# Turbulent Polymers

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# Drag Reduction

Experimental facts:

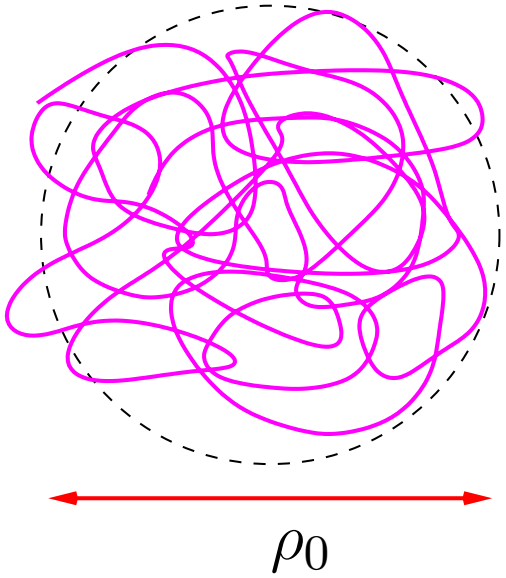
- Toms (1948) observed that the addition of $\sim 10$ ppm polymers to turbulent pipe flow reduced the pressure drop substantially.
- Typical example: polyethilene oxide in water: 18 ppm (by weight) reduces drag by 33%!!
- Qualitative understanding: classic review of Lumley (1969) uses dimensional analysis (polymer size, viscosity, etc.) to predict magnitude and onset of effect.
- Molecular scales matter! Mystery...
- ...and motivation for studying polymers in turbulent flows.

# Polymers

Polymers are long chains of molecules. Random walk at rest.

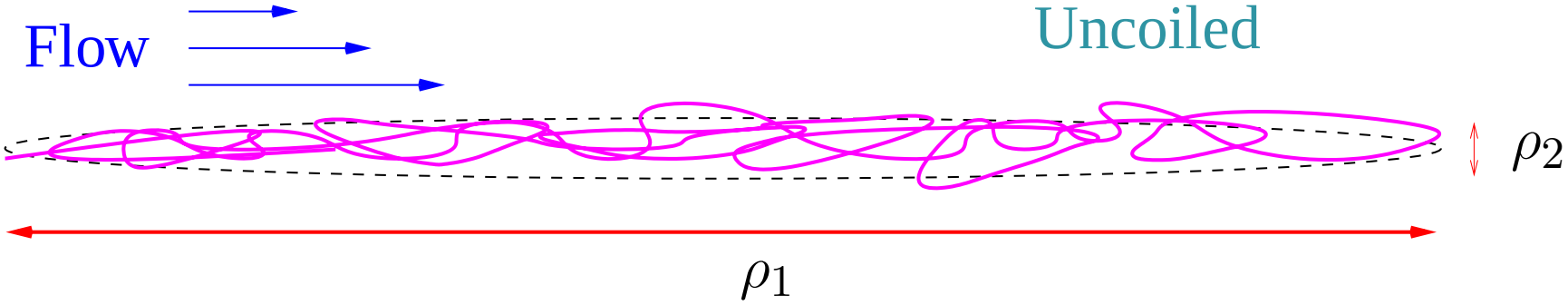
No flow

Coiled




Flow

Uncoiled



# Constitutive Models

Model polymers in dilute solution as a continuum:
Stress on the fluid.
How does the stress tensor $\mathbb{T}$ depend on the state of the fluid?

- Elastic:

$$T_{ij} = \eta\, \gamma_{ij}\,, \qquad \gamma \equiv \text{strain (deformation) tensor.}$$

- Viscous or Newtonian:

$$T_{ij} = \mu\, \dot{\gamma}_{ij}\,, \qquad \dot{\gamma} \equiv \boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T$$

- Viscoelastic:

$$T_{ij} = \int_{-\infty}^{t} G(t-t')\, \dot{\gamma}_{ij}(t')\, \mathrm{d}t'$$

# Maxwell Model

Exponentially-decaying memory:

$$T_{ij} = (\mu/\tau) \int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau}\, \dot{\gamma}_{ij}(t')\, \mathrm{d}t'$$

Reformulate as differential equation for $\mathbb{T}$:

$$\tau\, \dot{T}_{ij} = \mu\, \dot{\gamma}_{ij} - T_{ij}$$

Problem: not frame-indifferent!

Not good as a fluid relation. Remedied by introducing a frame-independent (Oldroyd) derivative

$$\dot{\mathbb{T}} \quad \Longrightarrow \quad \mathcal{D}\mathbb{T} \equiv \frac{\partial \mathbb{T}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\mathbb{T} - (\mathbb{T} \cdot \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T \cdot \mathbb{T})$$

# The Equations of Motion

Couple stress to Navier–Stokes for an incompressible fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla} p = \nu\nabla^2\boldsymbol{u} + \frac{s}{\tau}\,\boldsymbol{\nabla}\cdot\mathbb{A}\,;$$

$$\mathcal{D}\mathbb{A} = -\frac{1}{\tau}\left(\mathbb{A} - \rho_0^2\,\mathbb{I}\right)\,; \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0,$$

where $\mathbb{A}$ is equal to $\mathbb{T}$ up to constants, and can be regarded as the local deformation of the polymers, with $\mathbb{A} = \rho_0^2\,\mathbb{I}$ at rest.

Can be derived from a kinetic model of "Hookean dumbbells."

More generally: allow nonlinear saturation of the length of polymers (FENE-type models)

$$\mathcal{D}\mathbb{A} = -\frac{1}{\tau}\left(F(A)\,\mathbb{A} - \rho_0^2\,\mathbb{I}\right).$$

# Evolution of the Principal Axes

The polymer conformation tensor $\mathbb{A}$ can be diagonalized, with orthonormal eigenvectors $\boldsymbol{e}_\alpha$ and eigenvalues $(\rho^\alpha)^2$ that evolve according to

$$\frac{d\rho^\alpha}{dt} = \hat{\sigma}^\alpha \rho^\alpha - \frac{1}{\tau}\left(F(\|\rho\|^2)\rho^\alpha - \rho_0^2/\rho^\alpha\right),$$

$$\hat{\sigma}^\alpha(t,\boldsymbol{x}) \equiv \boldsymbol{e}_\alpha \cdot \boldsymbol{\nabla}\boldsymbol{u} \cdot \boldsymbol{e}_\alpha\,, \qquad d/dt = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\,.$$

The $\rho^\alpha$ are the lengths of the principal axes of the ellipsoid delineating the deformation of the polymer.

If the flow is smooth, the polymers tend to align with the dominant stretching direction, so we consider only the major axis:

$$\frac{d\rho}{dt} = \hat{\sigma}\rho - \frac{1}{\tau}F(\rho^2)\rho\,. \qquad \text{(Neglect } \rho_0\text{.)}$$

# Turbulence! (well. . . sort of)

For turbulence, a locally smooth regime is achieved when the viscous scale is much longer that the polymer length.

In that case, model the velocity field as a Gaussian random variable representing a smooth straining field $\hat{\sigma}(t)$ that changes rapidly; $\hat{\sigma}$ satisfies

$$\langle \hat{\sigma}(t)\hat{\sigma}(t')\rangle - \bar{\lambda}^2 = \delta(t-t')\,\Delta\,; \qquad \langle \hat{\sigma}(t)\rangle = \bar{\lambda}\,,$$

where the angle brackets denote an average over $\hat{\sigma}$.

The variable $\hat{\sigma}(t)$ is $\delta$-correlated in time, which means that it forgets about its previous state immediately. It has mean $\bar{\lambda}$ and standard deviation $\Delta$.

This “slightly” artificial situation has great analytical advantages.

# Steady-state Distribution

Because the distribution of $\hat{\sigma}(t)$ is Gaussian and $\delta$-correlated, can obtain a Fokker–Planck equation for the distribution $\mathcal{P}(t,\rho)$ of the major axis: (Chertkov, 2000)

$$\partial_t \mathcal{P} = \tfrac{1}{2}\Delta\,\partial_\rho \rho\,\partial_\rho \rho\,\mathcal{P} - \bar{\lambda}\,\partial_\rho\,\rho\,\mathcal{P} + \frac{1}{\tau}\,\partial_\rho\,F(\rho^2)\rho\,\mathcal{P}$$

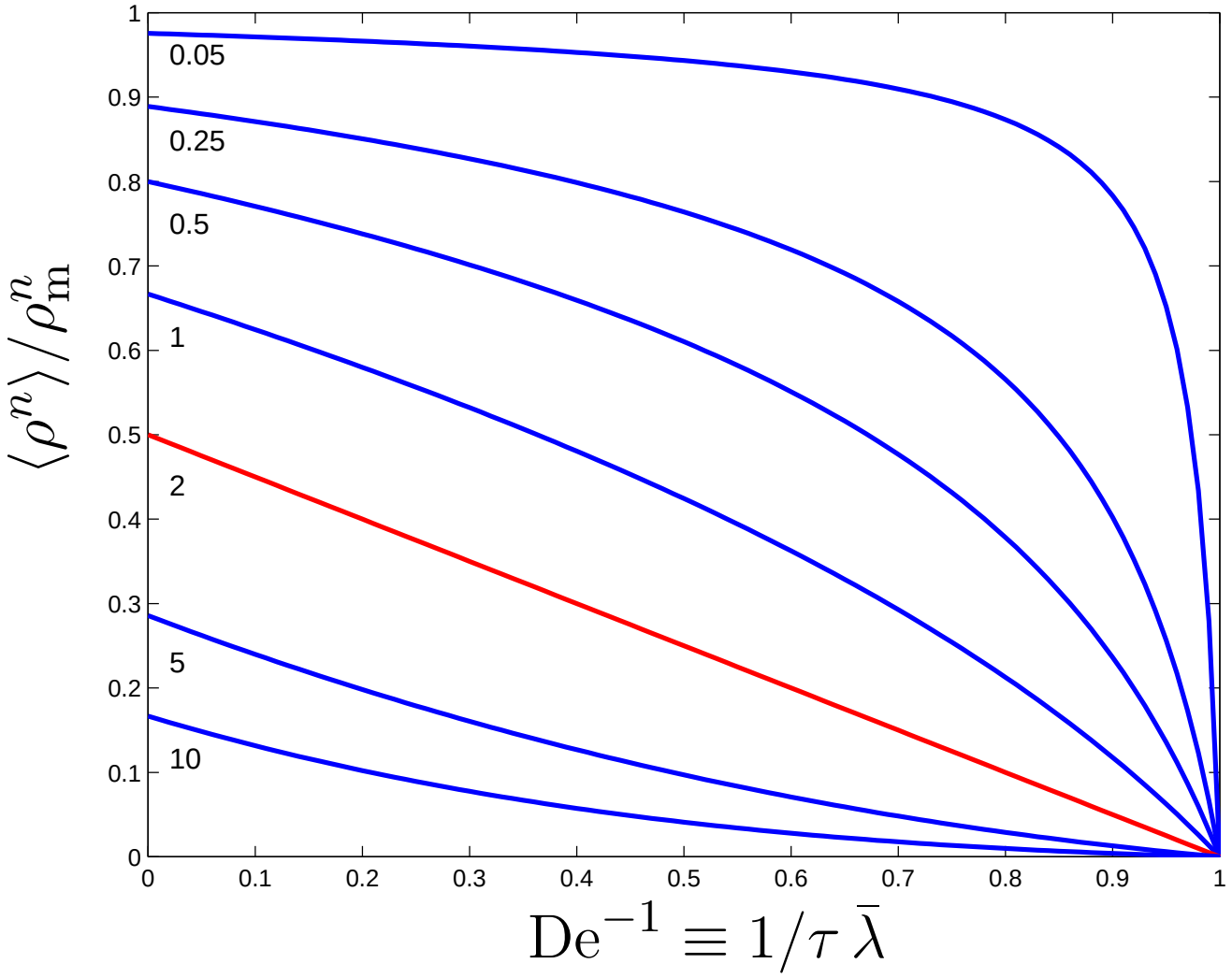
Assuming a FENE-type (Finite Extension Nonlinear Elastic) model which limits the length of the polymers to $\rho_{\mathrm{m}}$, can find equilibrium distribution (J.-L.T., in preparation)

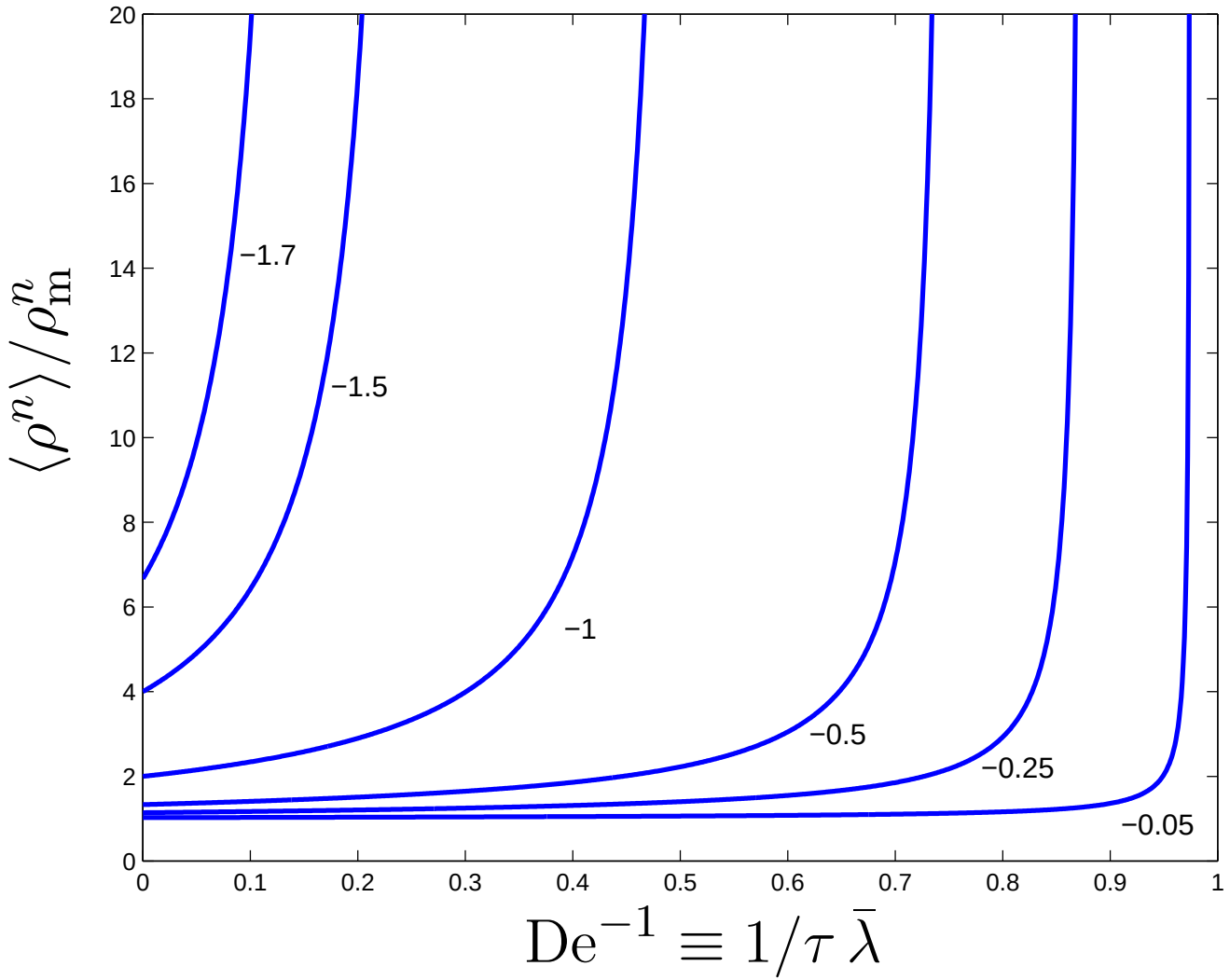
$$\mathcal{P}_{\mathrm{eq}}(\rho) = \frac{\Gamma(1+\bar{\lambda}/\Delta)}{2\,\Gamma(\bar{\lambda}/\Delta-\xi)\Gamma(1+\xi)}\,\rho^{-1+2(\bar{\lambda}/\Delta-\xi)}(1-\rho^2)^{\xi}$$

where $\rho$ is normalized by $\rho_{\mathrm{m}}$, and $\xi \equiv 1/\tau\Delta$.

# Moments of the Distribution

Coil-stretch transition at $\mathrm{De} = 1$.




20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
−1.7
−1.5
−1
−0.5
−0.25
−0.05
$\langle \rho^n \rangle / \rho_{\mathrm{m}}^n$
$\mathrm{De}^{-1} \equiv 1/\tau\,\bar{\lambda}$

# Ongoing Research

- Polymers are often delicate: try predict when they start to break.
- Model polymer as flexible chains instead of rods.
- Self-consistency: backreaction of the polymers on the fluid.
- Non-Gaussian statistics: Path integral formalism.
- Compressibility.
- Magnetic dynamo.
- Statistics of curvature.