Distribution of particle displacements in biomixing

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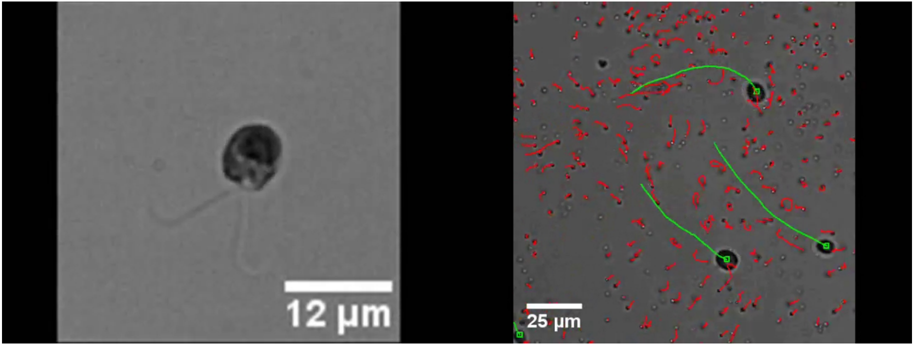
Chlamydomonas reinhardtii

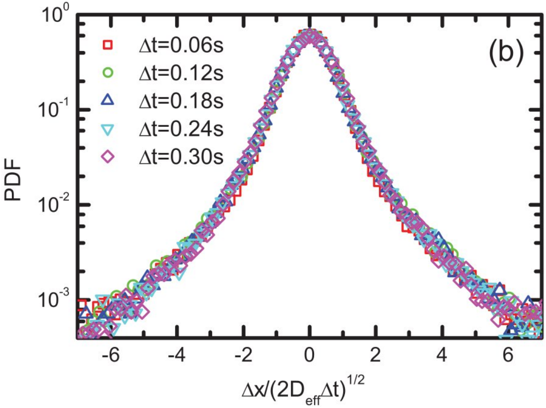


play movie

[Guasto, J. S., Johnson, K. A., & Gollub, J. P. (2010). *Phys. Rev. Lett.* **105**, 168102]

Probability density of displacements

Non-Gaussian PDF with 'exponential' tails:



[Leptos, K. C., Guasto, J. S., Gollub, J. P., Pesci, A. I., & Goldstein, R. E. (2009). *Phys. Rev. Lett.* **103**, 198103]

Probability density of displacements

Leptos *et al.* (2009) claim a reasonable fit of their PDF with the form

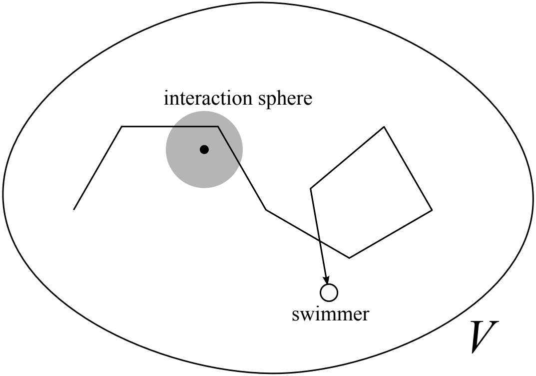
$$P_{\Delta t}(\Delta x) = \frac{1-f}{\sqrt{2\pi}\,\delta_g}\,\mathrm{e}^{-(\Delta x)^2/2\delta_g^2} + \frac{f}{2\delta_e}\,\mathrm{e}^{-|\Delta x|/\delta_e}$$

They observe the scalings $\delta_g \sim A_g(\Delta t)^{1/2}$ and $\delta_e \sim A_e(\Delta t)^{1/2}$, where A_g and A_e depend on ϕ.

They call this a diffusive scaling, since $\Delta x \sim (\Delta t)^{1/2}$. Their point is that this is strange, since the distribution is not Gaussian.

Commonly observed in diffusive processes that are a combination of trapped and hopping dynamics (Wang *et al.*, 2012).

Modeling: the interaction sphere



Model for effective diffusivity:

[Thiffeault, J.-L. & Childress, S. (2010). *Phys. Lett. A*, **374**, 3487–3490]

[Lin, Z., Thiffeault, J.-L., & Childress, S. (2011). *J. Fluid Mech.* **669**, 167–177]

Expected number of ‘dings’ (close interactions) after time t:

$$\langle M_t \rangle = n V_{\text{swept}}(R, L)\,(t/\tau)$$

n is the number density of swimmers, V_{swept} is the volume swept by the sphere of radius R moving a distance L, and τ is the time between turns.

Parameters in the Leptos et al experiment

- Velocity $U \sim 100\,\mu\mathrm{m/s}$;
- Volume fraction is less than 2.2%;
- Organisms of radius $5\,\mu\mathrm{m}$;
- Number density $n \lesssim 4.2 \times 10^{-5}\,\mu\mathrm{m}^{-3}$.
- Maximum observation time in PDFs is $t \sim 0.3\,\mathrm{s}$;
- A typical swimmer moves by a distance $Ut \sim 30\,\mu\mathrm{m}$.

Close encounters in the Leptos et al experiment

Combining this, we find the expected number of 'dings' after time t:

$$\boxed{\langle M_t \rangle \lesssim 1.6}$$

for the longest observation time, and interaction disk $R = 20\,\mu\mathrm{m}$.

Conclude: a typical fluid particle is only strongly affected by about one swimmer during the experiment.

The only displacements that a particle feels 'often' are the very small ones due to all the distant swimmers.

We thus expect the displacement PDF to have a central Gaussian core (since the central limit theorem will apply for the small displacements), but strongly non-Gaussian tails.

Probability of displacements

X_t is the displacement of a particle after a time t.

$$\mathbb{P}\{X_t \in [x, x + \mathrm{d}x]\} = \sum_{m=0}^{\infty} \mathbb{P}\{X_t \in [x, x + \mathrm{d}x]\,,\ M_t = m\}$$
$$= \sum_{m=0}^{\infty} \mathbb{P}\{X_t \in [x, x + \mathrm{d}x]\,|\,M_t = m\}\,\mathbb{P}\{M_t = m\}$$
$$= \sum_{m=0}^{\infty} \mathbb{P}\{X_m \in [x, x + \mathrm{d}x]\}\,\mathbb{P}\{M_t = m\}$$

The number of interactions obeys a Poisson distribution:

$$\mathbb{P}\{M_t = m\} \simeq \frac{1}{m!}\langle M_t\rangle^m\,\mathrm{e}^{-\langle M_t\rangle}$$

Small number of interactions

Define the probability density

$$\rho_{X_m}(x)\,\mathrm{d}x := \mathbb{P}\{X_m \in [x, x+dx]\}.$$

Normally we would now go to the large m limit and use large-deviation theory. But this doesn't hold here. Instead, keep only $m \leq 1$,

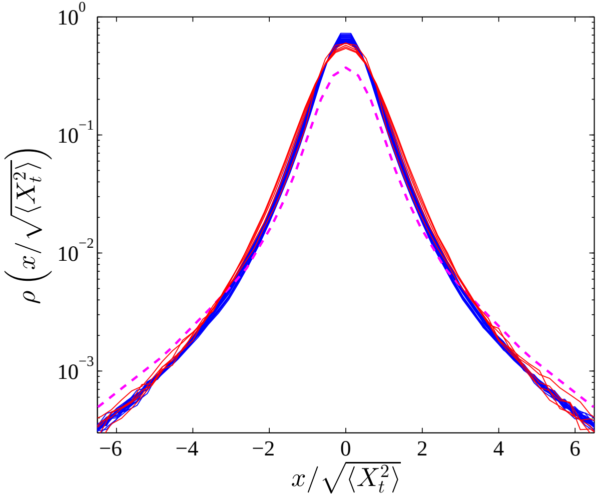
$$\begin{aligned}\rho_{X_t}(x) &= \sum_{m=0}^{\infty} \rho_{X_m}(x)\,\mathbb{P}\{M_t = m\} \\ &\simeq \mathbb{P}\{M_t = 0\}\,\rho_{X_0}(x) + \mathbb{P}\{M_t = 1\}\,\rho_{X_1}(x) + \dots\end{aligned}$$

i.e., most fluid particles feel 0 or 1 encounter with swimmers.

$\rho_{X_0}(x)$ is due to thermal noise (or the combined effect of distant swimmers), so is Gaussian.

$\rho_{X_1}(x)$ is the displacement probability after one close interaction with a swimmer, which has strongly non-Gaussian tails.

The combined PDF for stresslet swimmers



dashed: Leptos *et al.* (2009); blue: model; red: simulations.

(Because of normalization, there are no adjustable parameters.)

Conclusions

- Times in Leptos *et al.* (2009) are so short that the tails are not determined by asymptotic laws, such as the central limit theorem or large-deviation theory.
- Retaining only 0 and 1 close interactions gives a linear combination of a Gaussian and a distribition with non-Gaussian tails, as observed by Leptos *et al.* (2009).
- The Gaussian core arises because of the net effect of the distant swimmers, far from the test particle.
- The small discrepancy with experiments is probably due to noise (smear center a bit).
- Not yet sure if this model recovers the diffusive scaling, but there is a well-defined effective diffusivity.
- See Bruno Eckhardt's talk tomorrow for more!

References I

Blake, J. R. (1971). *J. Fluid Mech.* **46**, 199–208.

Darwin, C. G. (1953). *Proc. Camb. Phil. Soc.* **49** (2), 342–354.

Dombrowski, C., Cisneros, L., Chatkaew, S., Goldstein, R. E., & Kessler, J. O. (2004). *Phys. Rev. Lett.* **93** (9), 098103.

Drescher, K., Leptos, K., Tuval, I., Ishikawa, T., Pedley, T. J., & Goldstein, R. E. (2009). *Phys. Rev. Lett.* **102**, 168101.

Drescher, K. D., Goldstein, R. E., Michel, N., Polin, M., & Tuval, I. (2010). *Phys. Rev. Lett.* **105**, 168101.

Dunkel, J., Putz, V. B., Zaid, I. M., & Yeomans, J. M. (2010). *Soft Matter,* **6**, 4268–4276.

Eames, I., Belcher, S. E., & Hunt, J. C. R. (1994). *J. Fluid Mech.* **275**, 201–223.

Einstein, A. (1956). *Investigations on the Theory of the Brownian Movement*. New York: Dover.

Guasto, J. S., Johnson, K. A., & Gollub, J. P. (2010). *Phys. Rev. Lett.* **105**, 168102.

Hernandez-Ortiz, J. P., Dtolz, C. G., & Graham, M. D. (2005). *Phys. Rev. Lett.* **95**, 204501.

Ishikawa, T. (2009). *J. Roy. Soc. Interface,* **6**, 815–834.

Ishikawa, T. & Pedley, T. J. (2007). *J. Fluid Mech.* **588**, 399–435.

Ishikawa, T., Simmonds, M. P., & Pedley, T. J. (2006). *J. Fluid Mech.* **568**, 119–160.

Katija, K. & Dabiri, J. O. (2009). *Nature,* **460**, 624–627.

References II

Leptos, K. C., Guasto, J. S., Gollub, J. P., Pesci, A. I., & Goldstein, R. E. (2009). *Phys. Rev. Lett.* **103**, 198103.

Lighthill, M. J. (1952). *Comm. Pure Appl. Math.* **5**, 109–118.

Lin, Z., Thiffeault, J.-L., & Childress, S. (2011). *J. Fluid Mech.* **669**, 167–177.

Maxwell, J. C. (1869). *Proc. London Math. Soc.* **s1-3** (1), 82–87.

Oseen, C. W. (1910). *Ark. Mat. Astr. Fys.* **6** (29), 1–20.

Saintillian, D. & Shelley, M. J. (2007). *Phys. Rev. Lett.* **99**, 058102.

Thiffeault, J.-L. & Childress, S. (2010). *Phys. Lett. A,* **374**, 3487–3490.

Underhill, P. T., Hernandez-Ortiz, J. P., & Graham, M. D. (2008). *Phys. Rev. Lett.* **100**, 248101.

Visser, A. W. (2007). *Science,* **316** (5826), 838–839.

Wang, B., Kuo, J., Bae, S. C., & Granick, S. (2012). *Nature Materials,* **11**, 481–485.

Wu, X.-L. & Libchaber, A. (2000). *Phys. Rev. Lett.* **84**, 3017–3020.