Qualifying Talk

Modeling Shear Flow in Rayleigh–Bénard Convection

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Introduction

Shear flow:

$$\mathbf{v}(x,z,t) = \mathrm{U}(z,t) + \mathbf{v}'(x,z,t), \quad \text{where } \overline{\mathbf{v}'} = 0.$$

Turbulent convection $\mathbf{v}'(x,z,t) \longrightarrow U(z,t)$.

- Rayleigh-Bénard Convection
- Tokamak Edge

Sugama and Horton (1995): low-order model for L–H transitions. Uses turbulence theory to truncate the energy transfer terms between the two energies $\frac{1}{2}\left\langle \overline{\mathbf{v}'^2} \right\rangle$ and $\left\langle \mathrm{U}^2 \right\rangle$.

Motivation

Original aim:

See if such models could be justified from the PDE's by studying truncations.

But how do we choose the modes
to keep in the truncations?

$\longrightarrow$ **Energy-conserving approximations**

Outline

1. PDE's and Boundary Conditions

2. Conserved Quantities of the PDE's

3. Expansion into Normal Modes

4. Making Truncations Preserve the Invariants

5. Boundedness of Solutions and Heat Flux

6. Numerical Examples

Equations of the Rayleigh–Bénard System

Incompressible fluid heated from below and subject to gravity:

$$\begin{aligned}\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} &= -\nabla p + \nu\rho\nabla^2\mathbf{v} - \rho g\hat{\mathbf{z}}, \\ \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\Theta &= \kappa\nabla^2\Theta\end{aligned}$$

Stream function (2–D):

$$\mathbf{v} = (-\partial_z\chi, \partial_x\chi) = \nabla\chi \times \hat{\mathbf{y}}$$

Temperature deviation from conduction state:

$$\Theta = \Theta_{\text{upper}} + \left(1 - \frac{z}{\pi d}\right)\Delta T + T$$

Boussinesq approximation:

$$\rho \to \rho\left(1 - \alpha\left[\Theta - \Theta_{\text{avg}}\right]\right)$$

Rescale variables:

$$\begin{aligned}[x,z] &= d, & [t] &= \sqrt{d/g\alpha\Delta T},\\ [T] &= \Delta T, & [\chi] &= \sqrt{g\alpha\Delta T d^3},\end{aligned}$$

$$\begin{aligned}\frac{\partial\nabla^2\chi}{\partial t}+\left\{\chi\,,\nabla^2\chi\right\} &= \tilde{\nu}\,\nabla^4\chi+\frac{\partial T}{\partial x},\\ \frac{\partial T}{\partial t}+\{\chi\,,T\} &= \tilde{\kappa}\,\nabla^2 T+\frac{\partial\chi}{\partial x}.\end{aligned}$$

Poisson bracket:

$$\{a\,,b\}\equiv\frac{\partial a}{\partial x}\frac{\partial b}{\partial z}-\frac{\partial b}{\partial x}\frac{\partial a}{\partial z}.$$

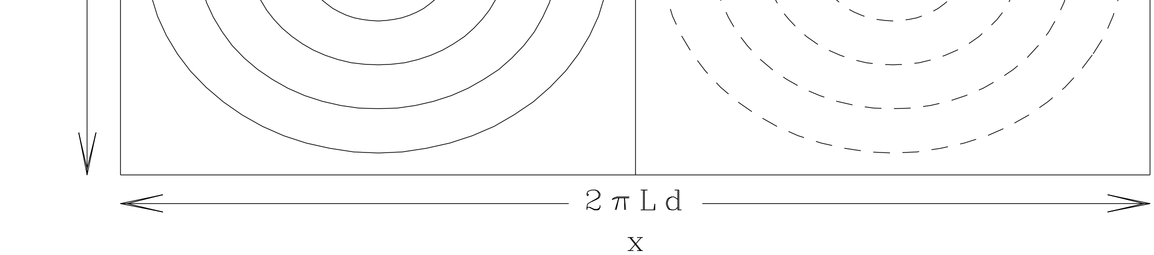
$$\begin{aligned}\sigma\equiv\nu/\kappa &= \text{Prandtl number},\\ R\equiv\frac{g\alpha\Delta T d^3}{\kappa\nu} &= \text{Rayleigh number},\end{aligned}$$

Boundary Conditions

Boundary Conditions are periodic in x, with period $2\pi Ld$, stress-free at walls:

$$\chi = \nabla^2\chi = \partial_x\chi = T = 0, \quad \text{for } z = 0 \text{ or } \pi d.$$

L is the aspect ratio. $k = 1/L$.



Conserved Quantities

In the dissipationless limit ($\tilde{\nu} = \tilde{\kappa} = 0$):

$$\frac{1}{2}\partial_t \left\langle \overline{(\nabla\chi)^2} \right\rangle = \left\langle \overline{T\partial_x\chi} \right\rangle,$$

$$\partial_t \left\langle \overline{zT} \right\rangle = \left\langle \overline{T\partial_x\chi} \right\rangle.$$

We obtain a conservation law:

$$\partial_t \left[\frac{1}{2}\left\langle \overline{(\nabla\chi)^2} \right\rangle - \left\langle \overline{zT} \right\rangle\right] = 0,$$

or

$$\partial_t \left[K + U\right] = \partial_t E = 0.$$

Other invariants:

$\frac{1}{2}\left[\left\langle \overline{(\nabla\chi)^2} \right\rangle - \left\langle \overline{T^2} \right\rangle\right]$, $\left\langle \overline{T\,\nabla^2\chi} \right\rangle$, and $\left\langle \overline{T} \right\rangle$.

Expansion into Normal Modes

$$\chi(x,z,t) = \sum_{m,n} \chi_{mn}(t)\, e^{i(mz+nkx)},$$
$$T(x,z,t) = \sum_{m,n} T_{mn}(t)\, e^{i(mz+nkx)}.$$

The boundary and reality conditions lead to:

$$\chi^r_{mn} = -\chi^r_{m,-n}\,, \qquad \chi^i_{mn} = \chi^i_{m,-n}\,,$$
$$\chi^r_{m0} = 0, \qquad \chi^r_{0n} = \chi^i_{0n} = 0,$$

and similarly for the T_{mn}'s.

This expansion is more general than traditional ones:

- It includes shear flow modes for the stream function (of the form $\chi^i_{m0}\, e^{imz}$).
- It allows for variable phase of the modes.

After inserting the mode expansions into the Boussinesq equations, we obtain a set of coupled nonlinear ODE's:

$$\rho_{mn}\frac{d}{dt}\chi_{mn} = -\tilde{\nu}\rho_{mn}^2\chi_{mn} - ik\,nT_{mn} - k\sum_{\substack{m'+m''=m\\ n'+n''=n}}(m'n''-m''n')\rho_{m''n''}\,\chi_{m'n'}\,\chi_{m''n''}\ ,$$

$$\frac{d}{dt}T_{mn} = -\tilde{\kappa}\rho_{mn}T_{mn} + i\,k\,n\,\chi_{mn} - k\sum_{\substack{m'+m''=m\\ n'+n''=n}}(m'n''-m''n')\,\chi_{m'n'}\,T_{m''n''}\ ,$$

with $\rho_{mn} = m^2 + k^2n^2$.

Truncations

$$\infty \text{ modes} \longrightarrow \text{Finite set of modes}$$

Set of pairs (m, n) corresponding to the inclusion of:

$$\chi_{mn} \ , \ \chi_{m,-n} \ , \ \chi_{-m,n} \ , \ \text{and } \chi_{-m,-n} \ ,$$

related by the boundary conditions.

Include the corresponding T_{mn}.

$$M \equiv \max\{m\}, \quad N \equiv \max\{n\}$$

In general, these truncations will not preserve all the invariants of the full PDE's.

Expansion of the Energies

The expansions for the kinetic and potential energy are:

$$K = \frac{1}{2}\sum_{m,n} \rho_{mn} \left|\chi_{mn}\right|^2$$
$$U = -2\sum_{p>0} \frac{(-1)^p}{p} T^i_{p0} \ .$$

Note that U depends only on the T^i_{p0} modes.

$$\dot{K} = \frac{1}{2}\sum_{m,n} \rho_{mn} \left(\chi^*_{mn}\dot{\chi}_{mn} + \chi_{mn}\dot{\chi}^*_{mn}\right) \ ,$$

$$\longrightarrow \chi^*_{mn}\chi_{m'n'}\chi_{m''n''}$$

Dissipationless: $\nu = \kappa = 0$

$$
\begin{aligned}
\dot{K} &= -4k \sum_{m,n>0} n \,\mathrm{Im}\, \chi_{mn} \, T^*_{mn} \ , \\
\dot{U} &= -4k \sum_{p,m,n>0} (-1)^p n \,\mathrm{Im}\, \chi_{mn} \, T^*_{m-p,n} \\
&\quad - 4k \sum_{p',m,n>0} (-1)^{p'} n \,\mathrm{Im}\, \chi_{mn} \, T^*_{m+p',n} \ .
\end{aligned}
$$

Need:

$$
T^*_{mn} + \sum_{p>0} (-1)^p \,\mathrm{sgn}(m-p) \, T^*_{|m-p|,n} + \sum_{p'>0} (-1)^{p'} \, T^*_{m+p',n} \ = 0.
$$

for energy conservation.

We separate the first sum in two parts to get rid of the sgn function and of the absolute value:

$$T^*_{mn} + \sum_{p=1}^{m-1} (-1)^p \, T^*_{m-p,n} - \sum_{p=m+1}^{M+m} (-1)^p \, T^*_{p-m,n} + \sum_{p'=1}^{M-m} (-1)^{p'} \, T^*_{m+p',n} \ ,$$

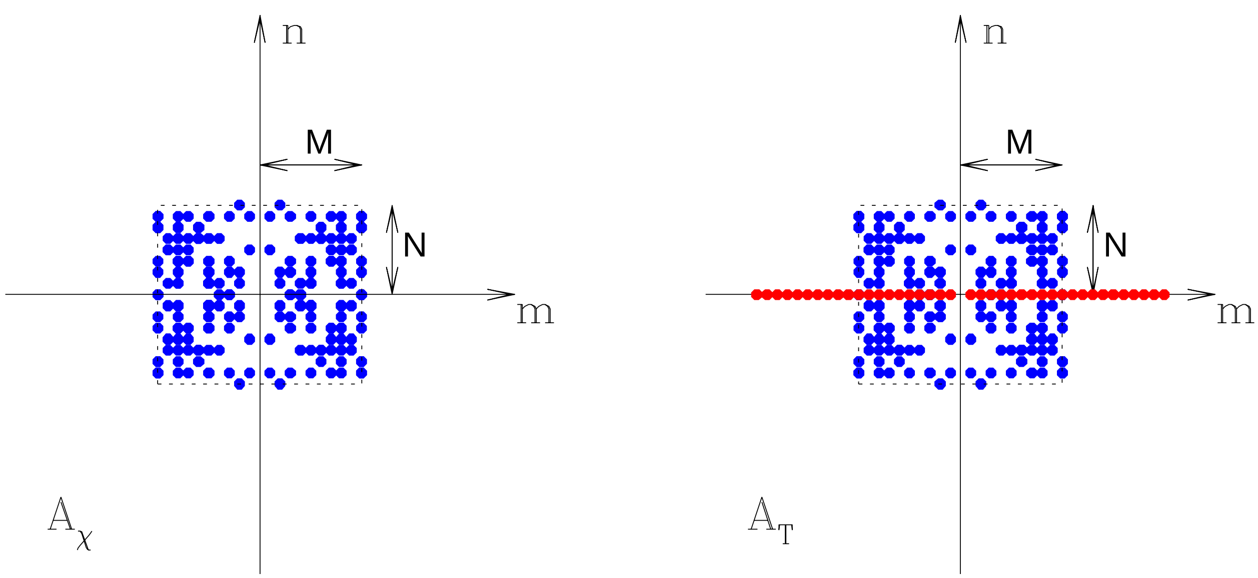
and relabel:

$$T^*_{mn} + \sum_{s=1}^{m-1} (-1)^{m-s} \, T^*_{sn} - \sum_{r=1}^{M} (-1)^{m+r} \, T^*_{rn} + \sum_{s=m+1}^{M} (-1)^{s-m} \, T^*_{sn} \ ,$$

$$= \sum_{s=1}^{M} (-1)^{m-s} \, T^*_{sn} - \sum_{r=1}^{M} (-1)^{m+r} \, T^*_{rn}$$

$$= 0.$$

For a given set of modes, include all modes of the form $T_{p0}, \quad p = 1, \ldots 2M$:



The effect of adding these modes is to preserve all of the invariants of the full PDE's.

Boundedness of Solutions

Non-negative quantity:

$$
\begin{aligned}
Q \;\equiv\; & 2 \sum_{m,n>0} \rho_{mn} \left|\chi_{mn}\right|^2 + \sum_{m>0} \rho_{m0} \left|\chi_{m0}\right|^2 \\
& + 2 \sum_{m,n>0} \left|T_{mn}\right|^2 + \sum_{m>0} \left(T^i_{m0} - \frac{2}{m}\right)^2 ,
\end{aligned}
$$

$$\frac{d}{dt} Q \leq -\min\{2\tilde{\nu}, \tilde{\kappa}\} Q + 4\tilde{\kappa} M_0 \ ,$$

$$M_0 \equiv \text{number of } T_{m0} \text{ modes.}$$

For $Q > 4\tilde{\kappa} M_0 / \min\{2\tilde{\nu}, \tilde{\kappa}\}$, $\frac{d}{dt} Q < 0$.

$\longrightarrow Q$ is bounded.

This is not true of arbitrary truncations.

Heat Flux

Total heat flux through a horizontal slice of fluid:

$$\begin{aligned}\overline{q_z}(z) &= \overline{q_z^{\mathsf{cv}}} + \overline{q_z^{\mathsf{cd}}} \\ &= \overline{v_z T} + \overline{\hat{\mathbf{z}} \cdot (-\nabla T)} \ .\end{aligned}$$

In a steady-state situation, should be independent of z.

For energy-conserving truncations:

$$\overline{q_z}(z) = \langle\, \overline{q_z}\, \rangle - 2 \sum_{m>0} \frac{\dot{T}^i_{m0}}{m} \cos mz \ ,$$

so that $\overline{q_z}(z) = \langle\, \overline{q_z}\, \rangle$ in a steady state.

Particular Truncations

The Lorenz model is energy-conserving:

$$\chi^r_{11}\,, \quad T^i_{11}\,, \quad T^i_{20}\,.$$

It does not allow for shear flow or variable phase.

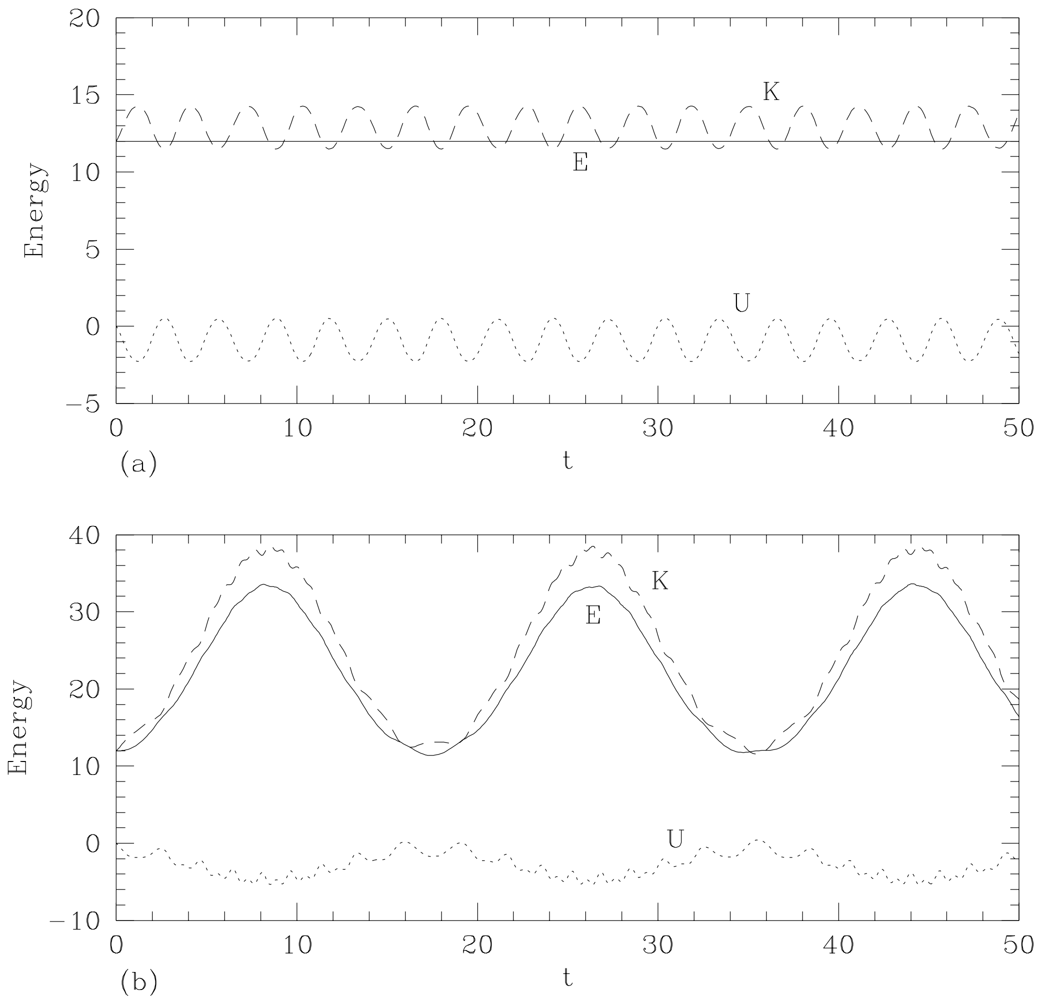
The 6–mode Howard and Krishnamurti model is used to study generation of shear flow (no variable phase):

$$\chi^i_{10}\,, \quad \chi^r_{11}\,, \quad \chi^i_{21}\,, \quad T^i_{11}\,, \quad T^r_{21}\,, \quad T^i_{20}\,.$$

It is *not* energy-conserving. It lacks the T^i_{40} mode.

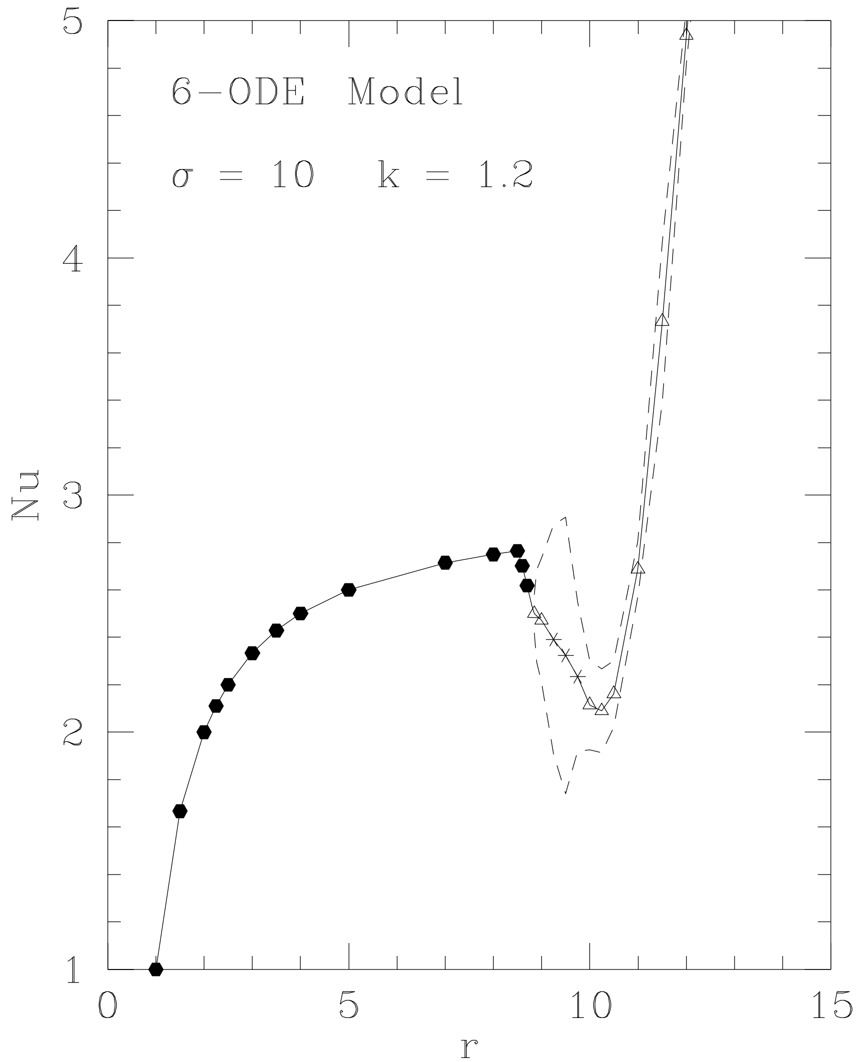
Numerical Demonstration

Integration of the 7–ODE (a) and 6–ODE (b) models in the dissipationless limit:

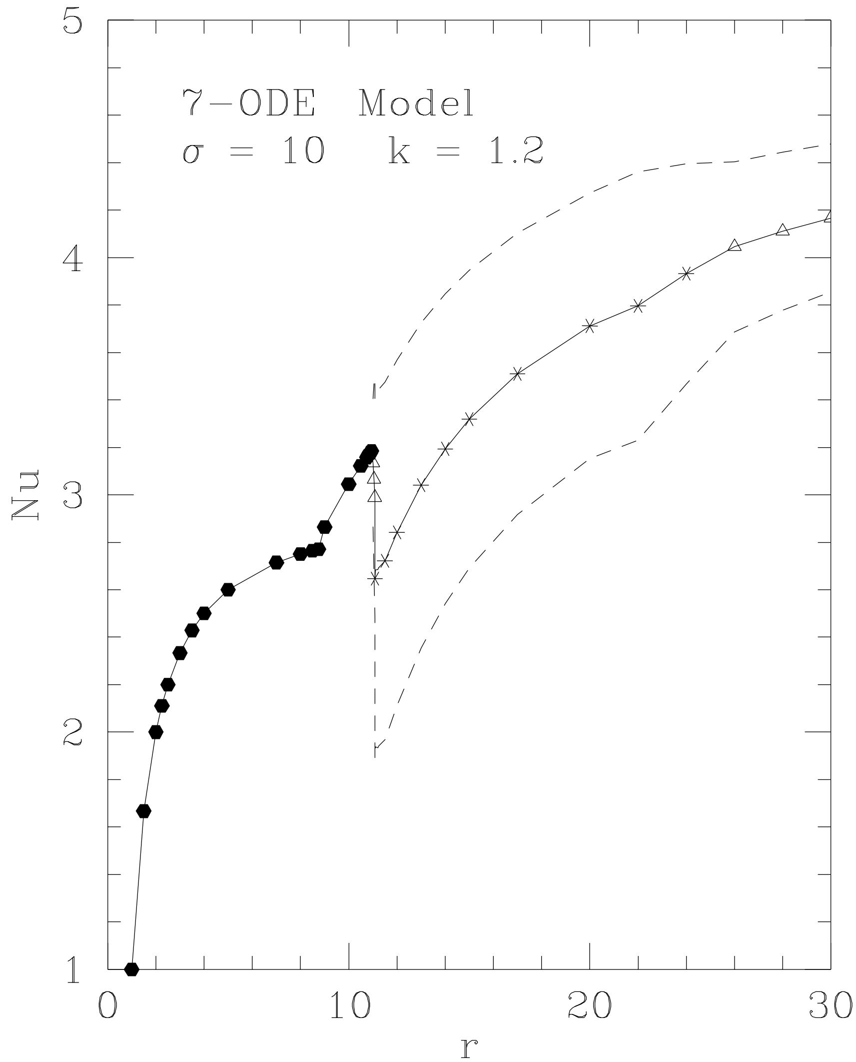


Energy is definitely not conserved by the 6–ODE model.

$$\mathrm{Nu} \equiv \frac{\text{Heat transported by conduction and convection}}{\text{Heat conduction of fluid at rest}}$$



Solid Dots: Steady-state
Triangles: Quasi-periodic
Asterisks: Chaotic



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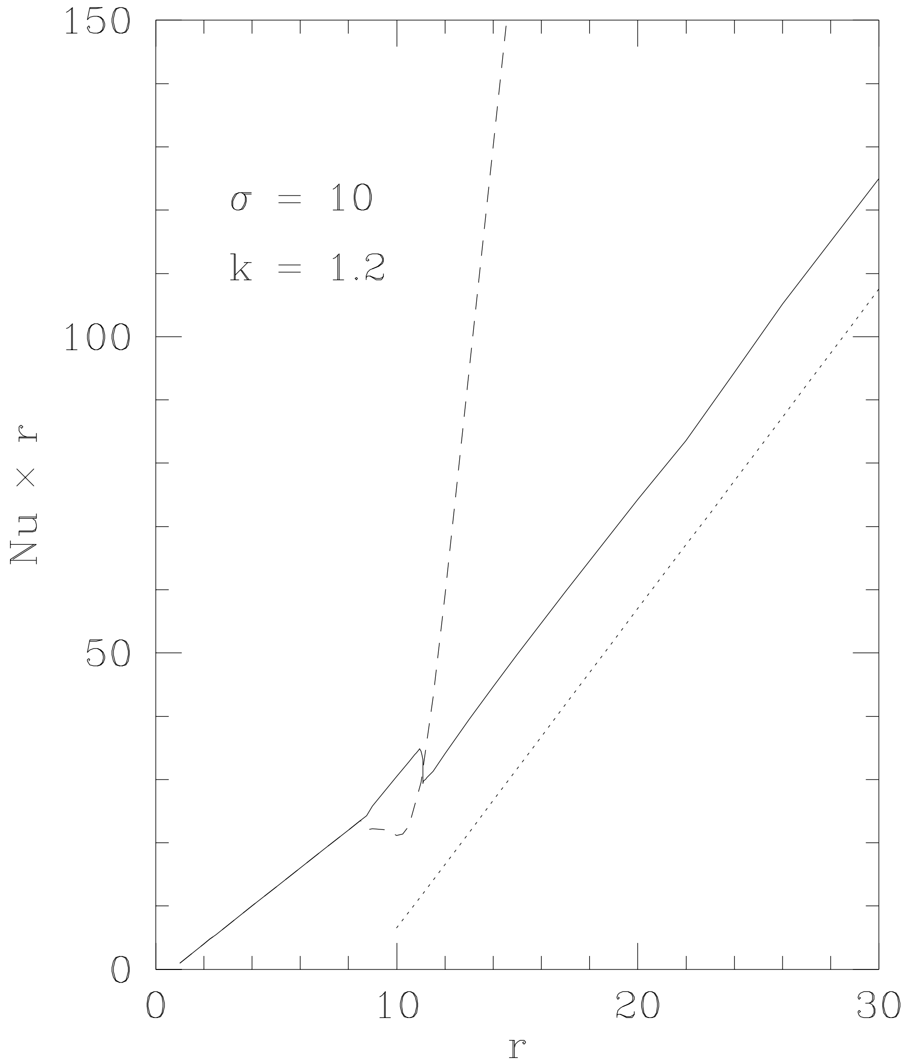
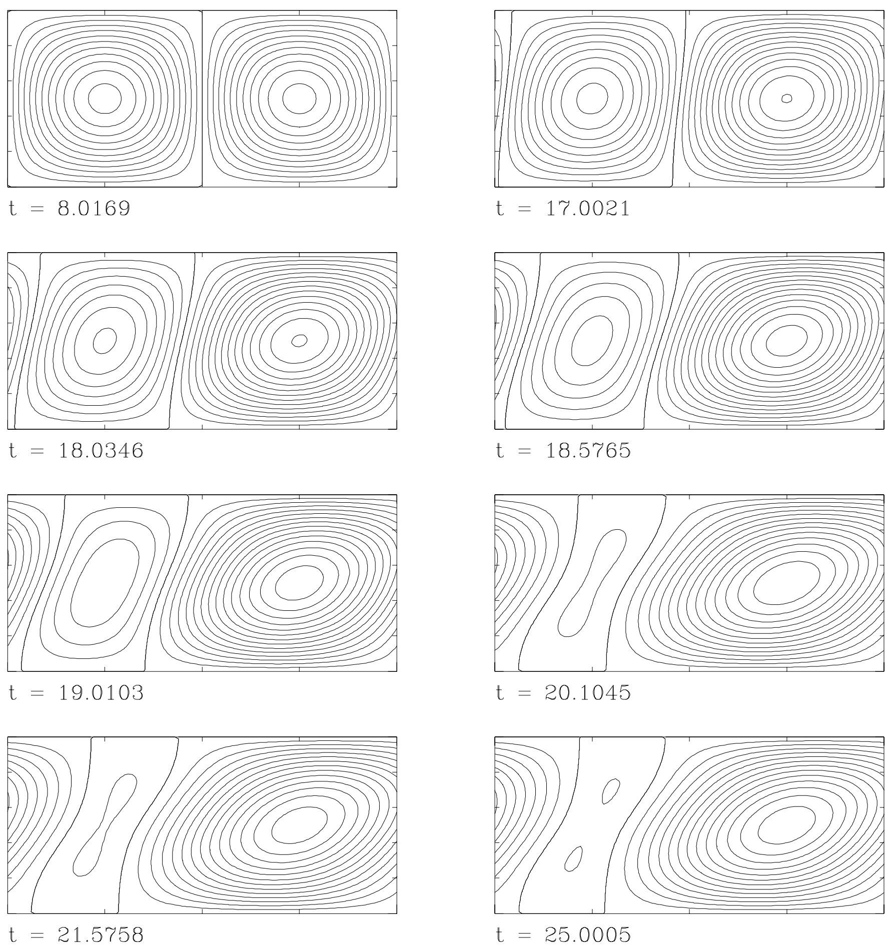


Solid: 7–ODE, energy-conserving

Dashed: 6–ODE

Dotted: experiment (slope of 5.05)

Transition to Shear Flow



$$\sigma = 1, \quad r = 3.4, \quad k = 1.2$$

For $t > 25$, steady-state tilted-cell convection.

Conclusions

- Used expansions that include shear flow and breaking of point-symmetry.

- General method for generating energy-conserving truncations.

Advantages of energy-conserving approximations:

1. The cascade of energy through the inertial range to the dissipation scale is modeled without extraneous terms in the energy equations.
2. Proper description of the heat flow in the steady-state limit, even with dissipation.
3. Boundedness of solutions.