# Braids of entangled particle trajectories

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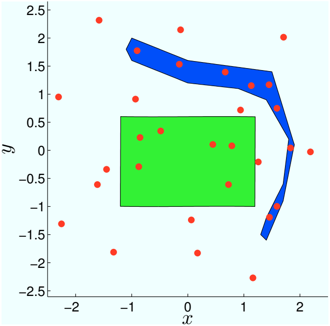
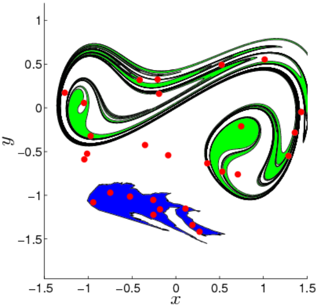
# Sparse trajectories and material loops



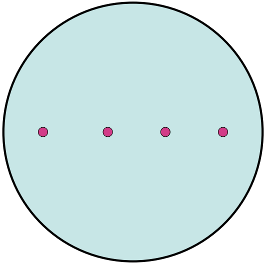




How do we efficiently detect trajectories that ‘bunch’ together?

[movie 1]

# Mathematical background: Punctured disks

Low-dimensional topologists have long studied transformations of surfaces such as the punctured disk:

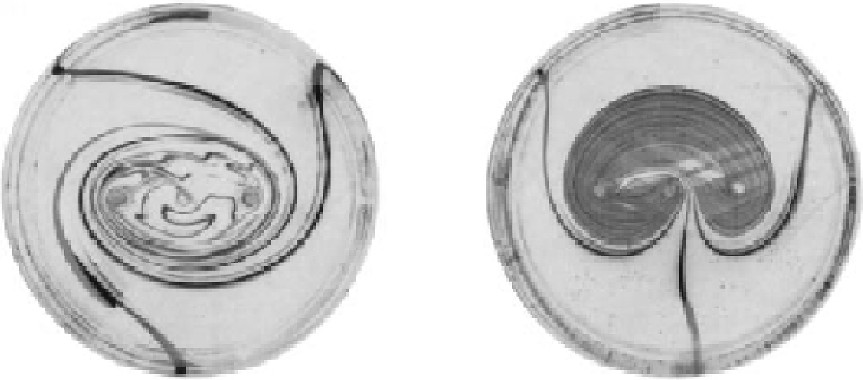


The central object of study is the homeomorphism: a continuous, invertible transformation whose inverse is also continuous.

For instance, this is a model of a two-dimensional vat of viscous fluid with stirring rods.

# Punctured disks in experiments

The transformation in this case is given by the solution of a fluid equation over one period of rod motion.

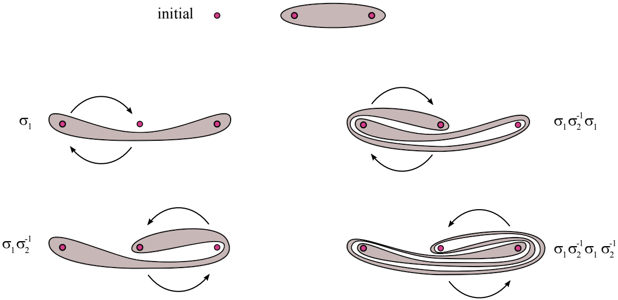


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

[movie 2] [movie 3]

# Growth of curves on a disk

On a disk with 3 punctures (rods), we can also look at the growth of curves:




We use the braid generator notation: $\sigma_i$ means the clockwise interchange of the $i$th and $(i+1)$th rod. (Inverses are counterclockwise.)

The motion above is denoted $\sigma_1\sigma_2^{-1}$.

# Growth of curves on a disk (2)

The rate of growth $h = \log\lambda$ is called the topological entropy.

But how do we find the rate of growth of curves for motions on the disk?

For 3 punctures it's easy: the entropy for $\sigma_1\sigma_2^{-1}$ is $h = \log\varphi^2$, where $\varphi$ is the Golden Ratio!

For more punctures, use Moussafir iterative technique (2006).

[Thiffeault, *Phys. Rev. Lett.* (2005); *Chaos* (2010); Gouillart et al., *Phys. Rev. E* (2006) 'ghost rods']

# Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

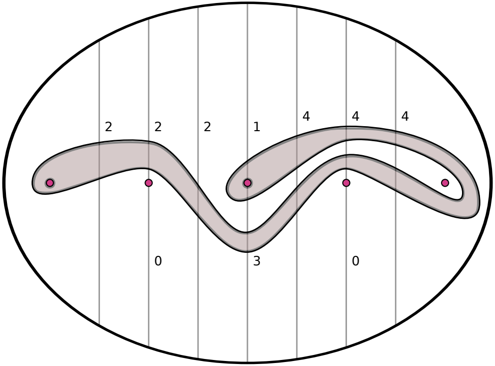
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

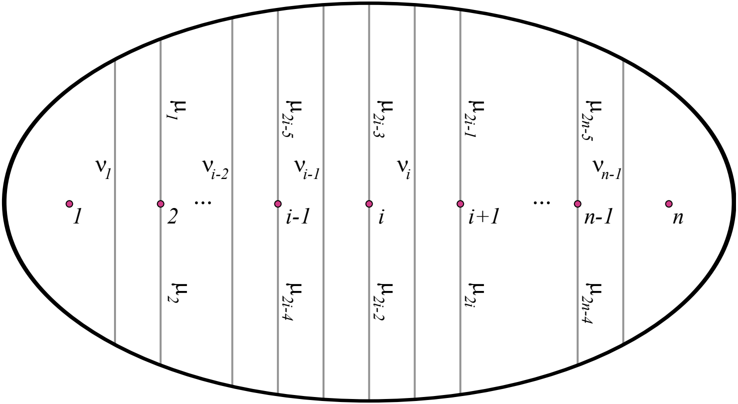
# Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



# Crossing numbers

Label the crossing numbers:



# Dynnikov coordinates

Now take the difference of crossing numbers:

$$a_i = \tfrac{1}{2}\left(\mu_{2i} - \mu_{2i-1}\right),$$
$$b_i = \tfrac{1}{2}\left(\nu_i - \nu_{i+1}\right)$$

for $i = 1, \ldots, n-2$.

The vector of length $(2n-4)$,

$$\mathbf{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2})$$

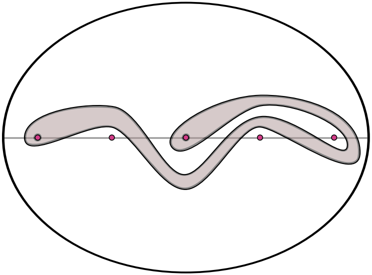
is called the Dynnikov coordinates of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can't do it with fewer than $2n-4$ numbers.

# Intersection number

A useful formula gives the minimum intersection number with the 'horizontal axis':

$$L(\mathbf{u}) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i| \,,$$

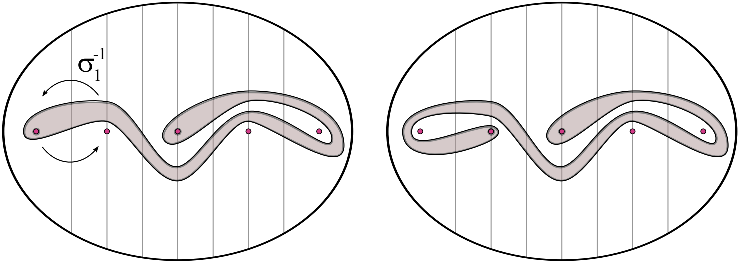


For example, the loop on the left has $L = 12$.

The crossing number grows proportionally to the the length.

# Solution to problem 2: Action on coordinates

Moving the punctures according to a braid generator changes some crossing numbers:




There is an explicit formula for the change in the coordinates!

# Action on loop coordinates

The update rules for $\sigma_i$ acting on a loop with coordinates $(\mathbf{a}, \mathbf{b})$ can be written

$$\begin{aligned}
a'_{i-1} &= a_{i-1} - b^+_{i-1} - \left(b^+_i + c_{i-1}\right)^+ \,, \\
b'_{i-1} &= b_i + c^-_{i-1} \,, \\
a'_i &= a_i - b^-_i - \left(b^-_{i-1} - c_{i-1}\right)^- \,, \\
b'_i &= b_{i-1} - c^-_{i-1} \,,
\end{aligned}$$

where

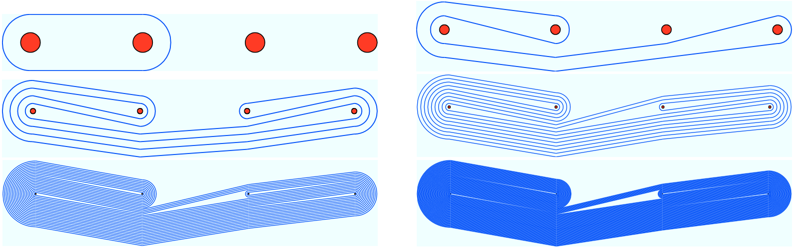
$$f^+ := \max(f, 0), \qquad f^- := \min(f, 0).$$

$$c_{i-1} := a_{i-1} - a_i - b^+_i + b^-_{i-1} \,.$$

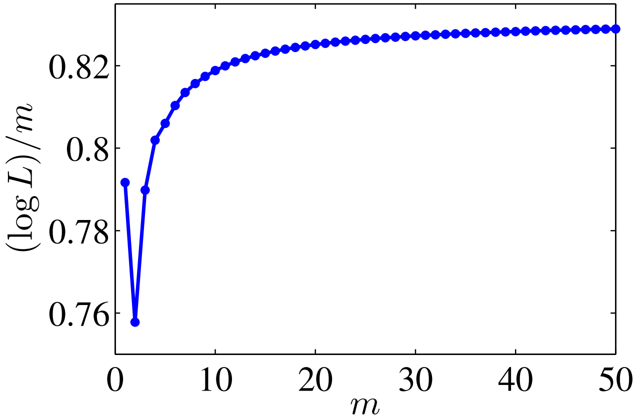
This is called a piecewise-linear action.
Easy to code up (see for example Thiffeault (2010)).

# Growth of $L$

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of $L$ and thus measure the entropy:

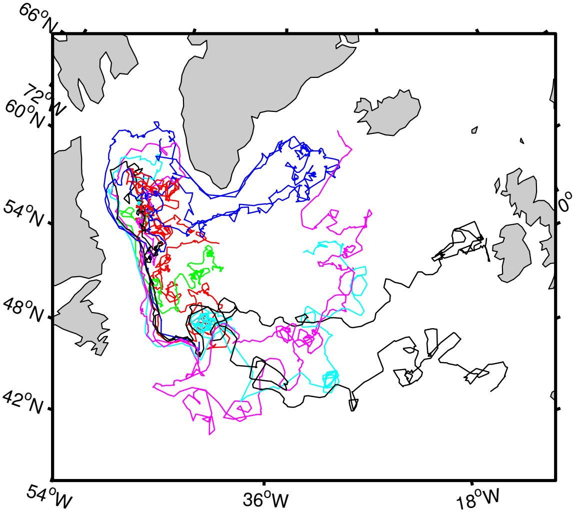


# Growth of $L$ (2)




$m$ is the number of times the braid acted on the initial loop.

# Oceanic float trajectories



# Oceanic floats: Data analysis

What can we measure?

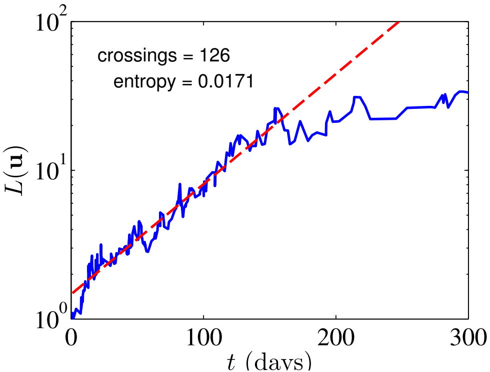
- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the $\sigma_i$ for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent).

# Oceanic floats: Entropy

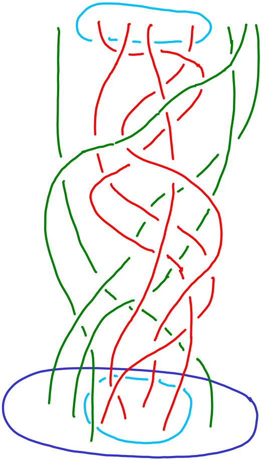
10 floats from Davis' Labrador sea data:




Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

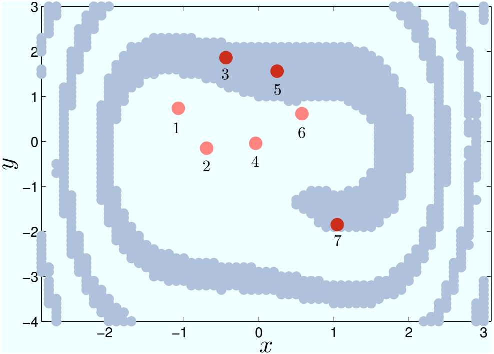
Source: WOCE subsurface float data assembly center (2004)

# Lagrangian Coherent Structures



- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- For now: regions are not ‘leaky.’

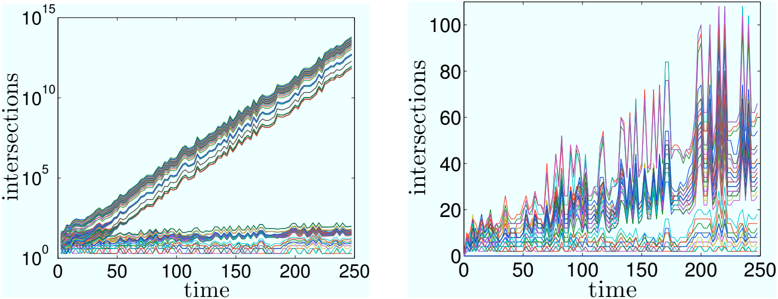
# Sample system: Modified Duffing oscillator




$$\dot{x} = y + \alpha \cos \omega t,$$
$$\dot{y} = x(1 - x^2) + \gamma \cos \omega t - \delta y\,,$$

+ rotation to further hide two regions. $\alpha = .1$, $\gamma = .14$, $\delta = .08$, $\omega = 1$.

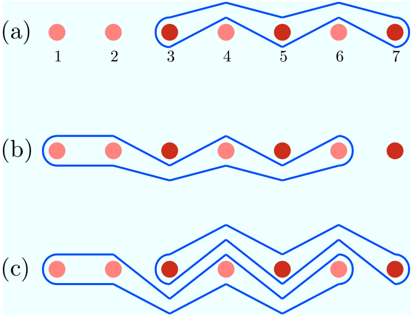
# Growth of a vast number of loops

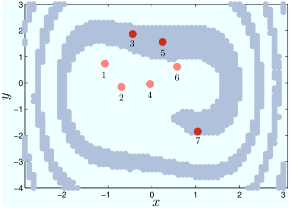



Left: semilog plot; Right: linear plot of slow-growing loops.

Clearly two types of loops!

# What do the slowest-growing loops look like?




[(c) appears because the coordinates also encode ‘multiloops.’]

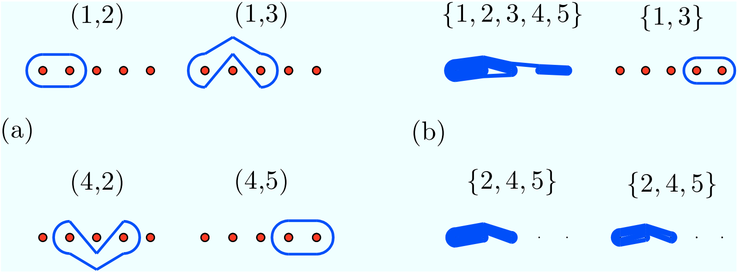
# Computational complexity

Here's the bad news:

- There are an infinite number of loops to consider.
- But we don't really expect hyper-convoluted initial loops (nor do we care so much about those).
- Even if we limit ourselves to loops with Dynnikov coordinates between $-1$ and 1, this is still $3^{2n-4}$ loops.
- This is too many... can only treat about 10–11 trajectories using this direct method.

# An improved method: Pair-loops

The biggest problem is that we only look at whether a loop grows or not. But there is a lot more information to be found in how a loop entangles the punctures as it evolves.




Consider loops that enclose two punctures at once. More involved analysis, but scales *much* better with $n$.

# Improvement

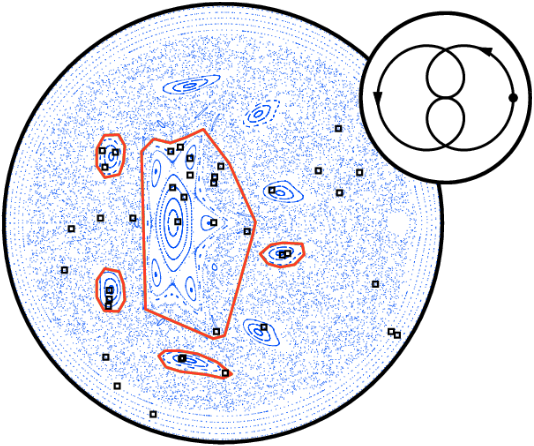
Run times in seconds:

| # of trajectories | 6 | 7 | 8 | 9 | 10 | 11 | 20 |
|---|---|---|---|---|---|---|---|
| direct method | 0.46 | 0.70 | 6.0 | 53 | 462 | 3445 | N/A |
| pair-loop method | 9.5 | 11.6 | 12.3 | 13 | 15 | 20 | 128 |

Bottleneck for the pair-loop method is finding the non-growing loops. (Should scale as $n^2$ for large enough $n$.)

The downside is that the pair-loop method is much more complicated. But in the end it accomplishes the same thing.

# A physical example: Rod stirring device



[movie 4]

# Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (stretching of material lines);
- This idea can also be used on fluid particles to estimate entropy;
- Need a way to compute entropy fast: loop coordinates;
- There is a lot more information in this braid: extract it! (Lagrangian coherent structures);
- Is this useful? We need a good physical problem to try it on!
- See Thiffeault (2005, 2010) and preprint by Allshouse & Thiffeault (arXiv:1106.2231).

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# References

Bestvina, M. & Handel, M. 1995 Train-Tracks for Surface Homeomorphisms. *Topology* **34**, 109–140.

Binder, B. J. & Cox, S. M. 2008 A Mixer Design for the Pigtail Braid. *Fluid Dyn. Res.* **49**, 34–44.

Bowen, R. 1978 Entropy and the fundamental group. In *Structure of Attractors*, volume 668 of *Lecture Notes in Math.*, pp. 21–29. New York: Springer.

Boyland, P. L. 1994 Topological methods in surface dynamics. *Topology Appl.* **58**, 223–298.

Boyland, P. L., Aref, H. & Stremler, M. A. 2000 Topological fluid mechanics of stirring. *J. Fluid Mech.* **403**, 277–304.

Boyland, P. L., Stremler, M. A. & Aref, H. 2003 Topological fluid mechanics of point vortex motions. *Physica D* **175**, 69–95.

Dynnikov, I. A. 2002 On a Yang–Baxter map and the Dehornoy ordering. *Russian Math. Surveys* **57**, 592–594.

Gouillart, E., Finn, M. D. & Thiffeault, J.-L. 2006 Topological Mixing with Ghost Rods. *Phys. Rev. E* **73**, 036311.

Hall, T. & Yurttaş, S. Ö. 2009 On the Topological Entropy of Families of Braids. *Topology Appl.* **156**, 1554–1564.

Kolev, B. 1989 Entropie topologique et représentation de Burau. *C. R. Acad. Sci. Sér. I* **309**, 835–838. English translation at arXiv:math.DS/0304105.

Moussafir, J.-O. 2006 On the Entropy of Braids. *Func. Anal. and Other Math.* **1**, 43–54. arXiv:math.DS/0603355.

Thiffeault, J.-L. 2005 Measuring Topological Chaos. *Phys. Rev. Lett.* **94**, 084502.

Thiffeault, J.-L. 2010 Braids of entangled particle trajectories. *Chaos*, **20**, 017516.

Thiffeault, J.-L. & Finn, M. D. 2006 Topology, Braids, and Mixing in Fluids. *Phil. Trans. R. Soc. Lond. A* **364**, 3251–3266.

Thurston, W. P. 1988 On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc.* **19**, 417–431.