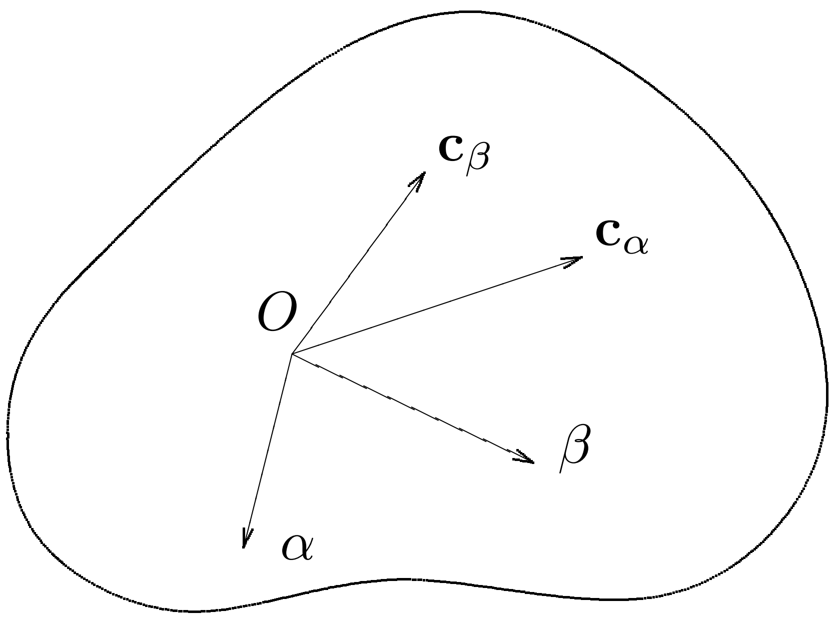
$$\begin{aligned} \{f\,,g\} \quad = \quad & -\ell\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \ell}\right) \\ & -\alpha\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \alpha}+\frac{\partial f}{\partial \alpha}\times\frac{\partial g}{\partial \ell}\right) \\ & -\beta\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \beta}+\frac{\partial f}{\partial \beta}\times\frac{\partial g}{\partial \ell}\right) \end{aligned}$$

where β is a 3-vector, the new bracket has Casimirs

$$C_1 = \alpha^2\,, \quad C_2 = \beta^2, \quad C_3 = \alpha\cdot\beta\,.$$

The angular momentum ℓ has disappeared from the Casimirs.

This can model a rigid body with two forces acting on it.



Note that knowing α and β completely specifies the orientation of the rigid body. In other words, by taking semidirect products we have reintroduced the Lagrangian information into the bracket.

Advection in an Ideal Fluid

For the ideal fluid, say low-β MHD with a second advected quantity, the pressure p, the Casimir is

$$C[\psi;p] = \int_D f(\psi,p)\,d^2x, \quad f \text{ arbitrary.}$$

This Casimir tells us we can label two contours. Locally this permits a unique labeling of the fluid elements as long as $\nabla\psi \times \nabla p$ does not vanish. However, globally there is still some ambiguity. Thus, in the infinite-dimensional case the semidirect product is not equivalent to recovering the full Lagrangian information, unless the contours do not close and are monotonic.

Beyond Semidirect: Cocycles

There are other ways to extend Lie algebras than the semidirect product. We have investigated brackets of the form

$$[\,\alpha\,,\beta\,]_\lambda = W_\lambda{}^{\mu\nu}\;[\,\alpha_\mu\,,\beta_\nu\,]$$

where λ is a component of an n-vector.

One example is the bracket derived by Hazeltine, Kotschenreuther, and Morrison (1985) for 2–D compressible reduced MHD, which has four fields. The Hamiltonian is

$$H[\omega;v;p;\phi] = \frac{1}{2}\left\langle|\nabla\phi|^2 + v^2 + \frac{(p-2\beta\,x)^2}{\beta} + |\nabla\psi|^2\right\rangle.$$

The bracket is rather large,

$$\begin{aligned}
\{A\,,B\} \;\; = \;\; & \left\langle \omega\,,\, \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta\omega}\right]\right\rangle \\
& + \left\langle v\,,\, \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta v}\right] + \left[\frac{\delta A}{\delta v},\frac{\delta B}{\delta\omega}\right]\right\rangle \\
& + \left\langle p\,,\, \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta p}\right] + \left[\frac{\delta A}{\delta p},\frac{\delta B}{\delta\omega}\right]\right\rangle \\
& + \left\langle \psi\,,\, \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta\psi}\right] + \left[\frac{\delta A}{\delta\psi},\frac{\delta B}{\delta\omega}\right] \right. \\
& \qquad \left. -\,\beta\left[\frac{\delta A}{\delta p},\frac{\delta B}{\delta v}\right] - \beta\left[\frac{\delta A}{\delta v},\frac{\delta B}{\delta p}\right]\right\rangle
\end{aligned}$$

The term proportional to β is an obstruction to the semidirect product structure, and it cannot be removed by a coordinate transformation. In the language of Lie algebra cohomology it is a *cocycle*.

Its Casimirs are

$$
\begin{aligned}
C_1[\psi] &= \int_D f(\psi)\, d^2x \\
C_2[p;\psi] &= \int_D p\, g(\psi)\, d^2x \\
C_3[v;\psi] &= \int_D v\, h(\psi)\, d^2x \\
C_4[\omega, v, p, \psi] &= \int_D \left(\omega k(\psi) + \frac{v\, p}{\beta}\, k'(\psi) \right)\, d^2x
\end{aligned}
$$

These do not allow a labeling of the fluid elements. Finding the invariant C_4 directly from the equations of motion would be tedious, but is straightforward from the bracket.

The meaning of invariants of the form of C_1, C_2, and C_3 has already been discussed: the total magnetic flux, pressure, and parallel velocity inside of any ψ-contour are preserved.

To understand C_4 we use the fact that $\omega = \nabla^2\phi$ and then integrate by parts to obtain

$$C_4[\omega, v, p, \psi] = \int_D \left(-\nabla\phi \cdot \nabla\psi + \frac{v\,p}{\beta} \right) k'(\psi)\, d^2x.$$

The quantity in parentheses is thus invariant inside of any ψ-contour.

It can be shown that this is a remnant of the conservation in the full MHD model of the cross helicity,

$$V = \int_D \mathbf{v} \cdot \mathbf{B}\, d^2x\,,$$

at second order in the inverse aspect ratio, while C_3 is a consequence of preservation of this quantity at first order. Here $\mathbf{B}$ is the magnetic field.

As for C_1 and C_2 they are, respectively, the first and second order remnants of the preservation of helicity,

$$W = \int_D \mathbf{A} \cdot \mathbf{B}\, d^2x,$$

where $\mathbf{A}$ is the magnetic vector potential.

Conclusions

- We gave an introduction to the reduction of physical systems based on their symmetries.

- The prototypical examples were shown, the rigid body and the 2–D ideal fluid.

- The semidirect product allows us to describe the group acting on larger systems. This led to the recovery of some or all of the Lagrangian information.

- For general extensions (not necessarily semidirect) things are different: the Lagrangian information is not necessarily a consequence of the Casimirs. However, for compressible reduced MHD the Casimirs represent constraints that are remnants of invariants of the full MHD equations from which the model is derived asymptotically.