

Stirring and Mixing

Topology, Optimization, and those Pesky Walls

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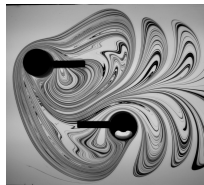
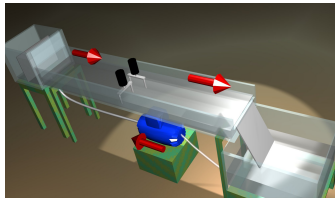
Stéphane Roux

CNRS / ENS Cachan

Stirring and Mixing of Viscous Fluids



- Viscous flows \Rightarrow
no turbulence! (laminar)
- Open and closed systems
- Active (rods) and passive



Understand the **mechanisms** involved.

Characterise and optimise the **efficiency** of mixing.

Stirring and Mixing: What's the Difference?

- **Stirring** is the mechanical motion of the fluid (**cause**);
- **Mixing** is the homogenisation of a substance (**effect, or goal**);
- Two extreme limits: **Turbulent** and **laminar** mixing, both relevant in applications;
- Even if turbulence is feasible, still care about energetic cost;
- For very viscous flows, use simple time-dependent flows to create **chaotic** mixing.
- Here we look at **rod stirring** and the impact of
 - the vessel **walls** on mixing rates;
 - the **topology** of the rod motions.

A Simple Example: Planetary Mixers

In food processing, **rods** are often used for stirring.

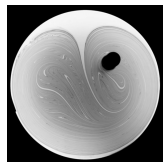
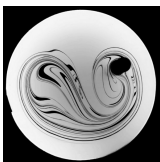


[movie 1] ©BLT Inc.

The Figure-Eight Stirring Protocol



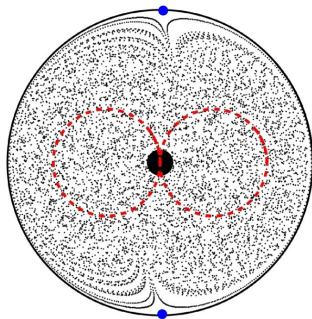
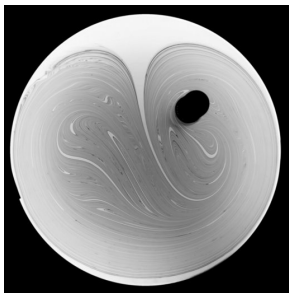
- Circular container of viscous fluid (sugar syrup);
- A rod is moved slowly in a 'figure-eight' pattern;
- Gradients are created by **stretching and folding**, the signature of chaos.



[movie 2] Experiments by E. Guillard and O. Dauchot (CEA Saclay).

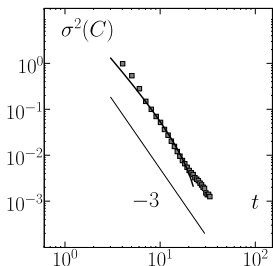
The Mixing Pattern

- Kidney-shaped mixed region extends to wall;
- Two **parabolic points** on the wall, one associated with injection of material;
- Asymptotically self-similar, so expect an **exponential decay** of the concentration ('**strange eigenmode**' regime).
(Pierrehumbert, 1994; Rothstein et al., 1999; Voth et al., 2003)

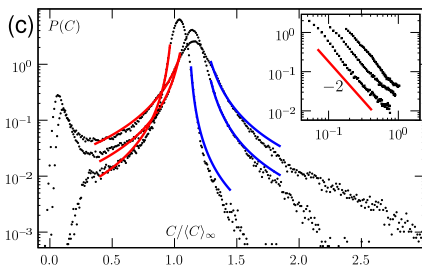


Mixing is Slower Than Expected

Concentration field in a well-mixed central region



$$\text{Variance} = \int |\theta|^2 dV$$

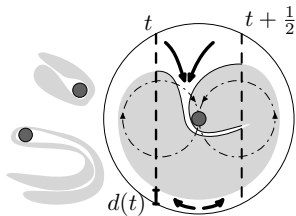
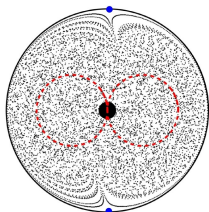


Concentration PDFs

⇒ Algebraic decay of variance \neq Exponential

The 'stretching and folding' action induced by the rod is an exponentially rapid process (**chaos!**), so why aren't we seeing exponential decay?

Walls Slow Down Mixing



- Trajectories are (almost) everywhere chaotic
⇒ but there is always poorly-mixed fluid near the walls;
- Re-inject unmixed (white) material along the unstable manifold of a parabolic point on the wall;
- No-slip at walls ⇒ width of “white stripes” $\sim t^{-2}$ (algebraic);
- Re-injected white strips contaminate the mixing pattern, in spite of the fact that stretching is exponential in the centre.

Hydrodynamics Near the Wall

We can characterize white strips in terms of hydrodynamics near the no-slip wall. x_{\parallel} and x_{\perp} denote respectively the distance along and \perp to the wall. No-slip boundary conditions impose

$$v_{\parallel} \sim x_{\perp}, \quad \text{near the wall: } x_{\perp} \ll 1.$$

Incompressibility

$$\frac{\partial v_{\parallel}}{\partial x_{\parallel}} + \frac{\partial v_{\perp}}{\partial x_{\perp}} = 0,$$

implies

$$v_{\perp} \simeq -a x_{\perp}^2.$$

Solve $\dot{x}_{\perp} = v_{\perp}$:

$$x_{\perp} \simeq \frac{x_0}{1 + at x_0}.$$

Hydrodynamics Near the Wall (continued)

Hence, the distance between the wall and a particle in the lower part of the domain (where $v_{\perp} < 0$) shrinks as

$$d(t) \simeq 1/at, \quad t \gg 1.$$

This scaling was derived in Chertkov & Lebedev (2003), and we verified it experimentally.

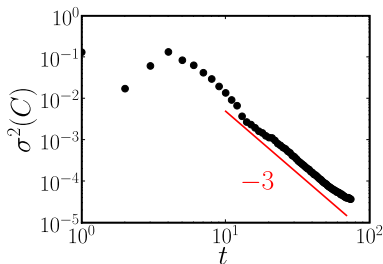
The amount of white that is 'shaved off' at each period is thus

$$d \sim T/at^2, \quad t \gg 1,$$

where T is the period. This is the origin of the power-law decay. Corrections due to the stretch/fold action are described in [Gouillart et al., *Phys. Rev. Lett.* **99**, 114501 (2007)].

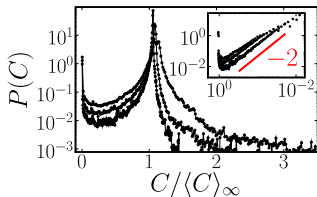
A Generic Scenario

- “Blinking vortex” (Aref, 1984) : numerical simulations



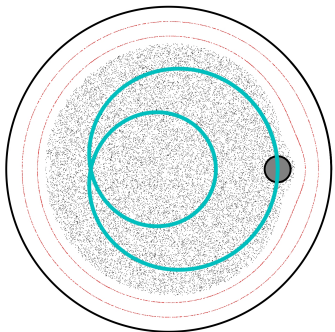
- 1-D Model: Baker's map + parabolic point

Reproduce statistical features of the concentration field;
Some analytical results possible.
(Guillart et al., 2007)

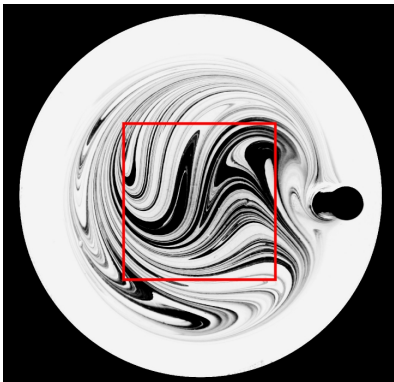


A Second Scenario

How do we mimic a slip boundary condition?

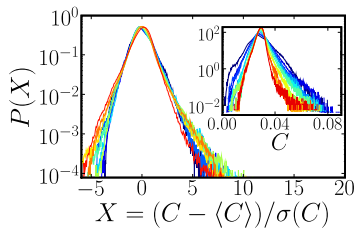
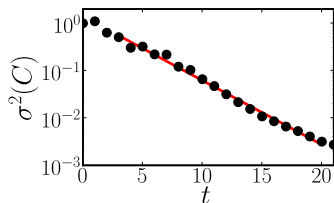
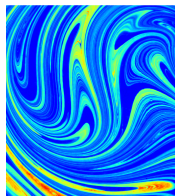
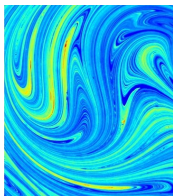
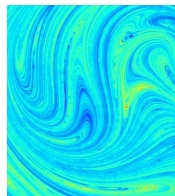


“Epitrochoid” protocol



Central chaotic region + regular region near the walls.

Recover Exponential Decay

 $t = 8$  $t = 12$  $t = 17$ 

... as well as 'true' self-similarity.

Another Approach: Rotate the Bowl!



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The Taffy Puller

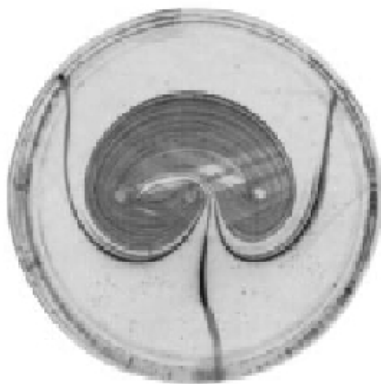
This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the precise nature of the rod motion!

[movie 3]



Experiment of Boyland, Aref, & Stremler



[movie 4] [movie 5]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Channel flow: Injection into mixing region



Injection
against flow



Injection
with flow

- Four-rod stirring device, used in industry;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.

Experiments by E. Guillard and O. Dauchot (CEA Saclay).

[movie 6] [movie 7] [movie 8]

Mathematical description

Focus on **closed systems**.

Periodic stirring protocols in two dimensions can be described by a **homeomorphism** $\varphi : \mathcal{S} \rightarrow \mathcal{S}$, where \mathcal{S} is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- \mathcal{S} is the **disc** with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: **Categorise all possible φ** .

φ and ψ are **isotopic** if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ' , where φ' is in one of the following three categories:

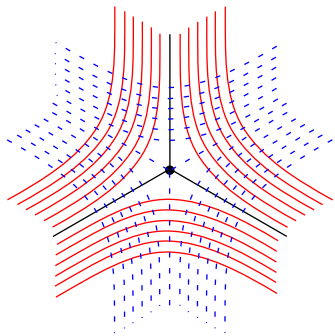
1. **finite-order**: for some integer $k > 0$, $\varphi'^k \simeq$ identity;
2. **reducible**: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. **pseudo-Anosov**: φ' leaves invariant a pair of transverse measured singular foliations, \mathcal{F}^u and \mathcal{F}^s , such that $\varphi'(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u)$ and $\varphi'(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1} \mu^s)$, for **dilatation** $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the **isotopy class** of φ .

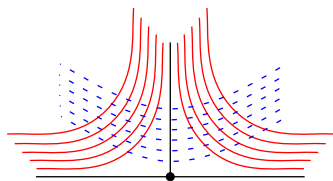
Number 3 is the one we want for good mixing

A singular foliation

The 'pseudo' in pseudo-Anosov refers to the fact that the foliations can have a finite number of **pronged singularities**.

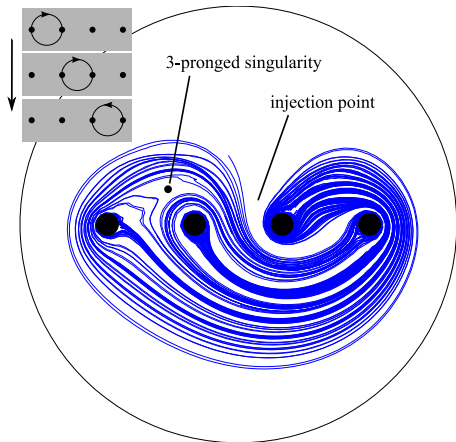


3-pronged singularity



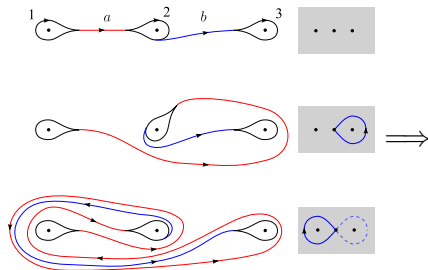
Boundary singularity

Visualising a singular foliation



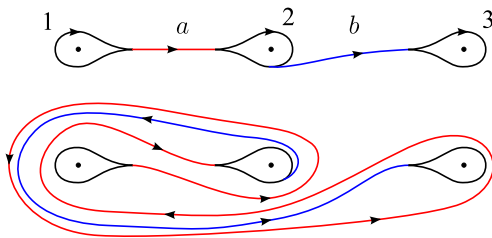
- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a **1-pronged** singularity.
- One **3-pronged** singularity in the bulk.
- One injection point (top): corresponds to **boundary** singularity;

Train tracks



Thurston introduced **train tracks** as a way of characterising the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\bar{2}\bar{a}\bar{1}ab\bar{3}\bar{b}\bar{a}1a, \quad b \mapsto \bar{2}\bar{a}\bar{1}ab$$

Easy to show that this map is **efficient**: under repeated iteration, cancellations of the type $a\bar{a}$ or $b\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

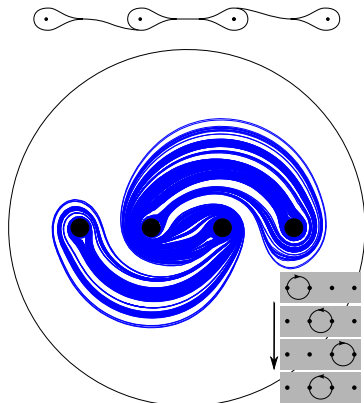
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the **topological entropy**, $\log \lambda$. This is a lower bound on the **minimal length of a material line** caught on the rods.

Find from the TT map by **Abelianising**: count the number of occurrences of a and b , and write as matrix:

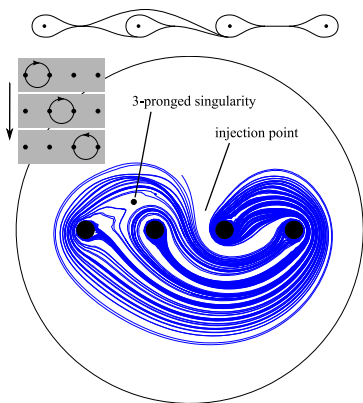
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the 'blob' is multiplied by 2.41 for each full stirring period.

Two types of stirring protocols for 4 rods



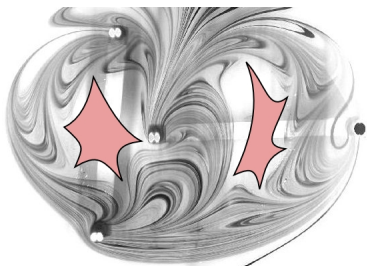
2 injection points



1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols.

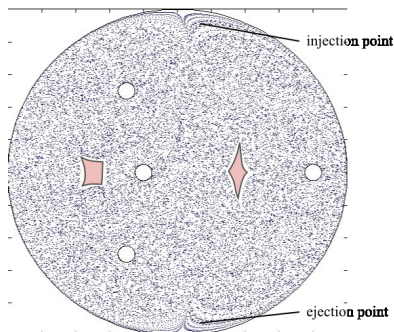
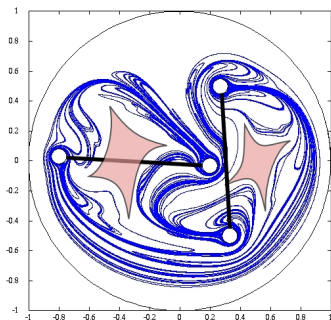
Back to the experiment



- Two 5-pronged singularities clearly visible;
 - Created by the “slicing” of the rods;
 - Only one injection point, at the top.
-
- Each 5-prong rotates unidirectionally;
 - They are never interchanged with each other;
 - Hence, the experimental picture greatly limits the possible pseudo-Anosovs that can occur.

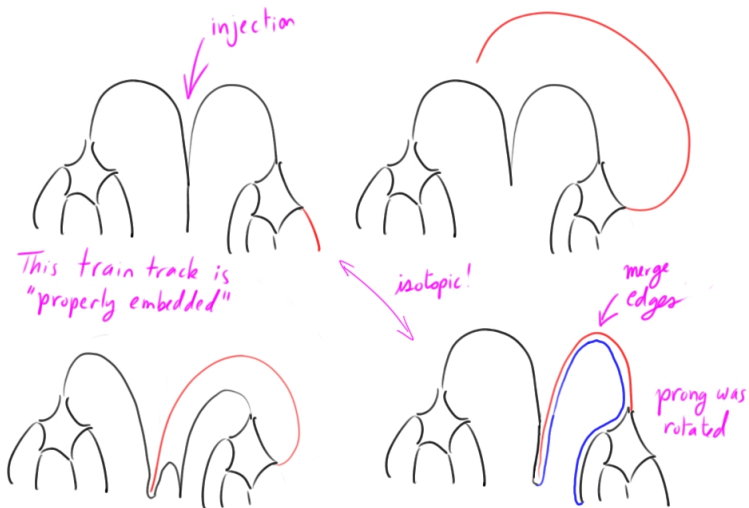
Two 4-pronged singularities

Same protocol, but in a closed container.



Varying the geometry changes the number of prongs: the pronged singularities rotate but lag behind the rods. Smaller rods will increase this lag, and thus the prongness.

A train track folding automaton

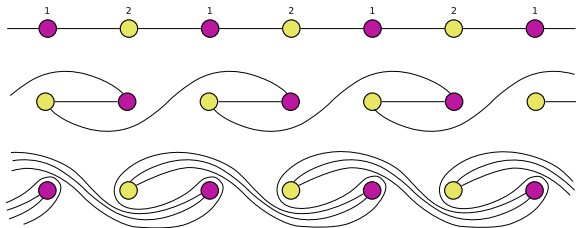


Train track automata (continued)

- Train track automata are a rigorous way of generating **all pseudo-Anosovs** associated with a train track.
- We know the train track type for our 4-rod experiment, just from watching the movie.
- The tiny automaton we built incorporates the constraints.
- Obtain a **train track map** by examining how edges are transformed and merged.
- For two k -prongs, the dilatation λ is the largest root of $x^{2k} - x^{2k-1} - 4x^k - x + 1$.
- **Decreases** with k , which indicates that smaller rods have less effect (shocking!).

Periodic Array of Rods

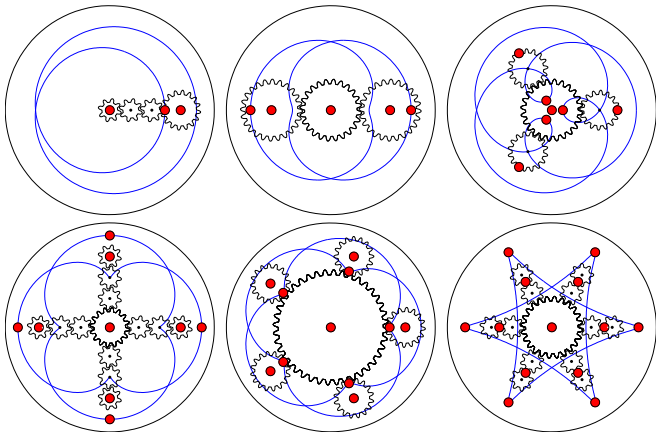
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).



- The entropy per 'switch' is $\log \chi$, where $\chi = 1 + \sqrt{2}$ is the **Silver Ratio!**
- This is **optimal** for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).
- Work with M. D. Finn (Adelaide).

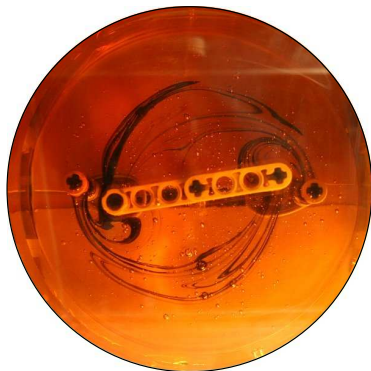
Silver Mixers!

- The designs with entropy given by the silver ratio can be realised with simple gears.
- All the rods move at once: very efficient.



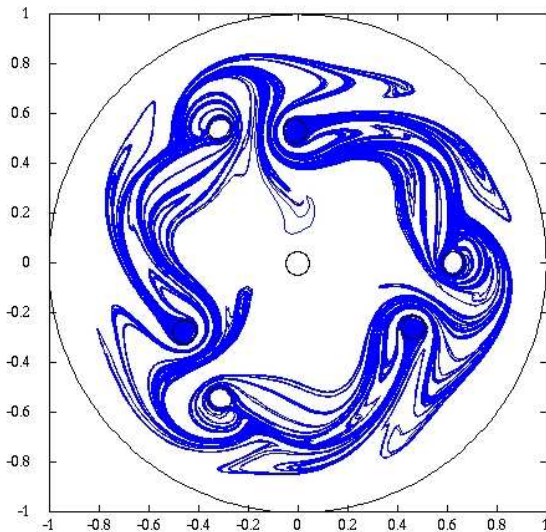
[movie 9]

Four Rods



[movie 10] [movie 11]

Six Rods



[movie 12]

Conclusions

- **Walls** can have a big impact and slow down mixing.
- It is sometimes possible to **shield** the walls from the mixing region, for instance by rotating the vessel.
- Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (**stretching of material lines**).
- Topology also predicts **injection** into the mixing region, important for **open flows**.
- Classify all rod motions and periodic orbits according to their topological properties.
- Train track automata allow exploration of possible pseudo-Anosovs, and can be used for rigorous proofs.
- We have an optimal design, the **silver mixers**.
- Need to also optimise other mixing measures, such as variance decay rate.

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