Measuring Topological Chaos

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- Two related goals:
  - Optimisation — faster, cheaper (chemical engineering)
  - Diagnostic — what is doing the mixing? (geophysics)
- For fluid dynamics, mixing is one of the best reasons to study chaos, since sensitivity to initial conditions leads directly to good mixing.

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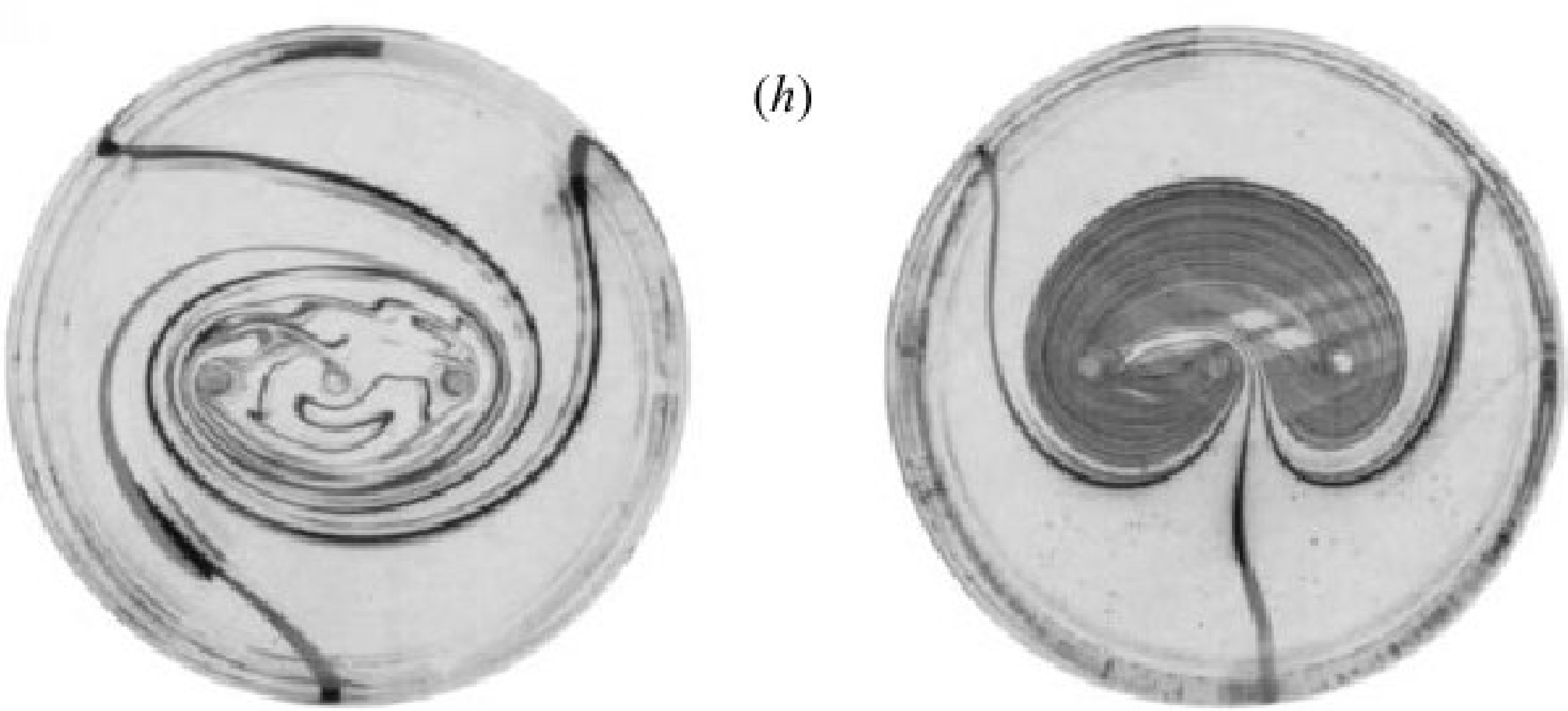
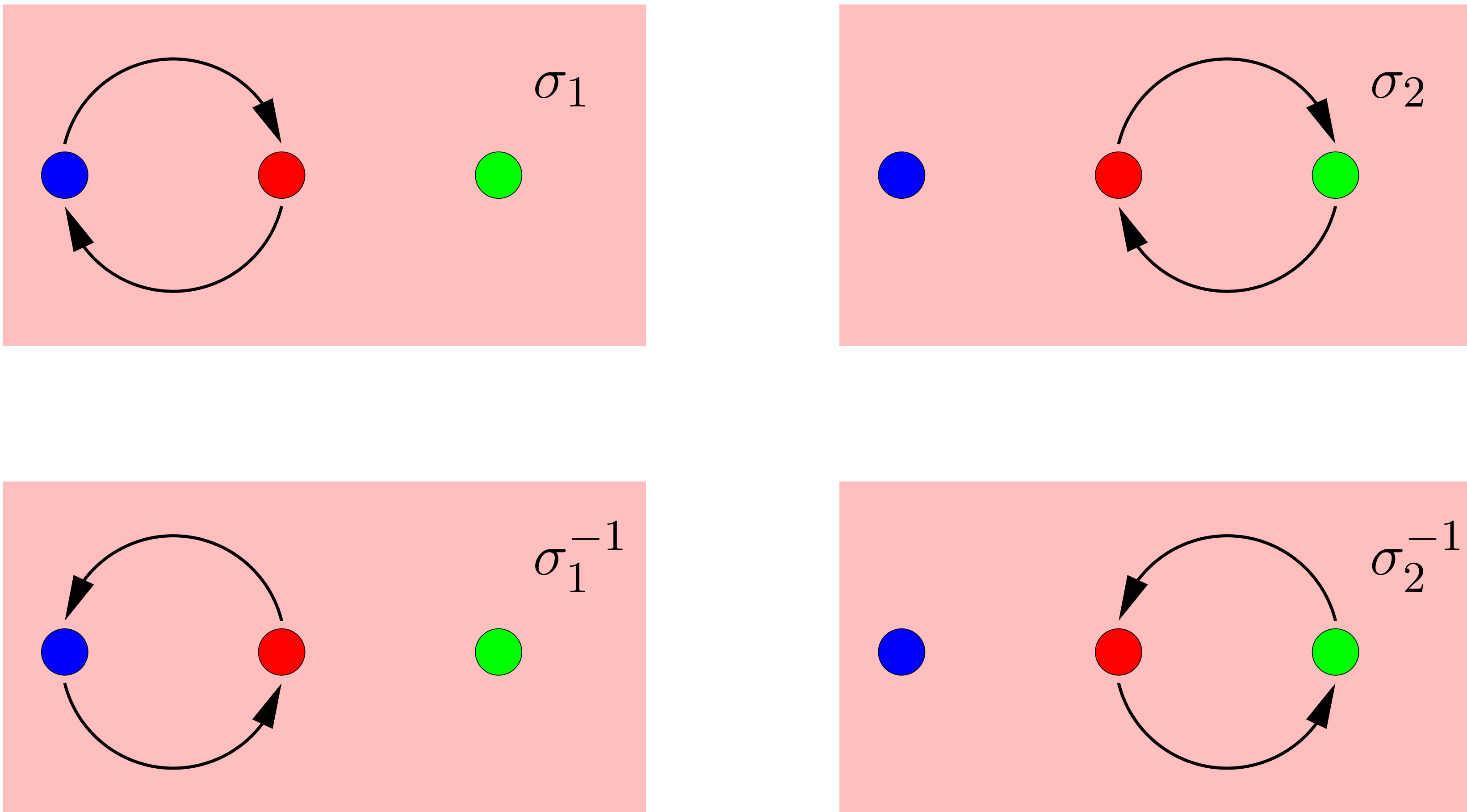
- Today: focus on chaos and topology.

Experiment of Boyland *et al.*



[P. L. Boyland, H. Aref, and M. A. Stremler, J. Fluid Mech. **403**, 277 (2000)]   (movie by Matthew Finn)

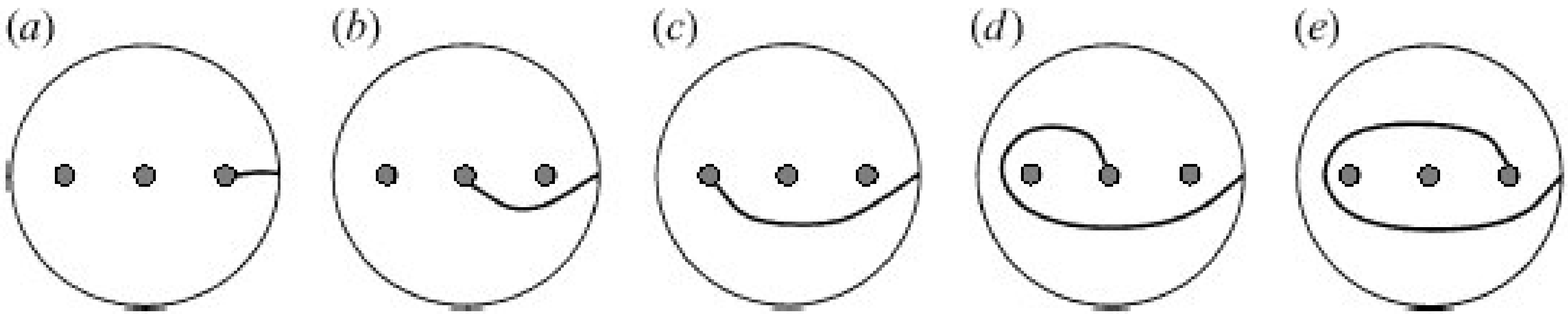
Four Basic Operations



σ_1 and σ_2 are referred to as the generators of the 3-braid group.

Two Stirring Protocols

$\sigma_1\sigma_2$ protocol



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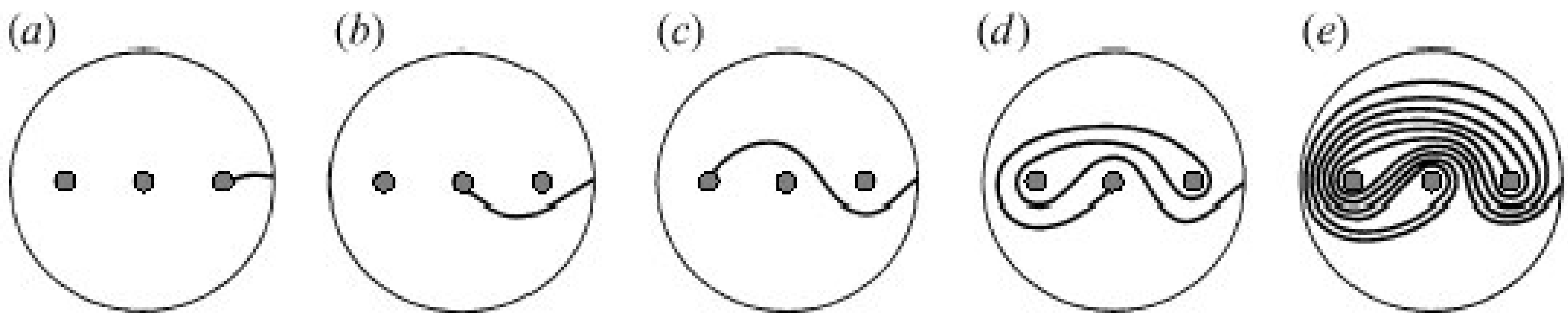

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Braiding

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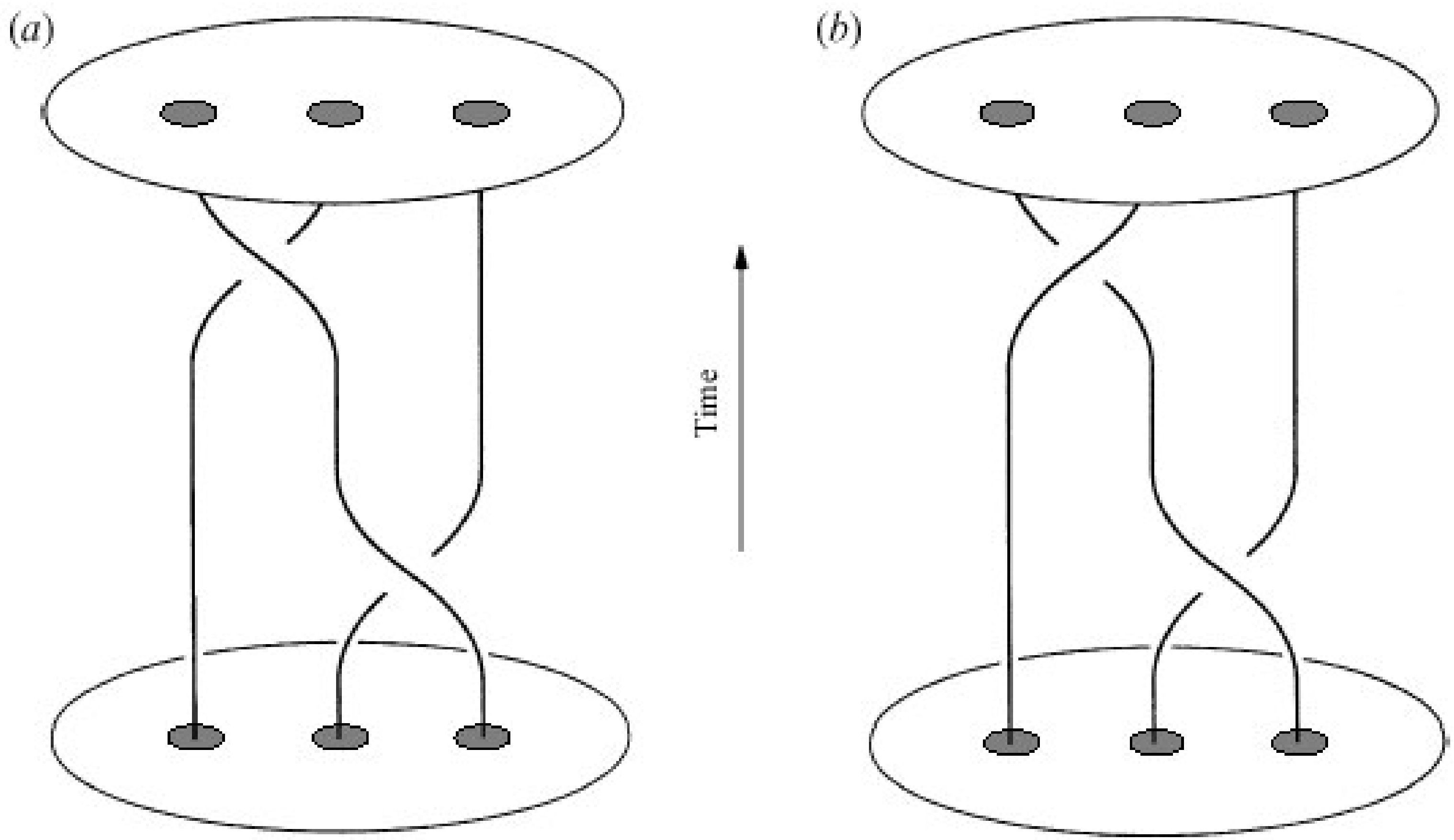

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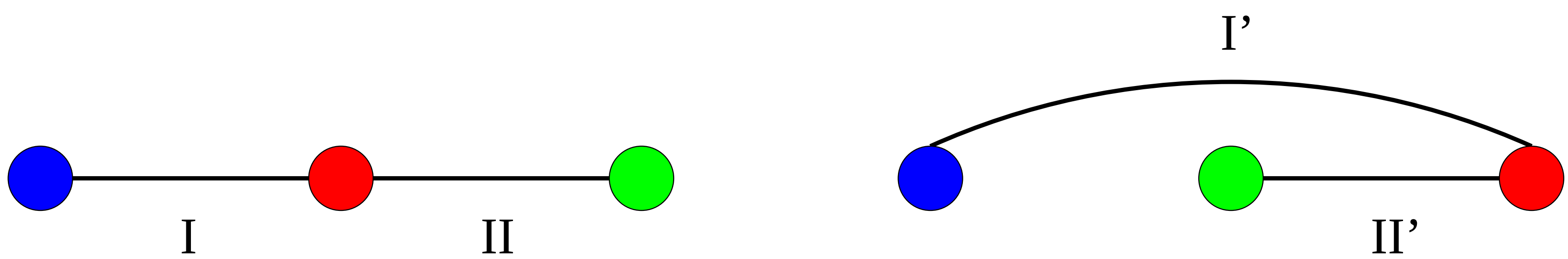
Matrix Representation of σ_2



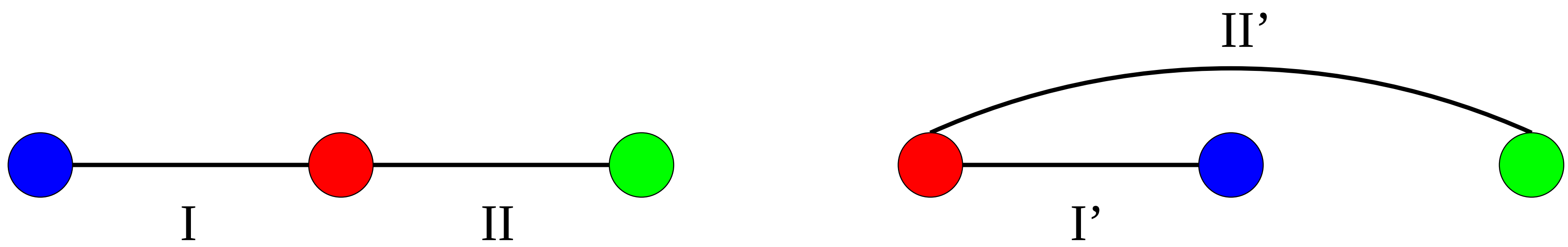
Let I and II denote the lengths of the two segments. After a σ_2 operation, we have

$$\begin{pmatrix} \mathrm{I}' \\ \mathrm{II}' \end{pmatrix} = \begin{pmatrix} \mathrm{I}+\mathrm{II} \\ \mathrm{II} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I} \\ \mathrm{II} \end{pmatrix} = \sigma_2 \begin{pmatrix} \mathrm{I} \\ \mathrm{II} \end{pmatrix}.$$

Hence, the matrix representation for σ_2 is

$$\sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Matrix Representation of σ_1^{-1}



Similarly, after a σ_1^{-1} operation we have

$$\begin{pmatrix} \mathrm{I}' \\ \mathrm{II}' \end{pmatrix} = \begin{pmatrix} \mathrm{I} \\ \mathrm{I} + \mathrm{II} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{I} \\ \mathrm{II} \end{pmatrix} = \sigma_1^{-1} \begin{pmatrix} \mathrm{I} \\ \mathrm{II} \end{pmatrix}.$$

Hence, the matrix representation for σ_1^{-1} is

$$\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Matrix Representation of the Braid Group

We now invoke the faithfulness of the representation to complete the set,

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

$$\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \qquad \sigma_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Our two protocols have representation

$$\sigma_1\sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \qquad \sigma_1^{-1}\sigma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

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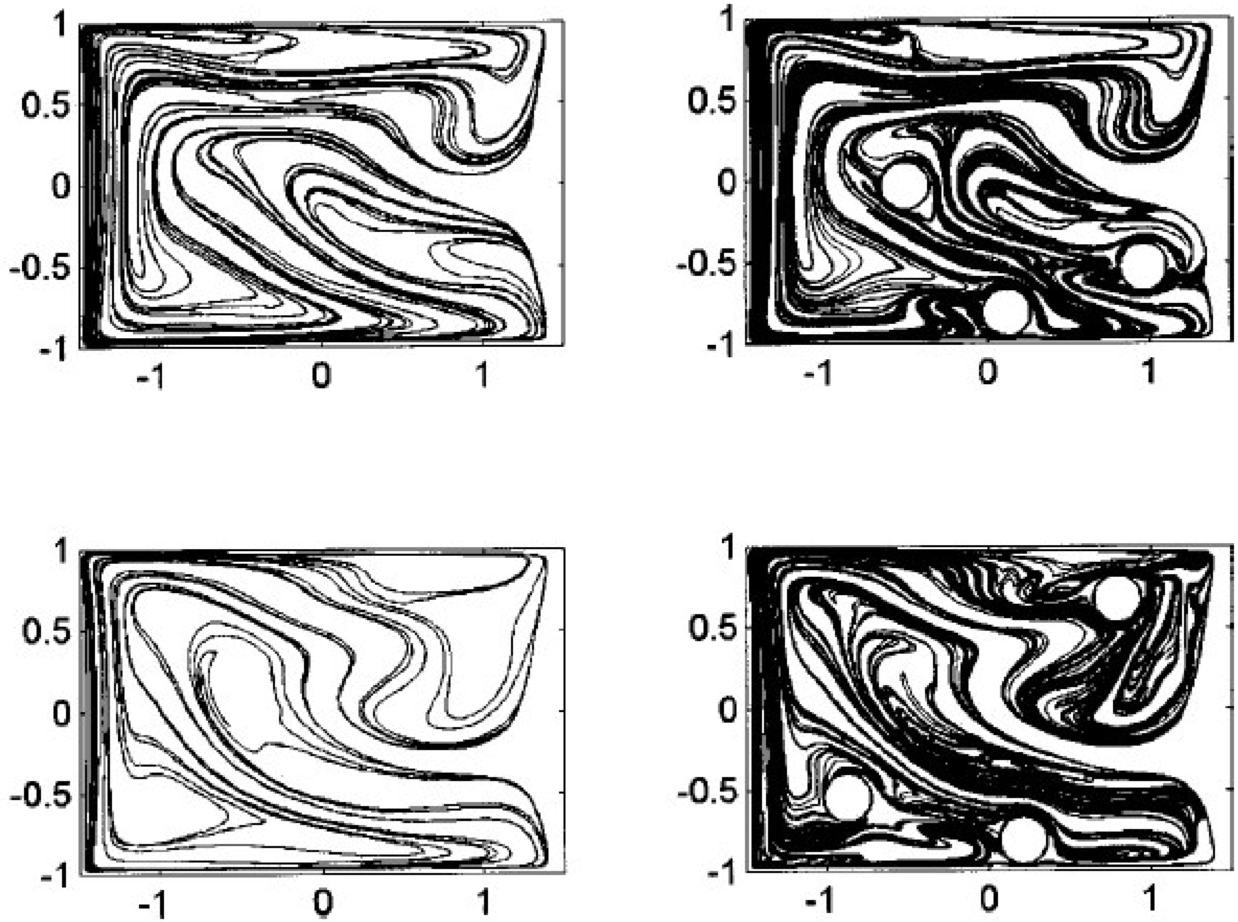
- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).

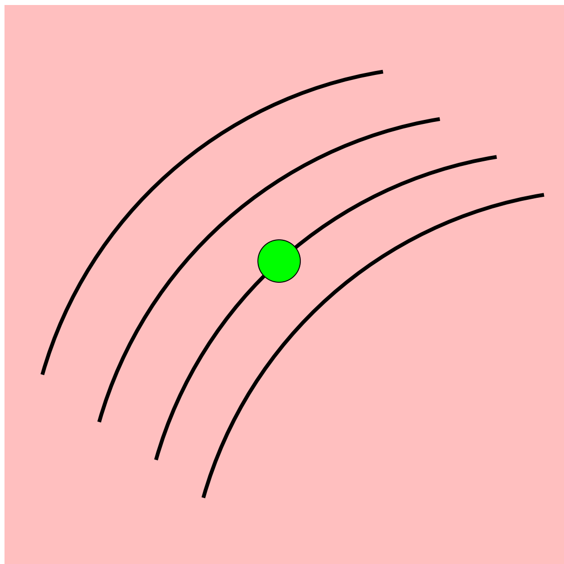
Freely-moving Rods in a Cavity Flow

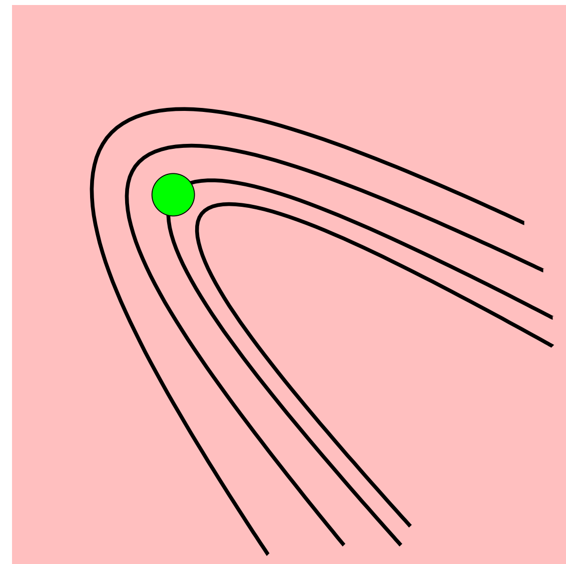


[A. Vikhansky, Physics of Fluids **15**, 1830 (2003)]

Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).



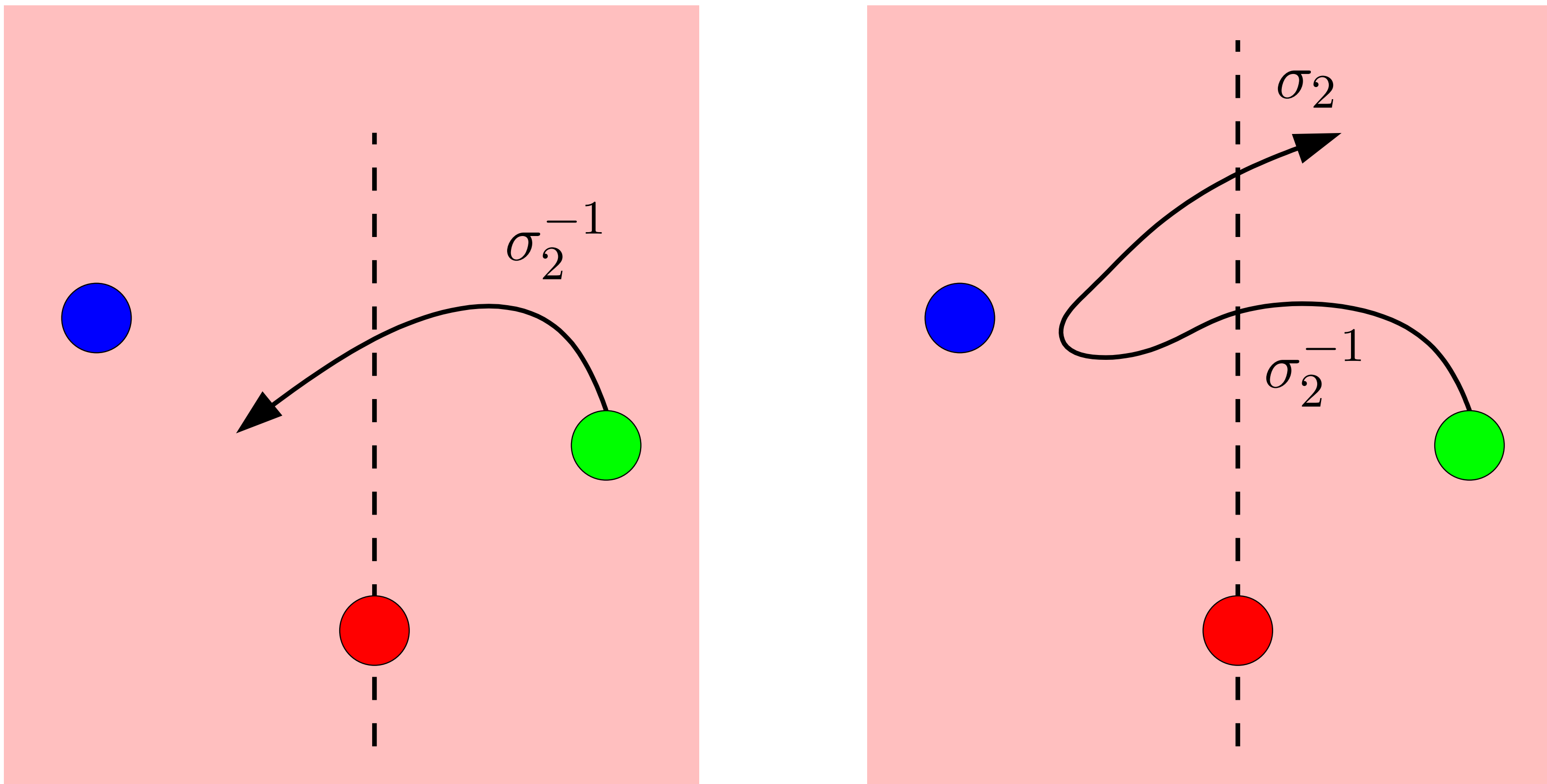


Material lines must bend around the orbit: it acts just like a “rod”!

The idea: pick any three fluid particles and follow them.

How do they braid around each other?

Detecting Braiding Events



In the second case there is no net braid: the two elements cancel each other.

Random Sequence of Braids

We end up with a sequence of braids, with matrix representation

$$\Sigma^{(N)} = \sigma^{(N)} \cdots \sigma^{(2)} \sigma^{(1)}$$

where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and N is the number of braiding events detected after a time t.

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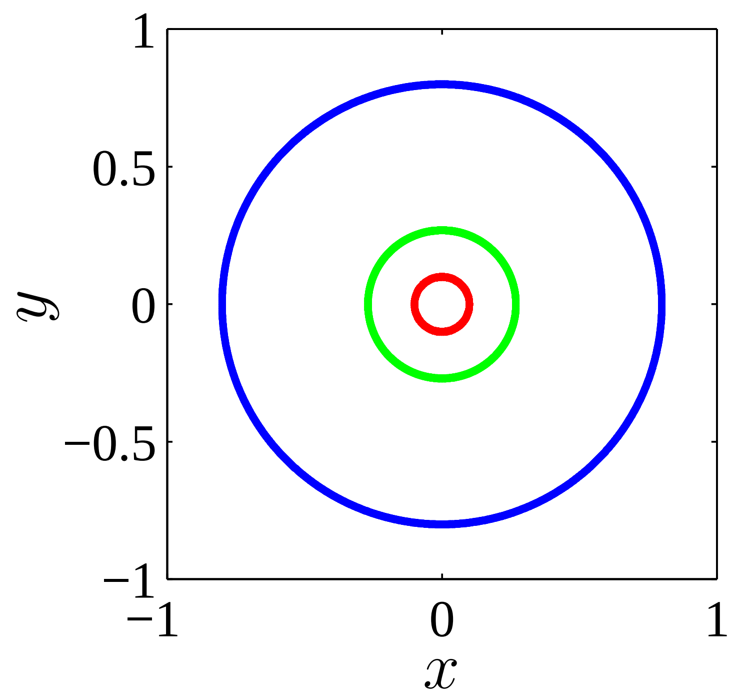
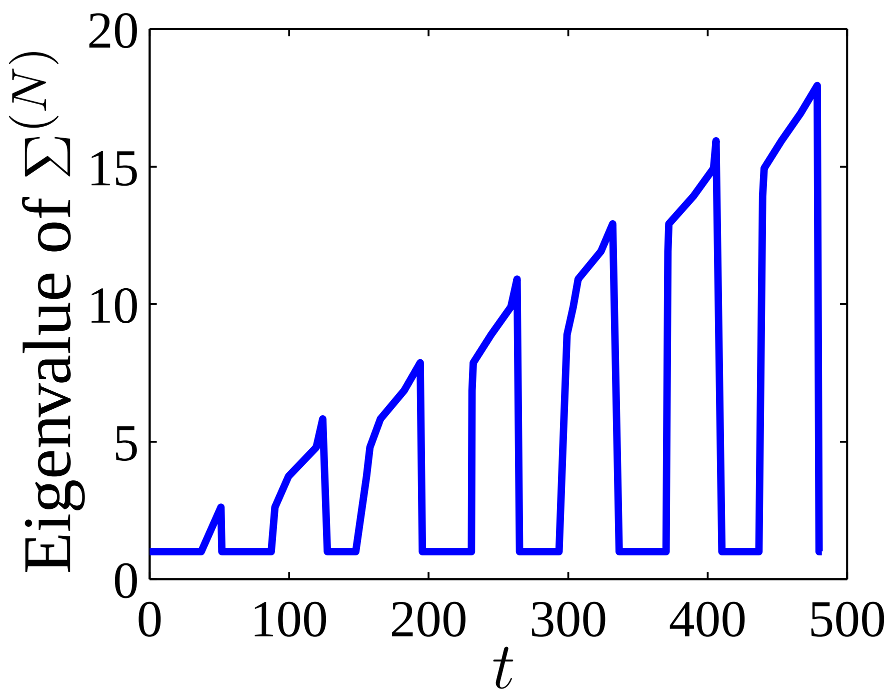
The largest eigenvalue of $\Sigma^{(N)}$ is a measure of the complexity of the braiding motion, called the braiding factor.

Random matrix theory says that the braiding factor can grow exponentially! We call the rate of exponential growth the braiding Lyapunov exponent or just braiding exponent.

Non-braiding Motion

First consider the motion of of three points in concentric circles with irrationally-related frequencies.



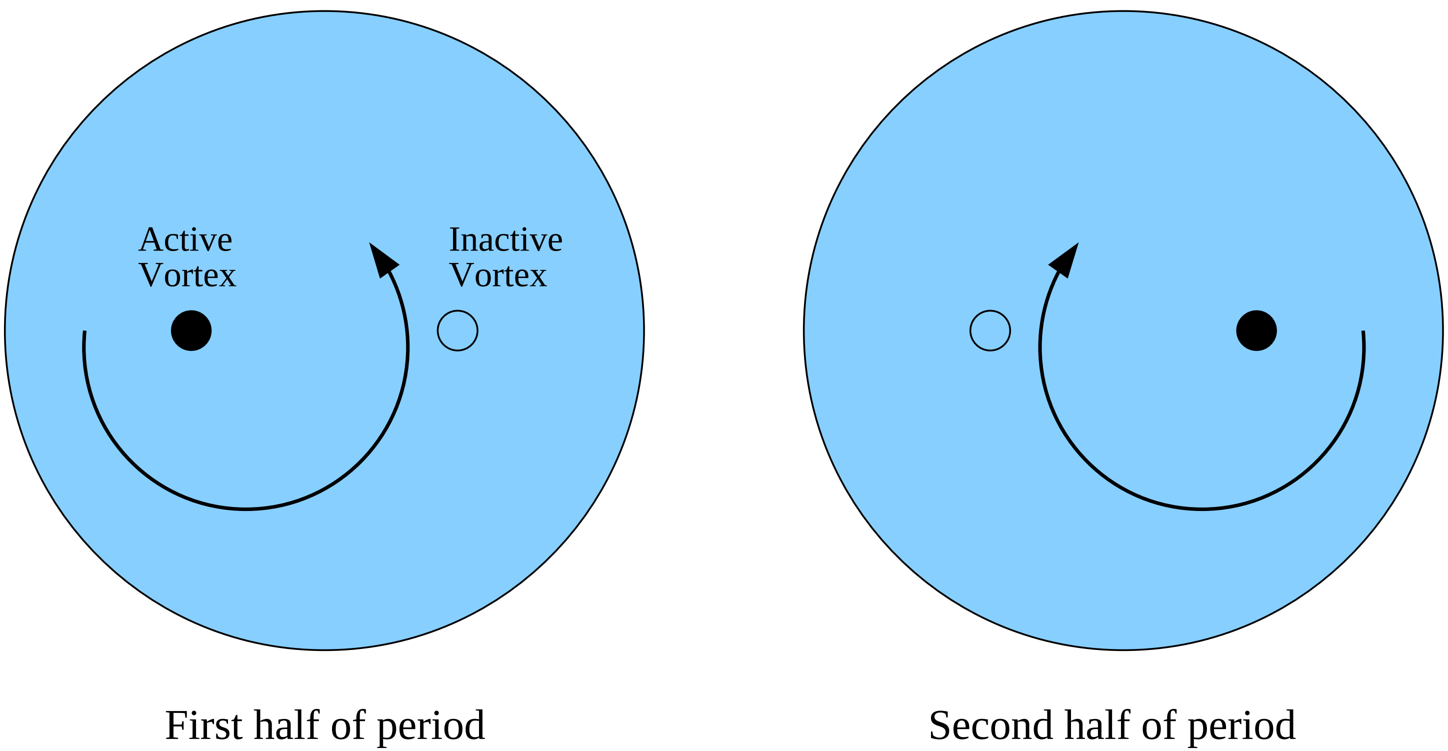


The braiding factor grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.

Blinking-vortex Flow

To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won't do).

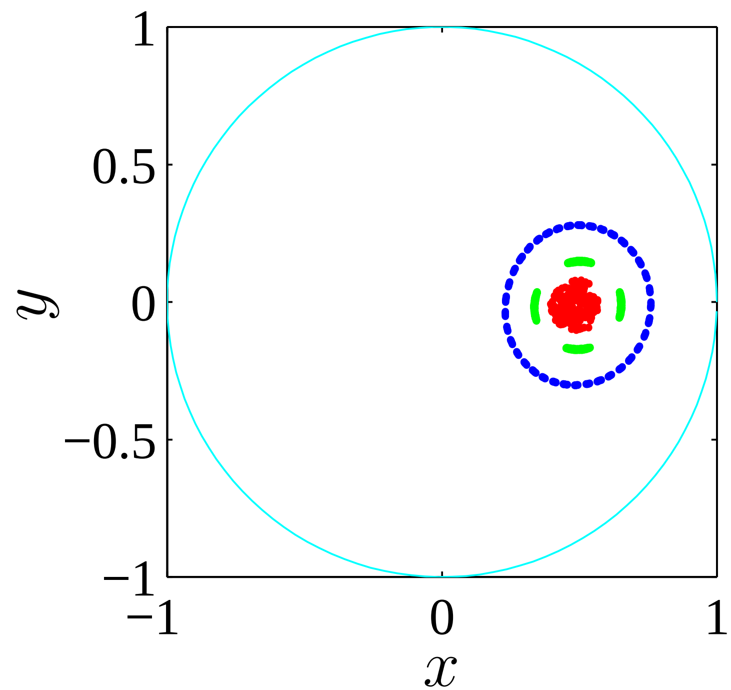
Aref's blinking-vortex flow is ideal.

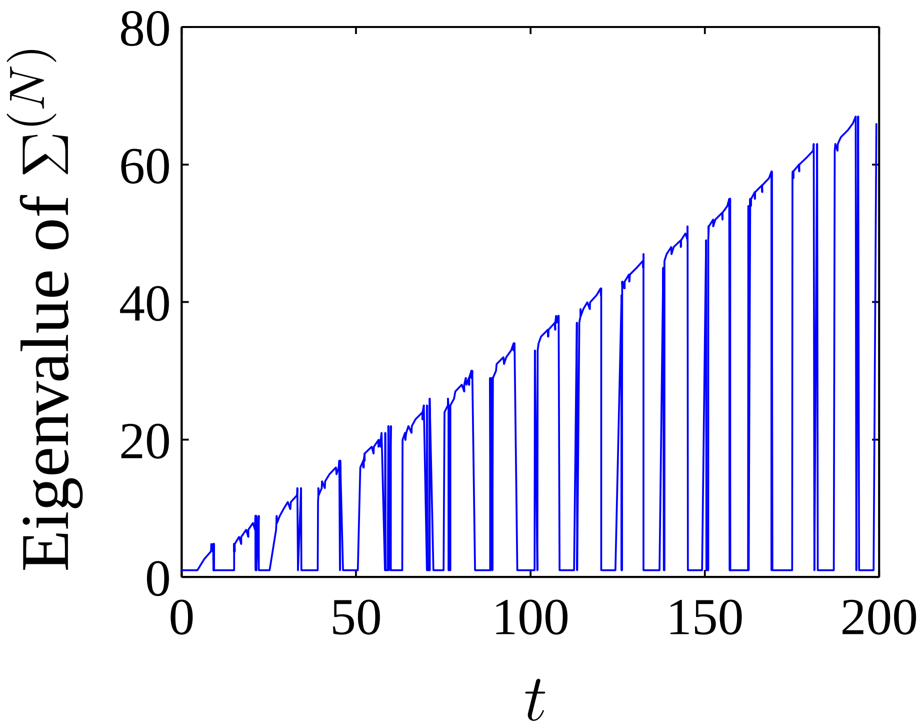


The only parameter is the circulation Γ of the vortices.

Blinking Vortex: Non-braiding Motion

For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.

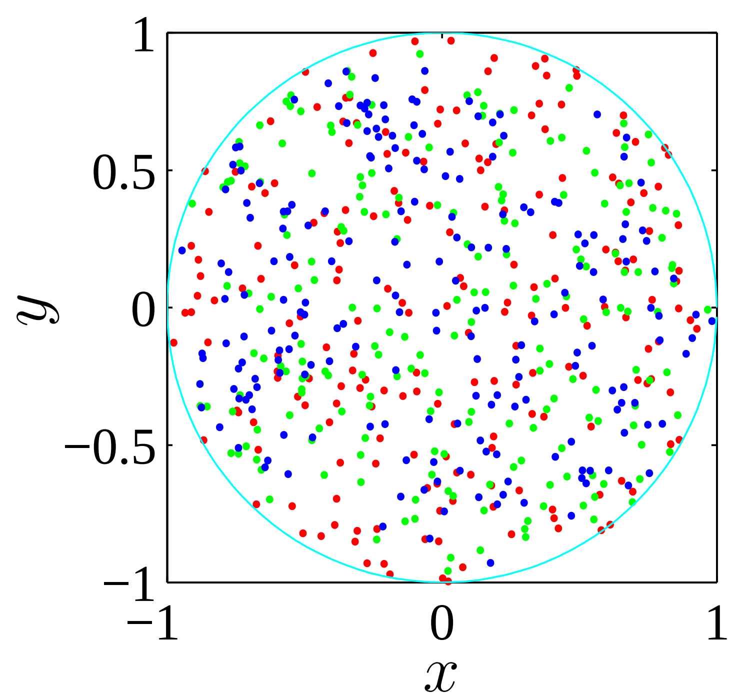


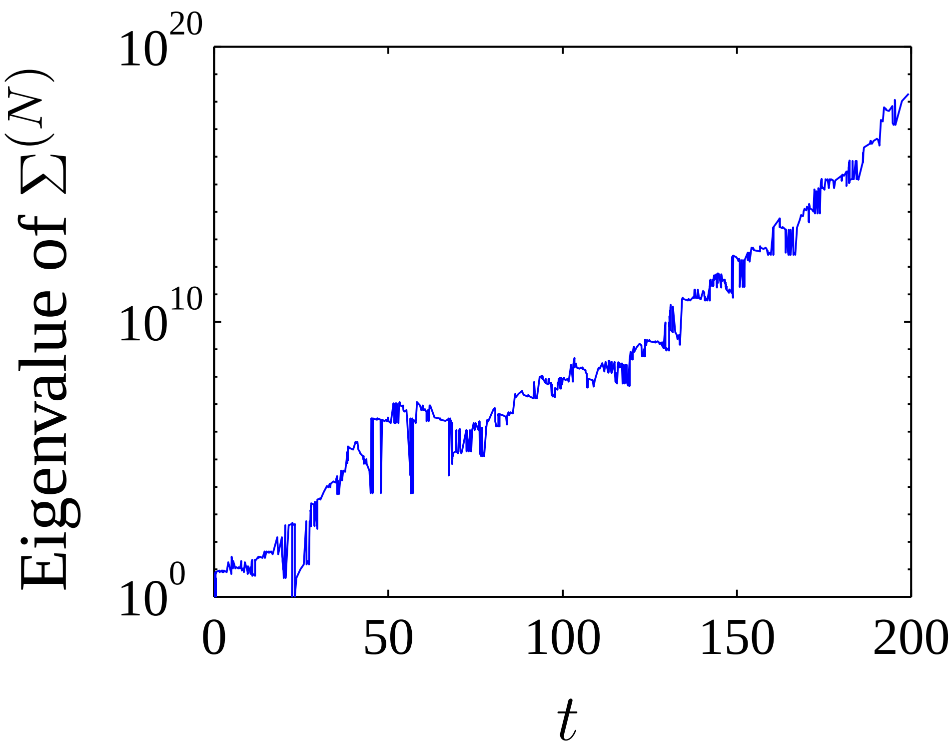


One of the orbits is chaotic, the other two are closed.

Blinking Vortex: Braiding Motion

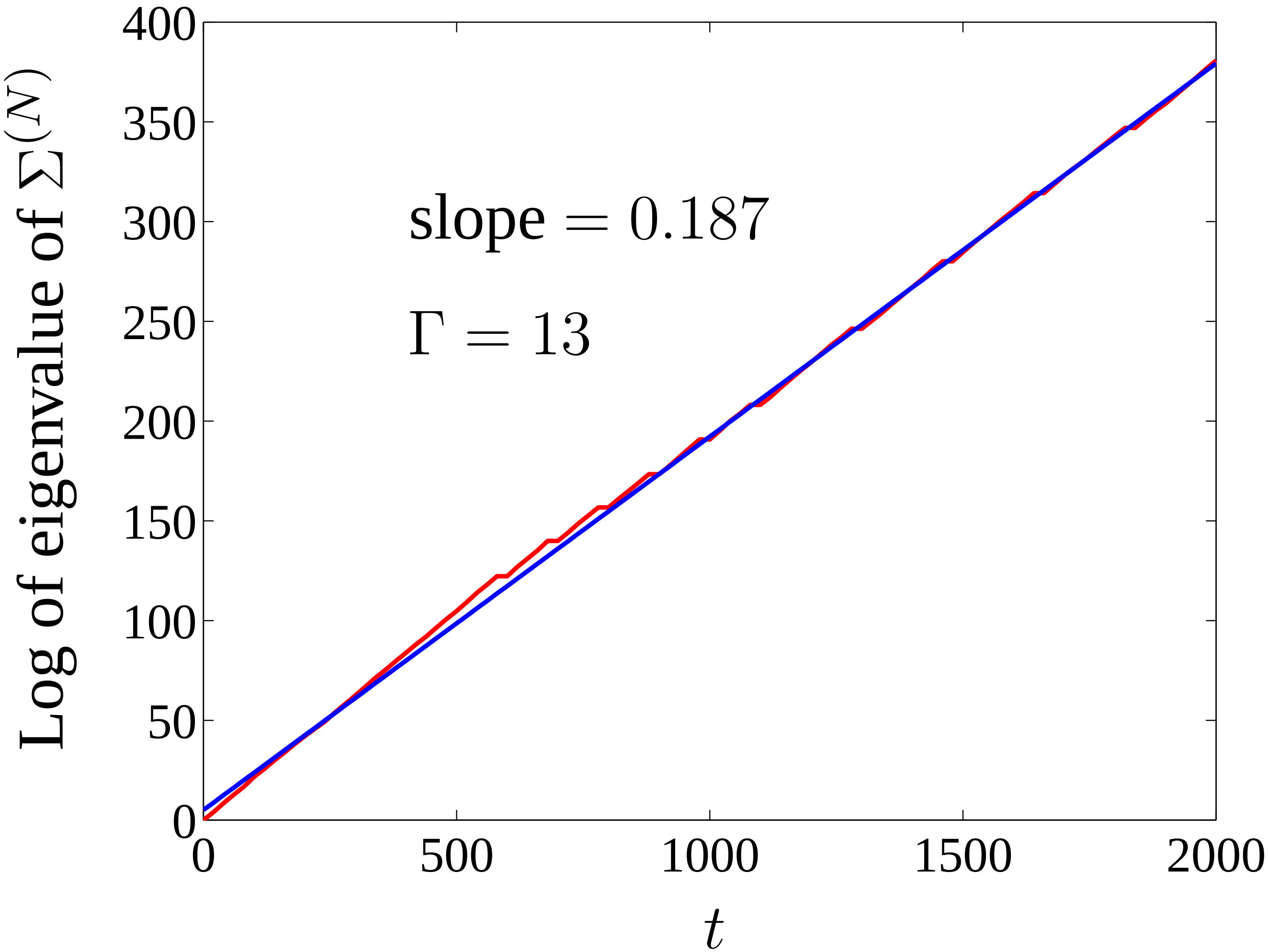
For $\Gamma = 13$, the blinking vortex is globally chaotic.





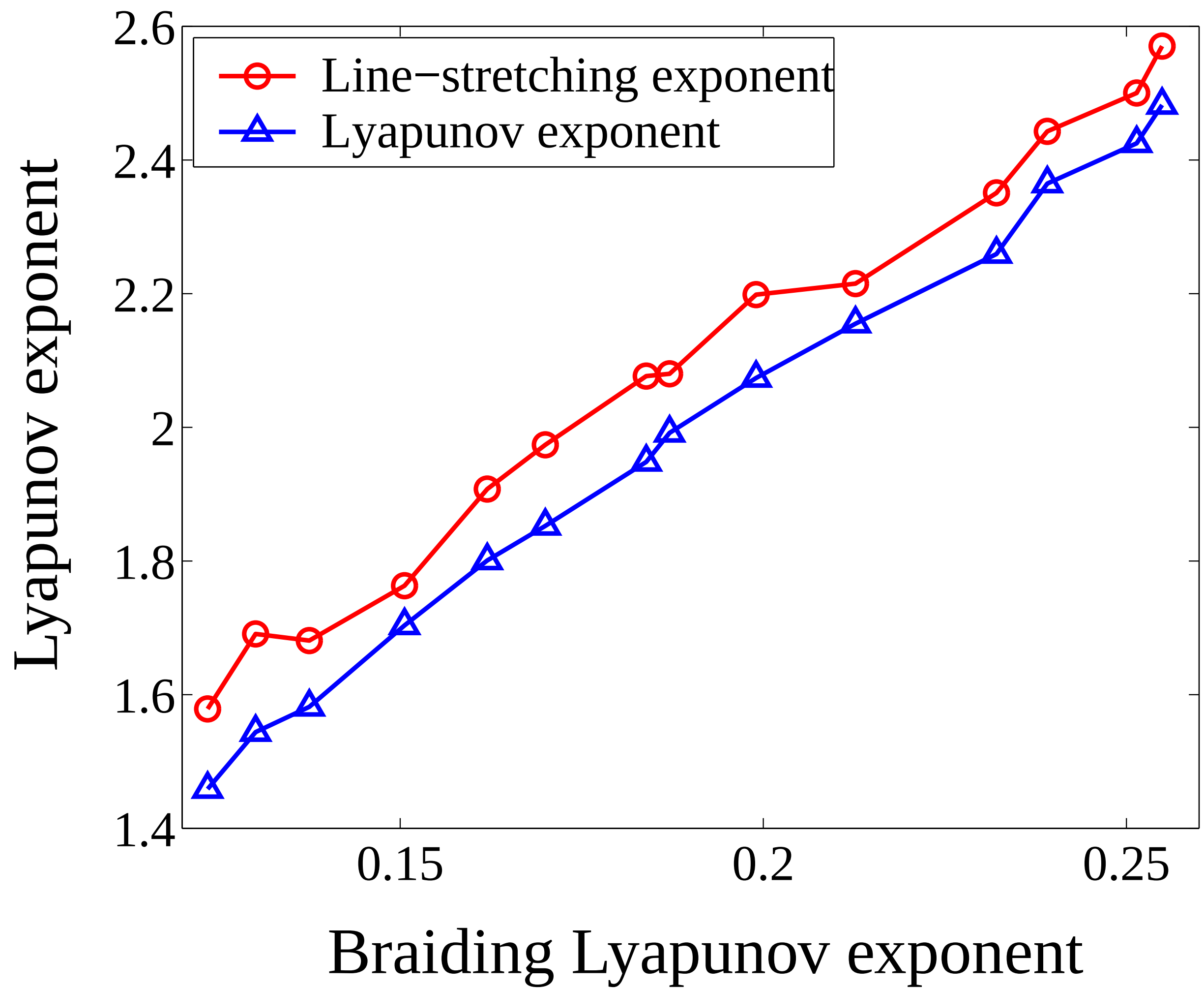
The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order 10^{20} rather than 80!

Averaging over many Triplets



Averaged over 100 random triplets.

Comparison with Lyapunov Exponents



Γ varies from 8 to 20.

Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- The complexity of a braid can be represented by the largest eigenvalue of a product of matrices—the braiding. factor.
- Any triplet of particles can potentially braid.
- The complexity of the braid is a good measure of chaos.
- No need for infinitesimal separation of trajectories or derivatives of the velocity field.
- For instance, can use all the floats in a data set (J. La Casce).
- Test in 2D turbulent simulations (F. Paparella).
- Higher-order braids!