# Shape Matters

## Homogenization for a confined Brownian microswimmer

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Analysis of Fluids and Related Topics Seminar
Princeton University
26 March 2020

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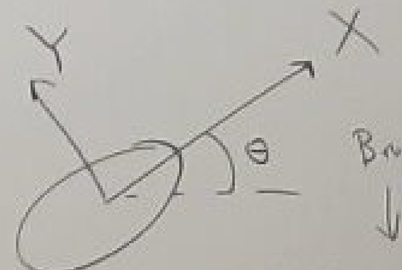
(joint w. Hongfei Chen)

## Brownian microswimmer:




Brownian
$\downarrow \mathbb{E}\, dW_i^2 = dt$

$$
\begin{aligned}
dX &= U\,dt + \sqrt{2D_X}\,dW_1 \\
dY &= \sqrt{2D_Y}\,dW_2 \\
d\theta &= \sqrt{2D_\theta}\,dW_3
\end{aligned}
$$

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Rotate (lab).

$$
\begin{aligned}
dx &= \left(U\,dt + \sqrt{2D_X}\,dW_1\right)\cos\theta - \sin\theta\sqrt{2D_Y}\,dW_2 \\
dy &= \left(\sqrt{2D_X}\,dW_1\right)\sin\theta + \cos\theta\sqrt{2D_Y}\,dW_2 \\
d\theta &= \sqrt{2D_\theta}\,dW_3
\end{aligned}
$$

Fokker-Planck (Kolm. forward) eq'n

$p(x, y, \theta, t)$

$$\frac{\partial p}{\partial t} + \nabla\cdot\left(\mu p - \mathbb{D}\nabla p\right) = 0$$

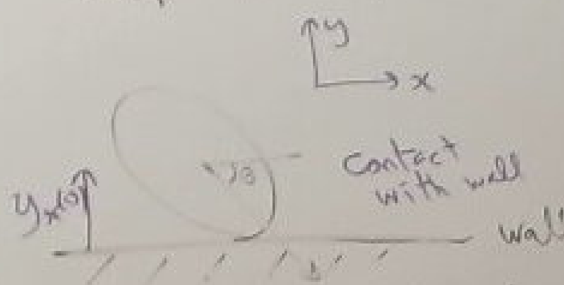
$$\mu = U\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} D_X\cos^2\theta + D_Y\sin^2\theta & \frac{1}{2}(D_X - D_Y)\sin 2\theta & 0 \\ '' & D_X\sin^2\theta + D_Y\cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_X = D_Y \quad \mathbb{D} = \begin{pmatrix} D_X & & \\ & D_Y & \\ & & D_\theta \end{pmatrix}$$

Green's func, etc. (Kurzthaler & Franosch, 2017)

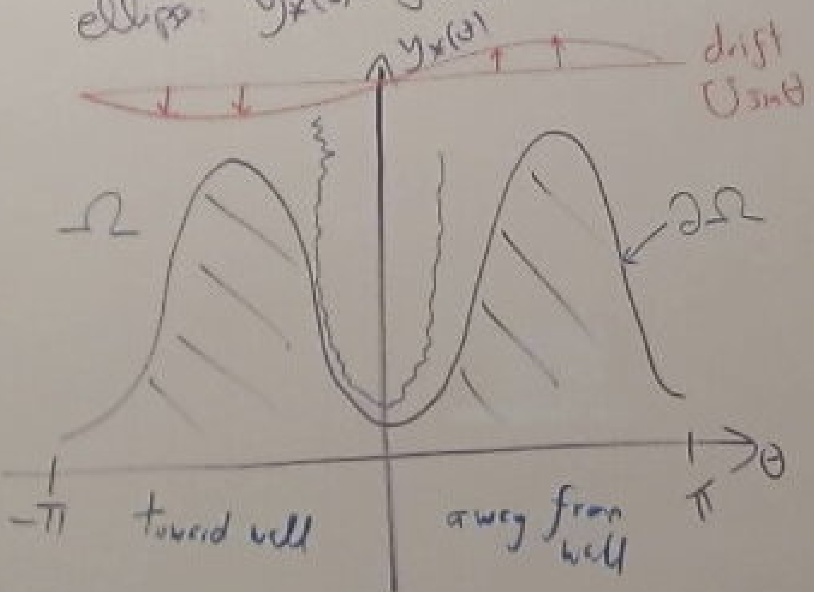
So far: no boundaries

Shape not important

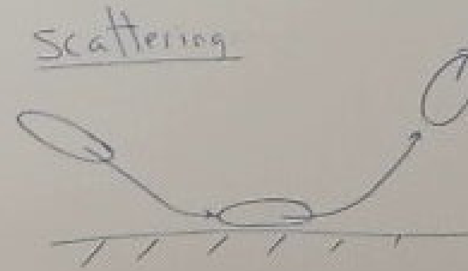



"wall distance function" $y_*(\theta)$

ellipse: $y_*(\theta) = \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}$




Scattering

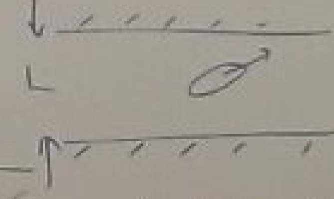


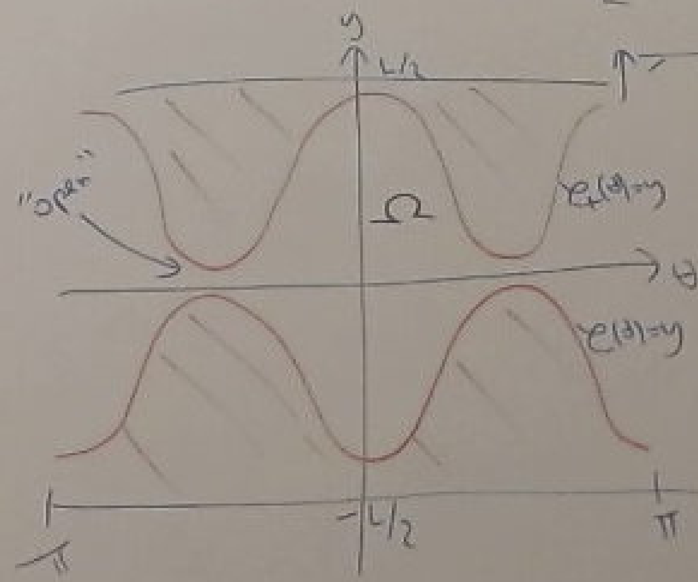
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mu \rho - \mathbb{D} \nabla \rho) = 0 \quad \text{in } \Omega$$

$$f = \mu \rho - \mathbb{D} \nabla \rho \quad \text{prob flux}$$

Boundary conditions $\quad f \cdot n = 0$, on $\partial\Omega$

Channel configuration:




$\ell_-(\theta) = y_*(\theta) - \frac{L}{2}$

$\ell_+(\theta) = -y_*(\theta + \pi) + \frac{L}{2}$

Left-right symmetric:

$y_*(\theta) = y_*(\theta - \pi)$

$y_*(\theta)$ related to Legendre transform of shape.

F-P eq'n in channel:

$n \cdot f = 0$ on $\partial\Omega$

$f = \mu p - D\nabla p$

$p(y,\theta)$

$$\partial_t p + \nabla\cdot(\mu p - D\nabla p) = 0$$

Geometry makes this hard!

Simplify: $D_\theta = \varepsilon \ll 1$

Use asymptotic expansion

$p = p_0 + \varepsilon p_1 + \dots$, $T = \varepsilon t$

$$p_0(y,\theta) = P(\theta, T)e^{\sigma(\theta)y}$$

$$\partial_T P + \partial_\theta(\mu(\theta) - \partial_\theta P) = 0$$

$$\sigma(\theta) = U\sin\theta / D_{yy}(\theta)$$

$$\mu(\theta) = \frac{\sigma(\theta)}{2\sinh\Delta}\left(e^{\Delta}\zeta_+' - e^{-\Delta}\zeta_-'\right)$$ ← shape!

$$\Delta(\theta) = \frac{1}{2}\sigma(\theta)\left(\zeta_+(\theta) - \zeta_-(\theta)\right)$$

Invariant density: $\partial_\theta(\mu P - \partial_\theta P) = 0$

$$\mu P - \partial_\theta P = \omega/2\pi$$

Can show: $\omega = 0$ for left-right symmetric swimmer

"detailed balance"

Otherwise: $\mathbb{E}\mu = \omega$ mean rotation rate as swimmer bounces off walls

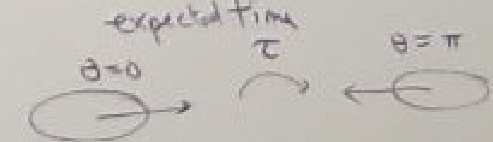
← elliptical swimmers stuck at walls.

$\theta = 0, \pi$

(or: $\theta = \pm\pi/2$

) shape matters!

# Mean reversal time:




Small $D_\theta$ again

$$\mu(\theta)\tau + \tau'' = -1$$

$$\tau(\pm\pi) = 0$$

We want $\tau(0)$. mean reversal time (MRT)

Without drift: Holcman & Schuss (2014) "narrow exit" limit

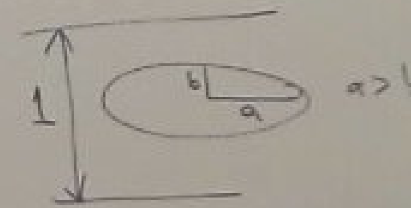
For L-R symmetric case,

$$\tau(0) = \frac{1}{4}\int_0^\pi \frac{d\theta}{\rho(\theta)}$$

invariant density

Integral controlled by "bottlenecks" of small $\rho(\theta)$.

(For example: narrow channel, sticking to walls)

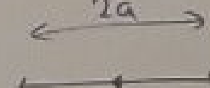



$\mu = 0$ (no swimming, diffusion only)

$$\tau(0) = \frac{(\pi - 2a)(\pi - \arccos a)}{D_\theta\sqrt{1-a^2}}$$

$a \geq 1$ can't move

(Valid for small $D_\theta$)




"needle" swimmer ($b=0$)

With swimming: harder!

$$\beta = \frac{Ua}{2D_r} \gg 1 \quad \text{Péclet number}$$

$$\tau(0) = \frac{\pi}{2\beta}e^\beta \quad (D_x = D_r)$$

exponential! Sticks to walls.

# Effective diffusion:

 $\mathbb{E}x^2 = 2D_{\text{eff}}\,t$

"Homogenize" the F-P eq'n.
(only in $x$!)

$x \mapsto x/\eta, \quad t \mapsto t/\eta^2$

... fairly standard derivation (see preprint)

$$\partial_t P = D_{\text{eff}}\, \partial^2 P$$

$$D_{\text{eff}} = \mathbb{E}\, D_{xx} + D_{\text{enhanced}}$$

$$\uparrow$$

$$\int_{-\pi}^{\pi} D_{xx}\, P\, d\theta$$

$$D_{\text{enh}} = -\int_{-\pi}^{\pi} \Xi\, w\, X\, d\theta$$

$w$ = weight depending on shape

$$\Xi(\theta) = U\cos\theta - U\sin\theta\, \frac{D_{xy}(\theta)}{D_{yy}(\theta)}$$

$X$ satisfies a cell problem:

$$\partial_\theta(\nu X - w\,\partial_\theta X) = -\Xi P$$

$\mu = \frac{\nu + w'}{w}$ ?

Can maybe solve for simple shapes?
But more importantly we can show bound:

$$D_{\text{enh}} \le \frac{1}{2}\tau(0)\left(\mathbb{E}|\Xi|\right)^2 \le \frac{1}{2}\tau(0)\, U^2$$

$(D_x \ge D_r/2)$ ?

"intuition"

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