Stirring and Mixing: A Mathematician's Viewpoint

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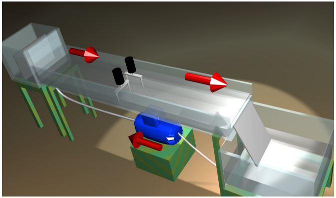
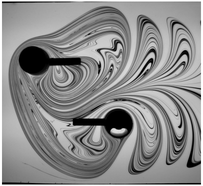
Stirring and Mixing of Viscous Fluids

- Viscous flows ⇒ no turbulence! (laminar)
- Open and closed systems
- Active (rods) and passive





Understand the mechanisms involved.
Characterise and optimise the efficiency of mixing.

Stirring and Mixing: What's the Difference?

- Stirring is the mechanical motion of the fluid (cause);
- Mixing is the homogenisation of a substance (effect, or goal);
- Two extreme limits: Turbulent and laminar mixing, both relevant in applications;
- Even if turbulence is feasible, still care about energetic cost;
- For very viscous flows, use simple time-dependent flows to create chaotic mixing.

Diffusion

The governing equation for the natural diffusion ("dispersal") of a substance (heat, dye, chemical. . .) is the diffusion equation:

$$\frac{\partial\theta}{\partial t} = \kappa\,\nabla^2\theta$$

- $\theta(\mathbf{x}, t)$ is the concentration of something we need to mix;
- κ is the diffusion coefficient;

The main problem is that natural (or molecular) diffusion is usually really slow. For example, the diffusion constant for heat is $\kappa = 2.4 \times 10^{-5}\,\mathrm{m}^2/\mathrm{s}$. If a room is $L = 10\,\mathrm{m}$ wide, the typical time for heat to diffuse across is $L^2/\kappa \simeq 1000$ hours (48 days).

This would make space heaters useless!

Advection and Diffusion

So what did we leave out? We omitted the effect of stirring, which creates a flow $\mathbf{u}(\mathbf{x}, t)$, giving the advection–diffusion equation:

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta = \kappa \nabla^2 \theta$$

The impact of the new term, called the advection or convection term, is tremendous.

Its role is to increase spatial gradients of θ, which makes the Laplacian term $\nabla^2\theta$ massive, even if κ is small.

This is why space heaters work: the rising hot air creates currents that help to 'stir' the air in a room.

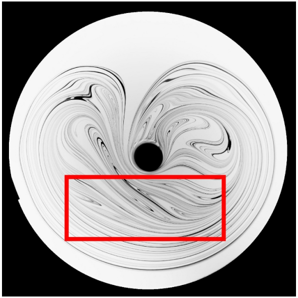
Thus, stirring causes mixing to occur **much** faster.

A Simple Example: Planetary Mixers

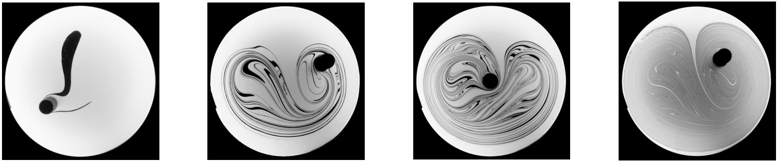
In food processing, rods are often used for stirring.

[movie 1] ©BLT Inc.

The Figure-Eight Stirring Protocol



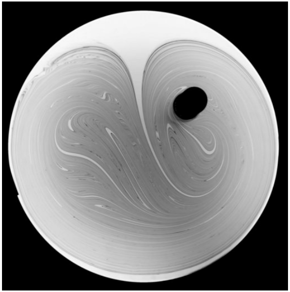
- Circular container of viscous fluid (sugar syrup);
- A rod is moved slowly in a 'figure-eight' pattern;
- Gradients are created by stretching and folding, the signature of chaos.

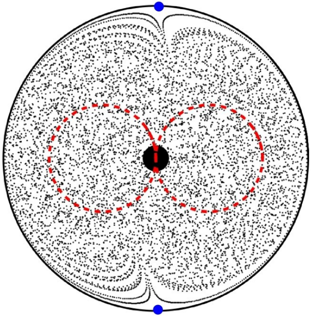


[movie 2] Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

The Mixing Pattern

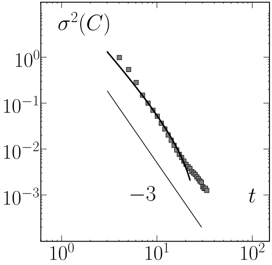
- Kidney-shaped mixed region extends to wall;
- Two parabolic points on the wall, one associated with injection of material;
- Asymptotically self-similar, so expect an exponential decay of the concentration ('strange eigenmode' regime).



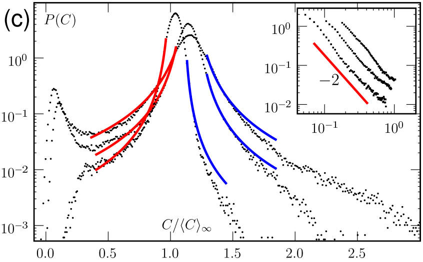


Mixing is Slower Than Expected

Concentration field in a well-mixed central region



Variance $= \int |\theta|^2 dV$

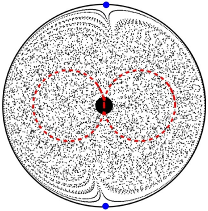


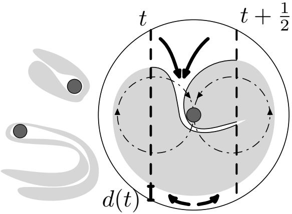
Concentration PDFs

$\Rightarrow$ Algebraic decay of variance $\neq$ Exponential

The 'stretching and folding' action induced by the rod is an exponentially rapid process (chaos!), so why aren't we seeing exponential decay?

Walls Slow Down Mixing

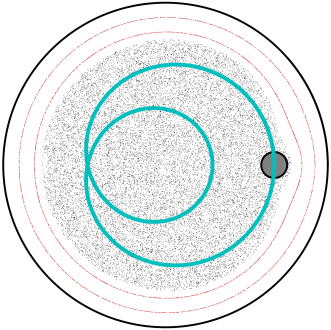




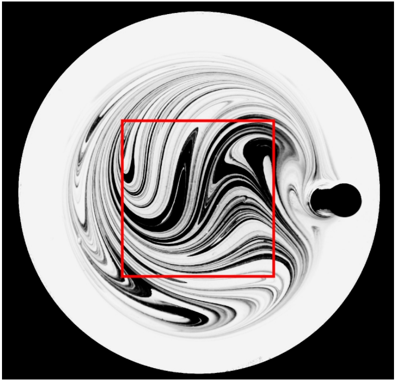
- Trajectories are (almost) everywhere chaotic
  ⇒ but there is always poorly-mixed fluid near the walls;
- Re-inject unmixed (white) material along the unstable manifold of a parabolic point on the wall;
- No-slip at walls ⇒ width of "white stripes" $\sim t^{-2}$ (algebraic);
- Re-injected white strips contaminate the mixing pattern, in spite of the fact that stretching is exponential in the centre;
- [Gouillart et al., *Phys Rev. Lett.* **99**, 114501 (2007)] → map model.

A Second Scenario

How do we mimic a slip boundary condition?

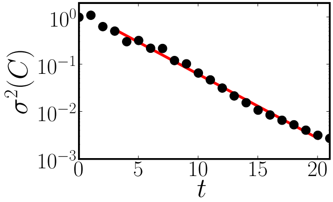


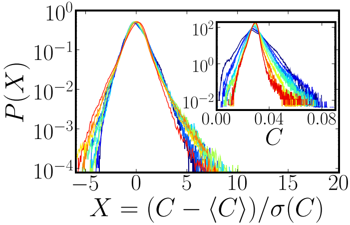
"Epitrochoid" protocol



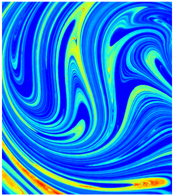
Central chaotic region + regular region near the walls.

Recover Exponential Decay

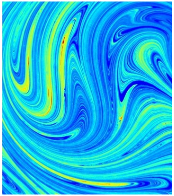




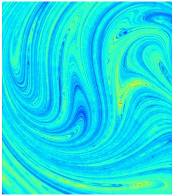
$t = 8$



$t = 12$



$t = 17$

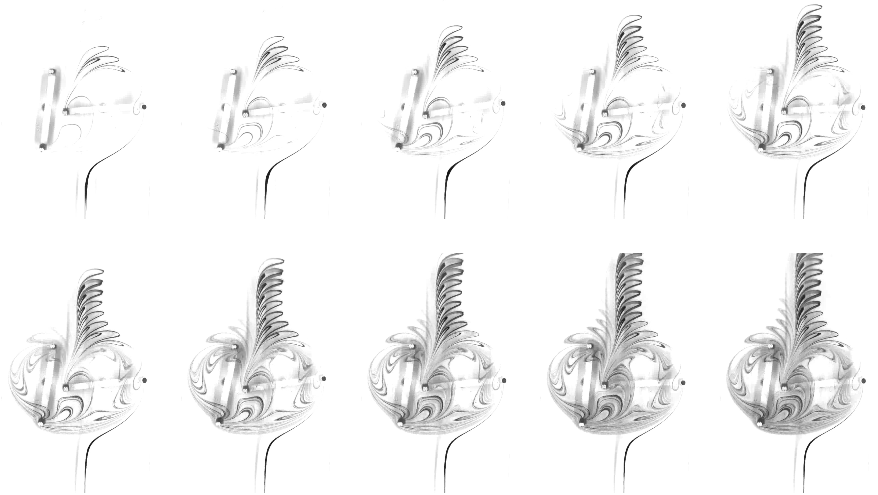


. . . as well as ‘true’ self-similarity.

Another Approach: Rotate the Bowl!

Self-similarity: Another Example

[movie 3]



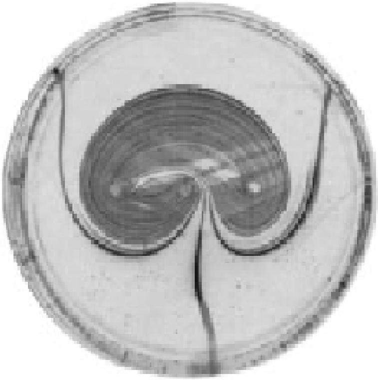
The Taffy Puller

This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the precise nature of the rod motion!

[movie 4]

Experiment of Boyland, Aref, & Stremler

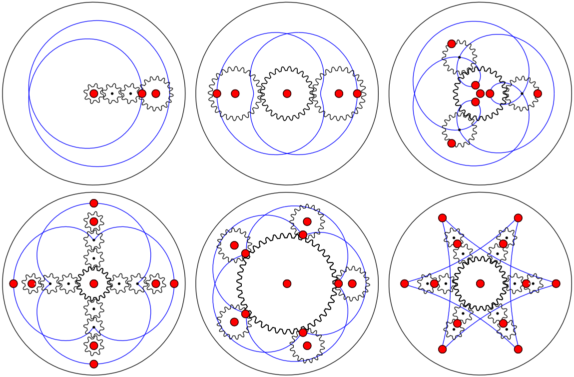


[movie 5] [movie 6]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

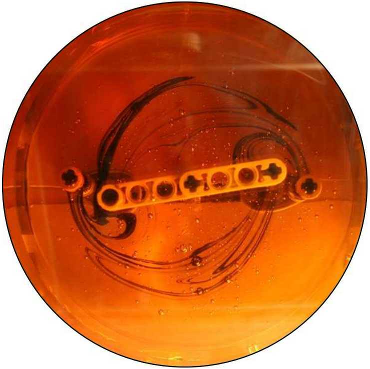
Topological Mixers

- The mathematical idea in the previous slide is called braiding, and is a consequence of the topology of the rod motion.
- There is an optimal rod motion from this viewpoint, and we have designed stirring devices that implements it:



Notice how every rod 'leapfrogs' the next one. [movie 7]

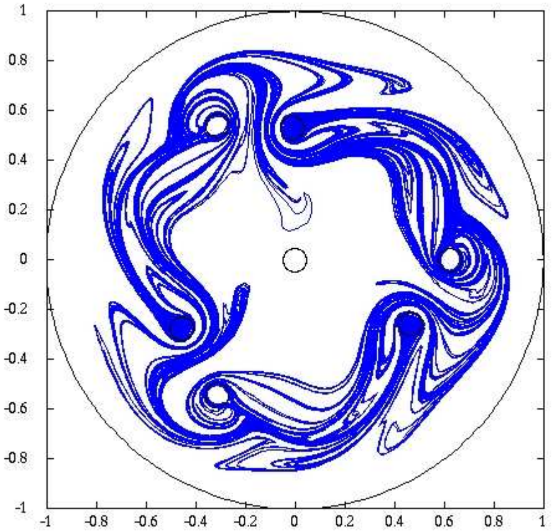
Four Rods



The central rod only plays a supporting role (literally).

[movie 8] [movie 9]

Six Rods



[movie 10]

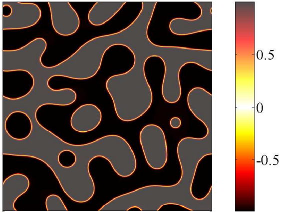
Multiphase Flows: Making Mayonnaise

“One day she... gave me a demonstration on how to make mayonnaise. I had no idea it was so technical... She whisked the mustard with one yolk for a few minutes, then started dribbling in the oil. As soon as any separation appeared she whisked even faster and continued whisking and oiling for long enough to make my wrist hurt, let alone hers. It was riveting, like watching an old master mixing his ochres with his burnt siennas.”

[M. Lipman, “Ireland: land of charm, humour, breathtaking vistas... and delicious homemade mayonnaise?”, *The Guardian*, 21 August 2006.]

Multiphase Flows: Stirring and Mixing

- Two immiscible fluids will phase-separate if left alone:



- Oil and vinegar do this, as do some metallic alloys.
- From the vinaigrette case, it is well known that you have to keep stirring to homogenise the mixture.
- How can we model this?

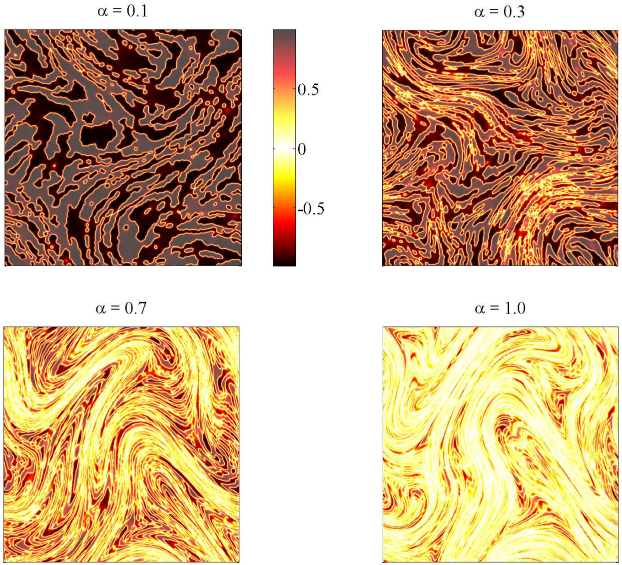
The Stirred Cahn–Hilliard Equation

- The passive stirring of a phase separated fluid is modelled by an advective term in the Cahn–Hilliard equation,

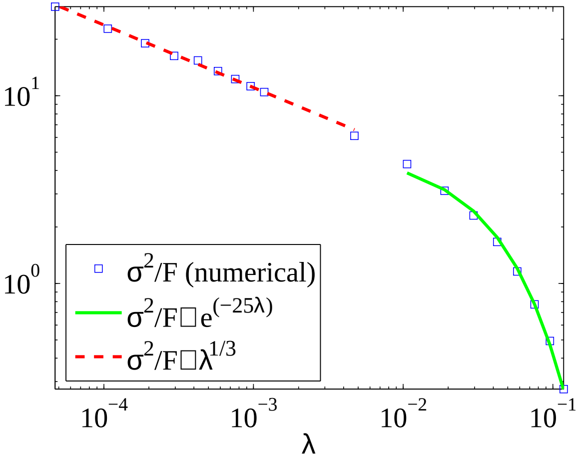
$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = D \, \nabla^2 \left(\theta^3 - \theta - \gamma \, \nabla^2 \theta \right).$$

- The CH equation is a classic model of phase-separating fluids: the separated state is more energetically favourable, so the system tends to it.
- Once again stirring can short-circuit this.
- Two co-existing regimes exist, depending on the strength of the stirring: Bubbles and filaments.

From Bubbles to Filaments



Efficiency of Stirring



Here σ^2/F is a measure of the homogeneity, for a steady stirring strength λ. Note that there is a sudden improvement at $\lambda \simeq 10^{-2}$ corresponding to the bubbles-to-filaments transition.

Conclusions

- There are many ways to stir: here we focused on rod stirring.
- Walls can have a big impact and slow down mixing.
- It is sometimes possible to shield the walls from the mixing region, for instance by rotating the vessel.
- Having rods undergo complex 'braiding' motions can lead to good mixer designs.
- For phase separating substances, an imposed flow not only arrests phase-separation, but can overcome it.
- For vigorous stirring, the phases are therefore well-mixed.
- The numerical simulations suggest the existence of a critical stirring amplitude for multiphase mixing.

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