Lyapunov Exponents and Transport in 2D Flows

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5 November 1999

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What is the Deal

We are interested in the advection-diffusion equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \frac{1}{\rho} \nabla \cdot (\rho \, D \nabla \phi)$$

where the Eulerian velocity field $\mathbf{v}(\boldsymbol{x}, t)$ is some prescribed time-dependent flow, which may or may not be be chaotic. The quantity ϕ represents the concentration of some passive scalar, ρ is the density, and D is the diffusion coefficient.

We assume that the Lagrangian dynamics are strongly chaotic ($\lambda L^2 / D \gg 1$).

Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates $\boldsymbol{x}$ satisfies

$$\frac{d\boldsymbol{x}}{dt}(\boldsymbol{\xi}, t) = \mathbf{v}(\boldsymbol{x}(\boldsymbol{\xi}, t), t),$$

where $\boldsymbol{\xi}$ are Lagrangian coordinates which label fluid elements. The usual choice is to take as initial condition $\boldsymbol{x}(\boldsymbol{\xi}, t = 0) = \boldsymbol{\xi}$, which says that fluid elements are labeled by their initial position.

$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, t)$ is thus the transformation from Lagrangian ($\boldsymbol{\xi}$) to Eulerian ($\boldsymbol{x}$) coordinates.

This transformation gets horrendously complicated as time evolves.

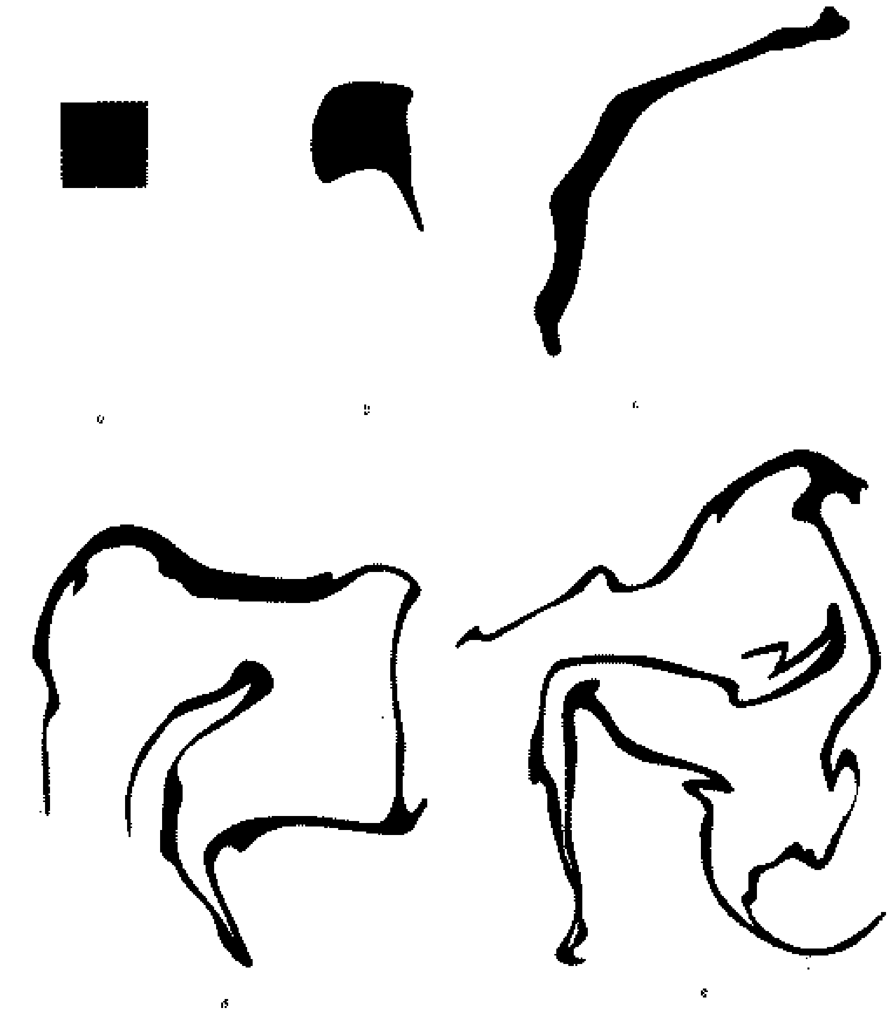
Lyapunov Exponents

The rate of exponential separation of neighbouring Lagrangian trajectories is measured by Lyapunov exponents

$$\lambda_\infty = \lim_{t\to\infty} \frac{1}{t} \ln \|(T_{\boldsymbol{x}}\mathbf{v})\mathbf{w}_0\|,$$

where $T_{\boldsymbol{x}}\mathbf{v}$ is the tangent map of the velocity field (the matrix $\partial\mathbf{v}/\partial\boldsymbol{x}$) and $\mathbf{w}_0$ is some constant vector.

Lyapunov exponents converge very slowly. So, for practical purposes we are always dealing with finite-time Lyapunov exponents.



(Welander, 1955)

The Idea

- Can we characterize the spatial and temporal evolution of finite-time Lyapunov exponents in a generic manner?
- Can we quantify the impact of these exponents on diffusion?

  Tang and Boozer brought the fancy tools of differential geometry to bear on this problem.
- Results: a generic functional form for the time evolution of finite-time Lyapunov exponents, and a relation between their spatial dependence and the shape of the stable manifolds.

A little differential geometry . . .

The Jacobian of the transformation from Lagrangian (ξ)to Eulerian (ξ) coordinates

$$J^i{}_j \equiv \frac{\partial x^i}{\partial \xi^j}$$

The Jacobian tells us how tensors transform:

- Covariant:

$$\tilde{V}_j = J^k{}_j \, V_k,$$

- Contravariant:

$$\tilde{W}^i = J^i{}_k \, W^k.$$

Measuring distances

The distance between two infinitesimally separated points in Eulerian space is given by

$$ds^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = \delta_{ij}\, dx^i dx^j \,.$$

Therefore, in Lagrangian coordinates distances are given by

$$ds^2 = \delta_{ij} \left(\frac{dx^i}{d\xi^k} d\xi^k \right) \left(\frac{dx^j}{d\xi^\ell} d\xi^\ell \right) = (J^i{}_k \, \delta_{ij} \, J^j{}_\ell)\, d\xi^k d\xi^\ell \,.$$

The Metric Tensor

The tensor δ_{ij} is a metric in the Eulerian (Euclidean) space. The tensor

$$g_{k\ell}(\boldsymbol{\xi}, t) \equiv \sum_i J^i{}_k \, J^i{}_\ell = \left(J^T J\right)_{k\ell}$$

is the same metric tensor but in the Lagrangian coordinate system.

Since the metric tells us about the distance between two neighbouring Lagrangian trajectories, its eigenvalues are related to the finite-time Lyapunov exponents.

2-D Incompressible Flow

We will now restrict ourselves to a 2-D, incompressible velocity field $\mathbf{v}$. This means that

$$\det g = (\det J)^2 = 1.$$

Now, g is a positive-definite symmetric matrix, which implies that it has real positive eigenvalues, $\Lambda(\boldsymbol{\xi}, t) \geq 1$ and $\Lambda^{-1}(\boldsymbol{\xi}, t) \leq 1$, and orthonormal eigenvectors $\hat{\mathbf{e}}(\boldsymbol{\xi}, t)$ and $\hat{\mathbf{s}}(\boldsymbol{\xi}, t)$:

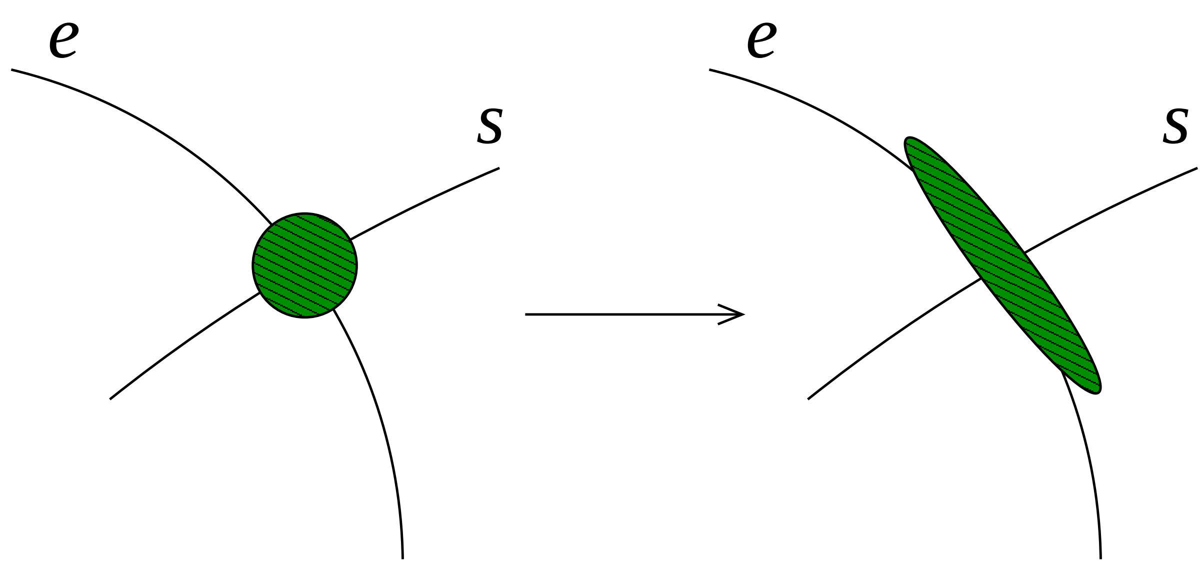
$$\boxed{g_{k\ell}(\boldsymbol{\xi}, t) = \Lambda\, e_k\, e_\ell + \Lambda^{-1}\, s_k\, s_\ell}$$

The finite-time Lyapunov exponents are given by

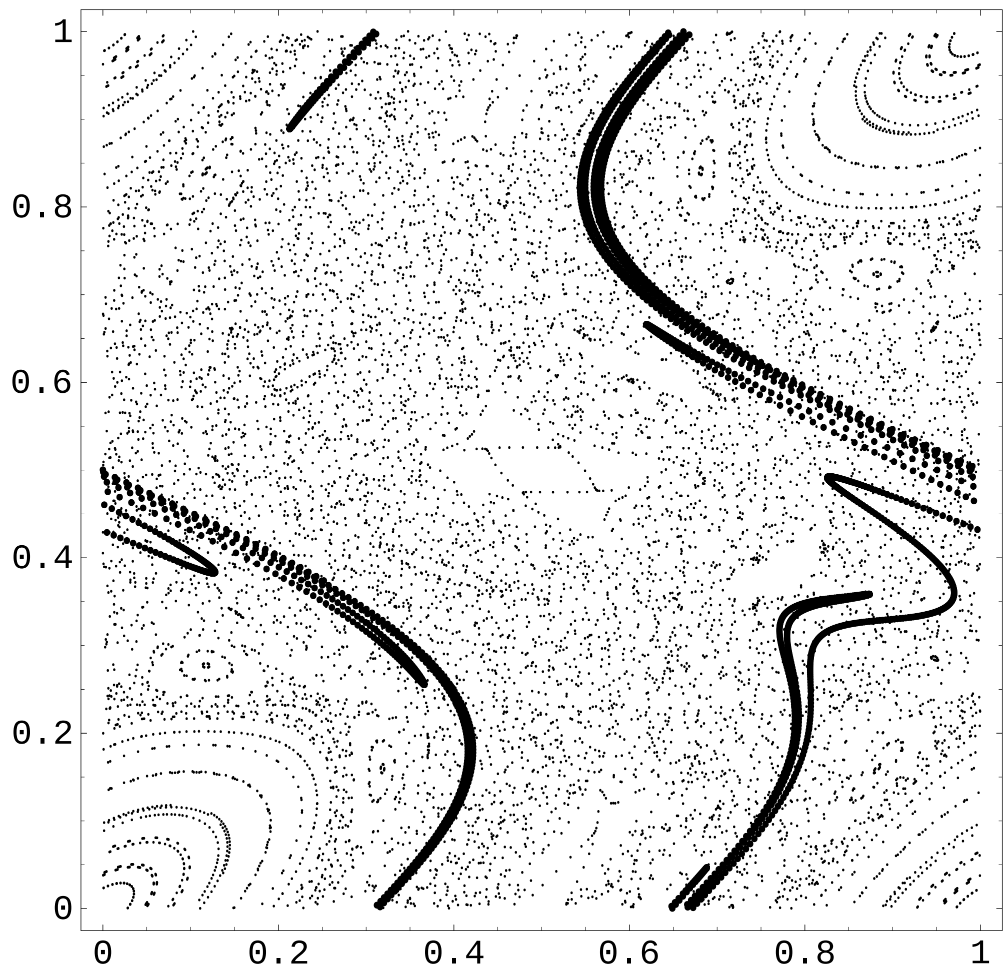
$$\boxed{\lambda(\boldsymbol{\xi}, t) = \ln \Lambda(\boldsymbol{\xi}, t)/2\, t}$$

Stable and Unstable Directions

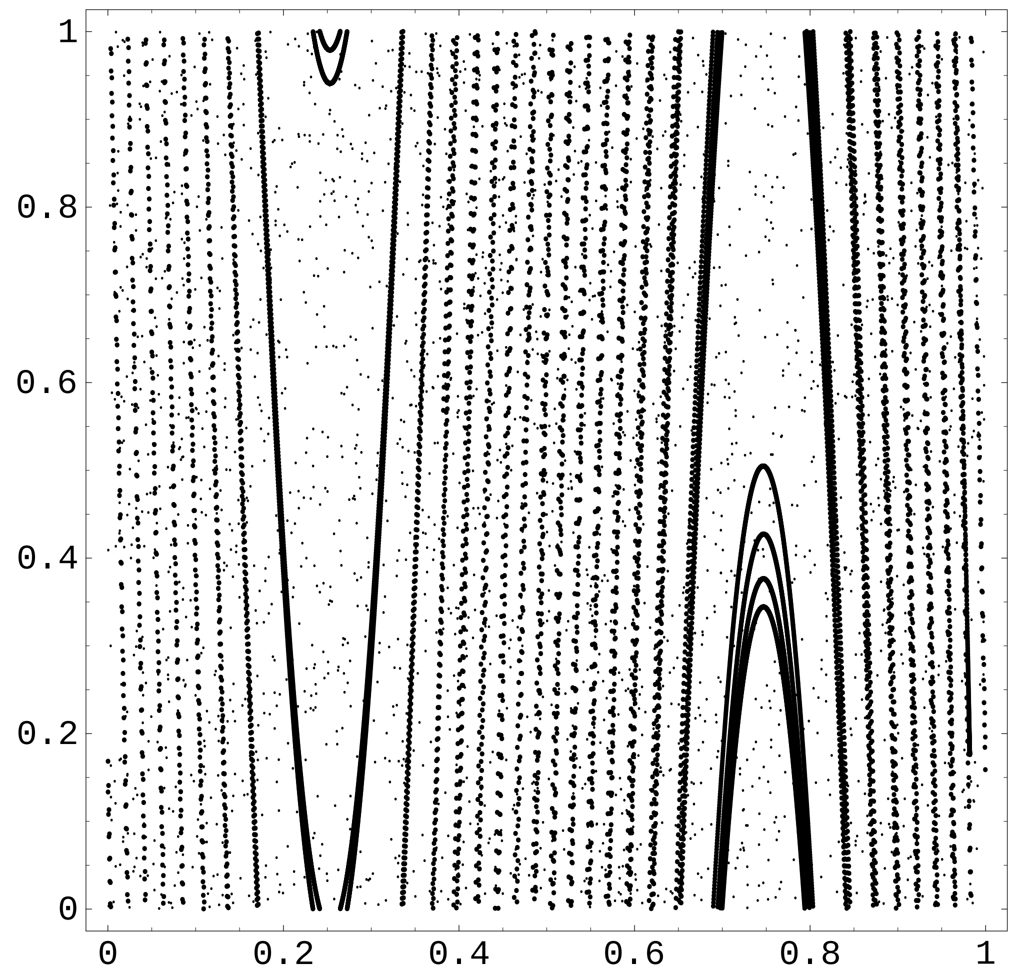
At a fixed coordinate $\boldsymbol{\xi}$:



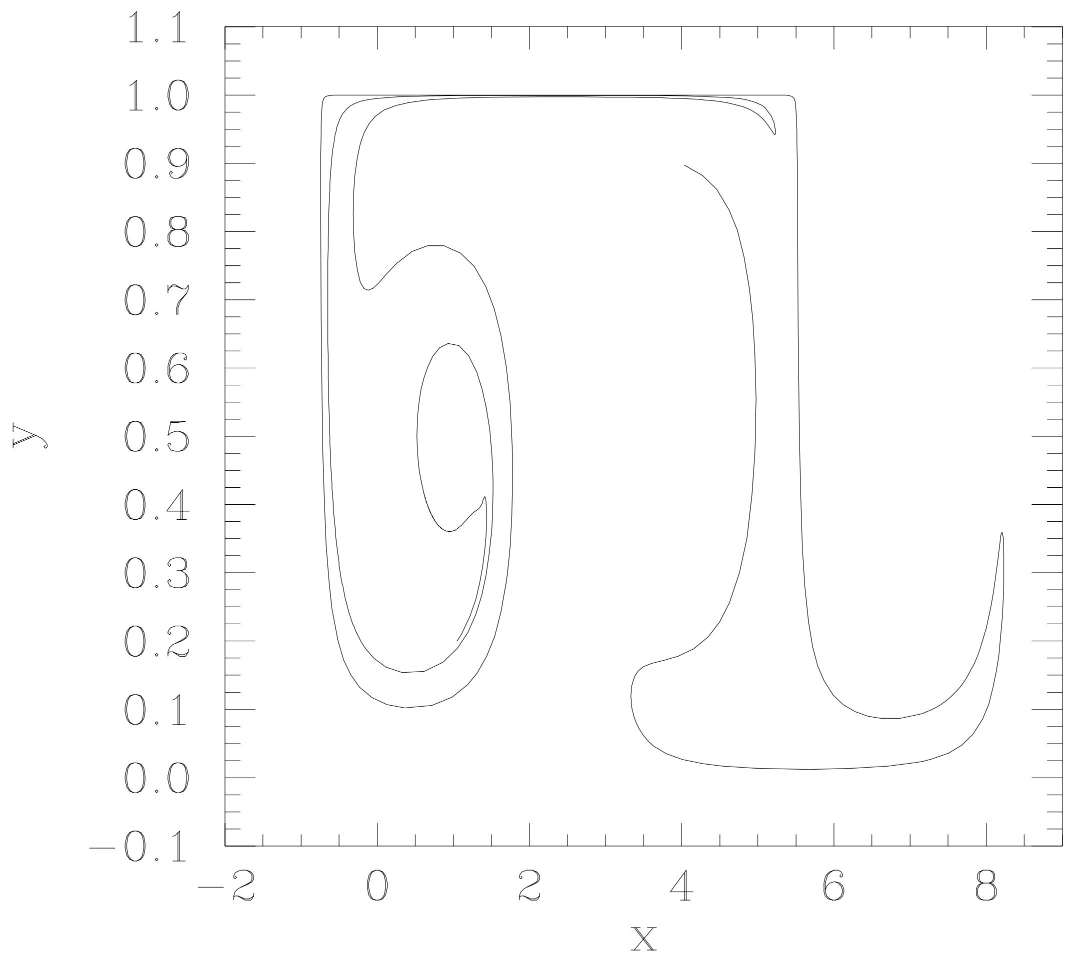
The stable and unstable manifolds $\hat{\mathbf{e}}(\xi, t)$ and $\hat{\mathbf{s}}(\xi, t)$ converge exponentially to their asymptotic values $\hat{\mathbf{e}}_\infty(\xi)$ and $\hat{\mathbf{s}}_\infty(\xi)$, whereas Lyapunov exponents converge logarithmically.



$\hat{\mathbf{s}}_\infty$-line for the standard map with $k = 1.5$.



$\hat{\mathbf{s}}_\infty$-line for the standard map with $k = 50$.



$$\psi(\boldsymbol{x}, t) = Ak^{-1}(\sin kx \sin \pi y + \epsilon \cos \omega t \cos kx \cos \pi y)$$

Oscillating convection rolls ($A = k = \epsilon = \omega = 1$).

The Advection-diffusion Equation

Under the coordinate change to Lagrangian variables the diffusion term becomes

$$\nabla \cdot (D\nabla\phi) = \frac{\partial}{\partial x^i}(D\delta^{ij}\frac{\partial\phi}{\partial x^j}) = \frac{\partial}{\partial \xi^i}(Dg^{ij}\frac{\partial\phi}{\partial \xi^j}).$$

In Lagrangian coordinates the diffusivity becomes Dg^{ij}: it is no longer isotropic.

The advection-diffusion equation is thus just the diffusion equation,

$$\boxed{\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial \xi^i}(Dg^{ij}\frac{\partial\phi}{\partial \xi^j}),}$$

because by construction the advection term drops out.

Diffusion along $\hat{\mathbf{s}}_\infty$ and $\hat{\mathbf{e}}_\infty$

The diffusion coefficients along the $\hat{\mathbf{s}}_\infty$ and $\hat{\mathbf{e}}_\infty$ lines are

$$D^{ss} = s_{\infty i}(Dg^{ij})s_{\infty j} = D\exp(2\lambda\, t),$$

$$D^{ee} = e_{\infty i}(Dg^{ij})e_{\infty j} = D\exp(-2\lambda\, t).$$

We see that D^{ee} goes to zero exponentially quickly. Hence, *essentially all the diffusion occurs along the* $\hat{\mathbf{s}}_\infty$ *line.*

Riemann Curvature Tensor

Differential geometry tells us if a metric describes a flat space, then its Riemann curvature tensor must vanish in every coordinate system.

After some tedious algebra, we find this implies that the quantity

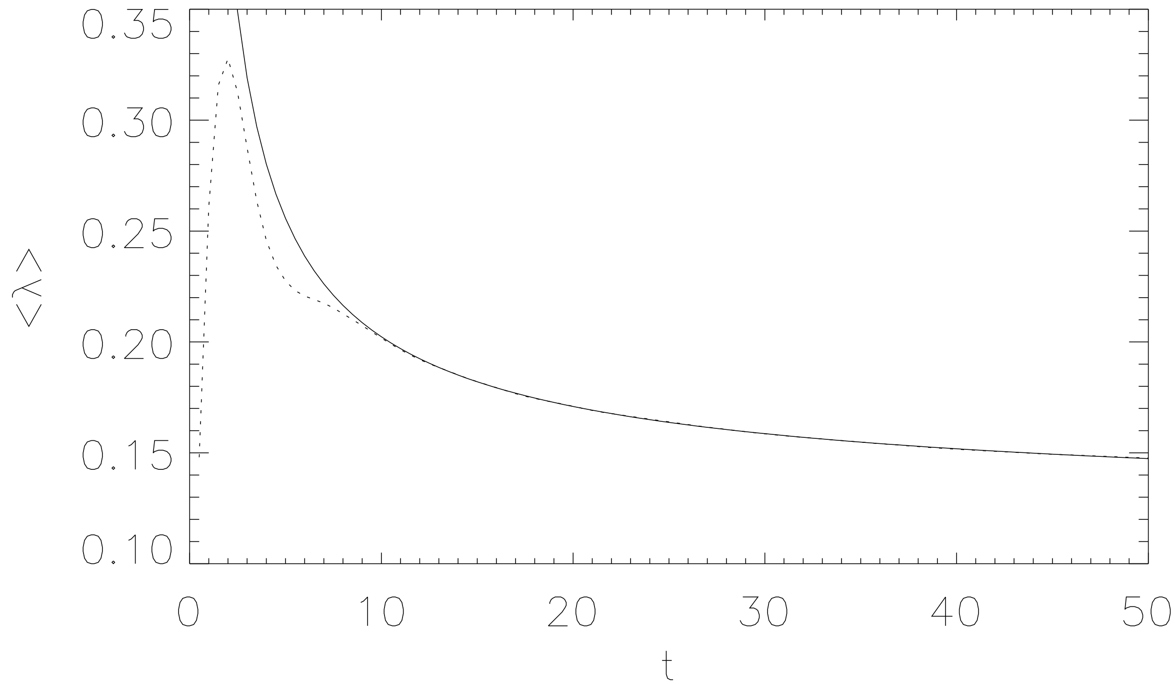
$$\boxed{\hat{\mathbf{s}}_\infty \cdot \nabla_0 \lambda(\xi, t)\, t + \nabla_0 \cdot \hat{\mathbf{s}}_\infty}$$

converges to 0 exponentially. Hence, it can be shown that the finite-time Lyapunov exponents must have the form

$$\boxed{\lambda(\xi, t) = \frac{\tilde{\lambda}(\xi)}{t} + \frac{f(\xi, t)}{\sqrt{t}} + \lambda_\infty,}$$

where $\hat{\mathbf{s}}_\infty \cdot \nabla_0 f = 0$ (the $1/\sqrt{t}$ factor comes from known results on the variance of the exponents).

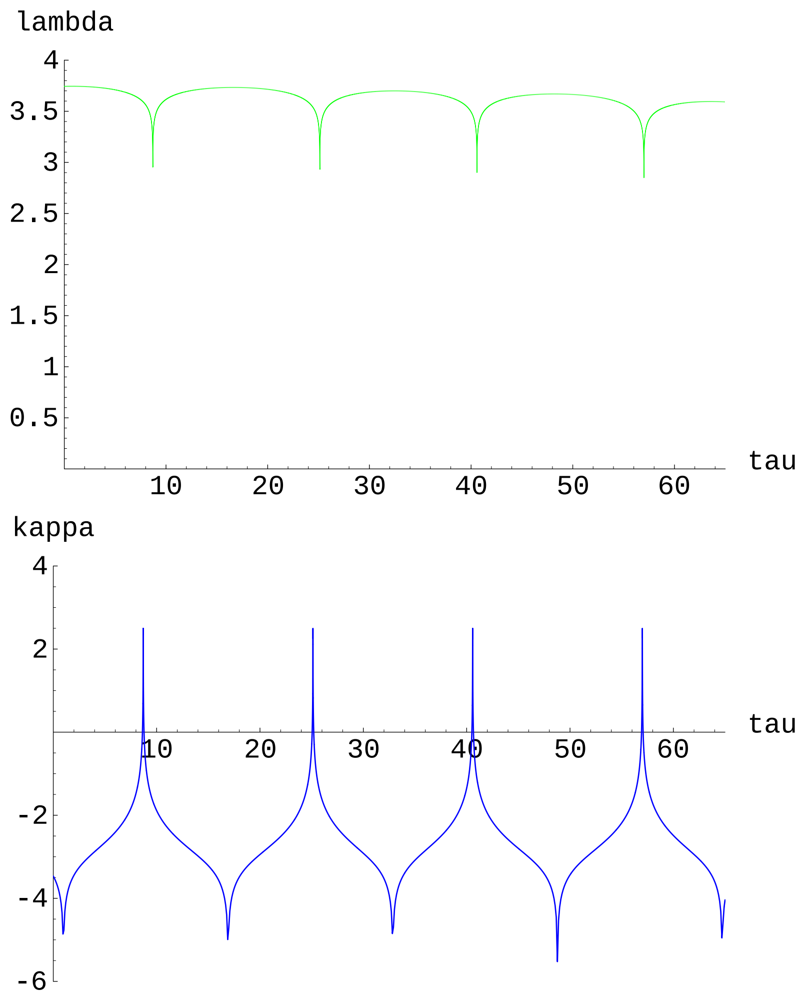
Example:



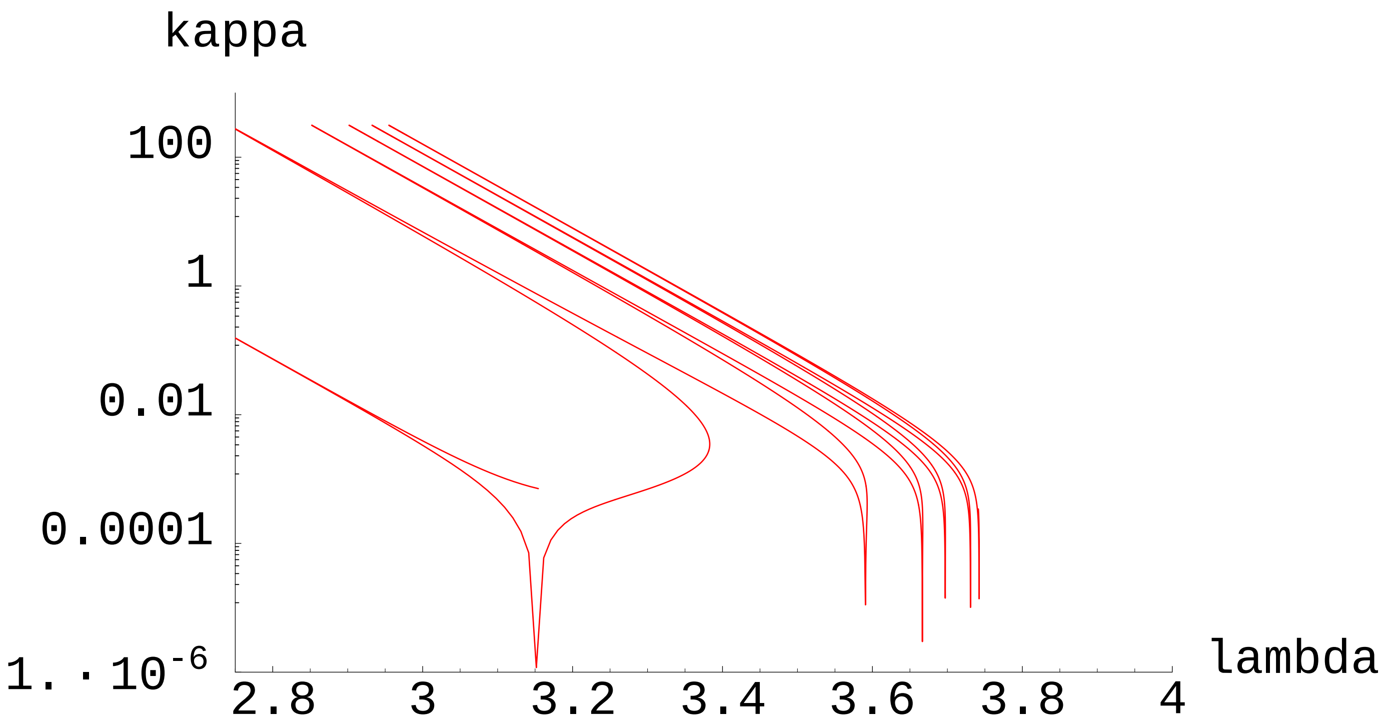
Dotted: Numerical

Solid: $0.305/t + 0.175/\sqrt{t} + 0.117$

Allows us to determine $\lambda_\infty = 0.117$ rapidly and accurately.



Standard map, 5th iteration, $k = 50$ (curvature $\kappa \equiv (\hat{\mathbf{s}}_\infty \cdot \nabla_0)\hat{\mathbf{s}}_\infty$).



Standard map, 5th iteration, $k = 50$.

Conclusions

- Diffusion occurs overwhelmingly along the stable direction.
- The spatial dependence of Lyapunov exponents along $\hat{\mathbf{s}}$ lines is contained in the smooth function $\tilde{\lambda}(\xi)$, which decays as $1/t$.
- The notoriously slow convergence of Lyapunov exponents is embodied in the function $f(\xi, t)$, which is constant on $\hat{\mathbf{s}}$ lines and decays as $1/\sqrt{t}$.
- Relation between $\hat{\mathbf{s}}_\infty(\xi)$, $\kappa \equiv (\hat{\mathbf{s}}_\infty \cdot \nabla_0)\hat{\mathbf{s}}_\infty$, and $\tilde{\lambda}(\xi)$.
- Sharp bends in the $\hat{\mathbf{s}}$ line lead to locally small finite-time Lyapunov exponents (diffusion is hindered).
- Test on flows.