Topological approaches to problems of stirring and mixing

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The Taffy Puller

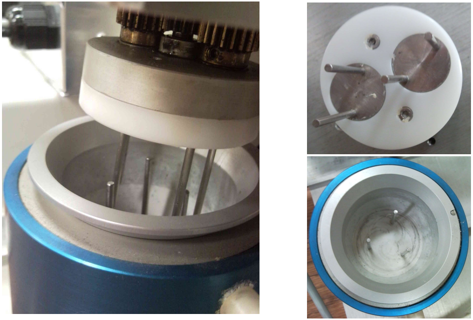
This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the topological nature of the rod motion.

[movie 1]

The mixograph

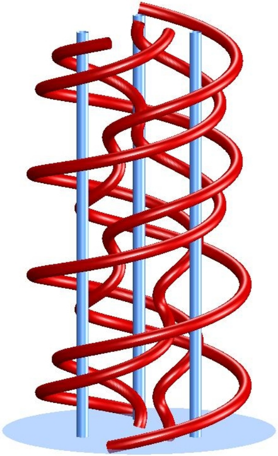
Experimental device for kneading bread dough:



[Department of Food Science, University of Wisconsin. Photos by J-LT.]

The mixograph as a braid

Encode the topological information as a sequence of generators of the Artin braid group B_n.



Equivalent to the 7-braid $\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$

Experiment of Boyland, Aref & Stremler

[movie 2] [movie 3]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]
(Simulations by M. D. Finn.)

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained by solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods.

Goal: Topological characterization of φ.

Three main ingredients

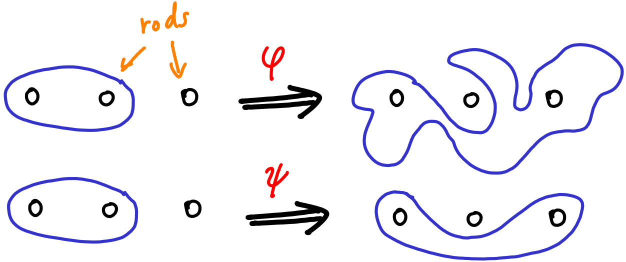
1. The Thurston–Nielsen classification theorem (idealized φ);
2. Handel's isotopy stability theorem (link to real φ);
3. Topological entropy (quantitative measure of mixing).

Isotopy

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

(Defines isotopy classes.)

Convenient to think of isotopy in terms of material loops. Isotopic maps act the same way on loops (up to continuous deformation).



(Loops will always mean essential loops.)

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

1. finite-order: for some integer $k > 0$, $\psi^k \simeq$ identity;
2. reducible: ψ leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: ψ leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\psi(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\,\mu^{\mathrm{u}})$ and $\psi(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1}\mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterize the isotopy class of φ.

TN classification theorem (cartoon)

φ is isotopic to a homeomorphism ψ, where ψ is in one of the following three categories:

1. finite-order (i.e., periodic);
2. reducible (can decompose into different bits);
3. pseudo-Anosov: ψ stretches all loops at an exponential rate $\log \lambda$, called the topological entropy. Any loop eventually traces out the unstable foliation.

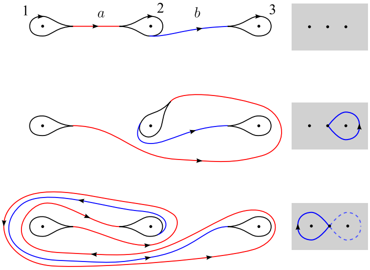
Number 3 is the one we want for good mixing

The Topological program

- Consider a motion of stirring elements, such as rods.
- Determine if the motion is isotopic to a pseudo-Anosov mapping.
- Compute topological quantities, such as foliation, entropy, etc.
- Analyze and optimize.

Train tracks: Computing entropy and foliations

'Figure-8' motion: $\sigma_2^{-2}\sigma_1^2$

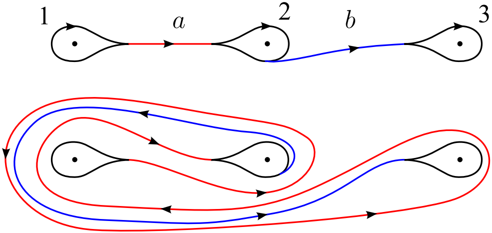


$\Longrightarrow$

exp. by E. Gouillart and O. Dauchot

Thurston introduced train tracks as a way of characterizing the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

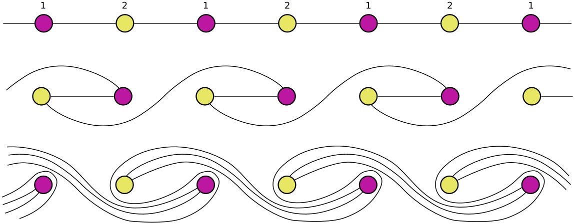
Find from the TT map by Abelianizing: count the number of occurences of a and b, and write as matrix:

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = (1 + \sqrt{2})^2 \simeq 5.83$. Hence, asymptotically, the length of the 'blob' is multiplied by 5.83 for each full stirring period.

Optimization

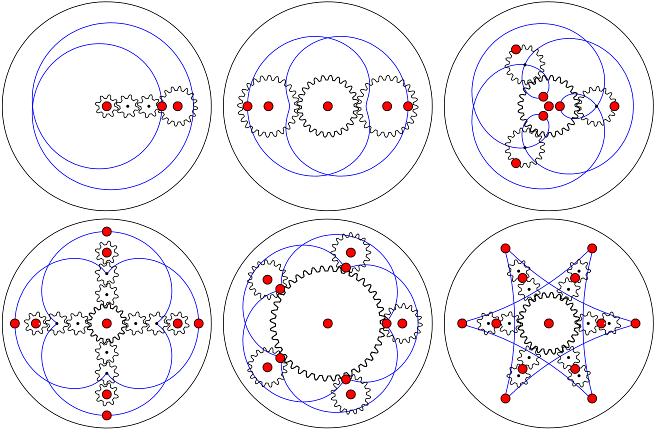
- Consider periodic lattice of rods.
- Move all the rods such that they execute the Boyland et al. (2000) rod motion (Thiffeault & Finn, 2006; Finn & Thiffeault, 2011).



- The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).

Silver Mixers

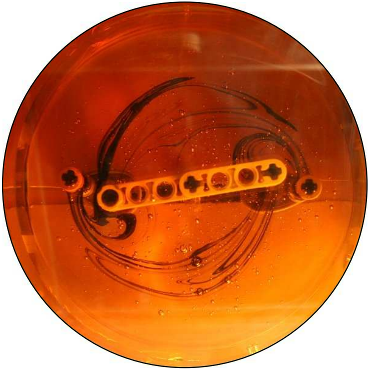
- The designs with dilatation given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.



[movie 4]

Silver Mixers: Building one out of Legos

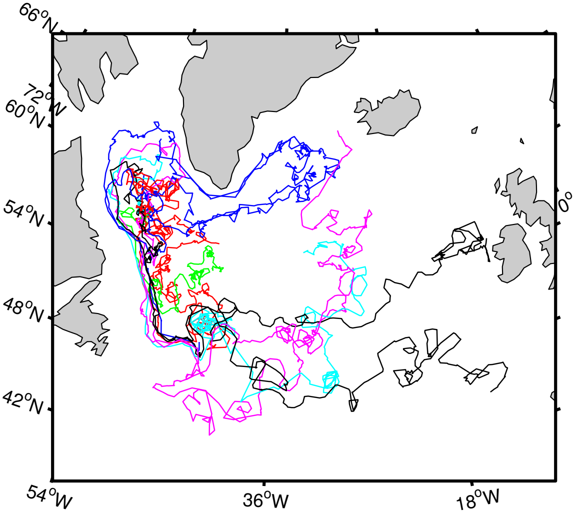
Four Rods



[movie 5] [movie 6]

See [M. D. Finn and J.-L. Thiffeault, *SIAM Review* **53**, 723–743 (2011)] for proofs, heavily influenced by [Boyland & Harrington (2011)]'s work on π_1-stirrers.

Oceanic float trajectories



Oceanic floats: Data analysis

What can we measure?

- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the braid group generators σ_i for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent).

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

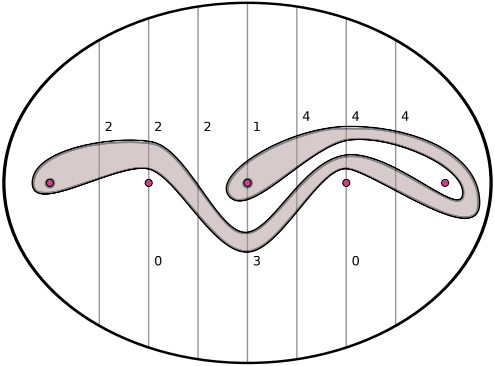
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

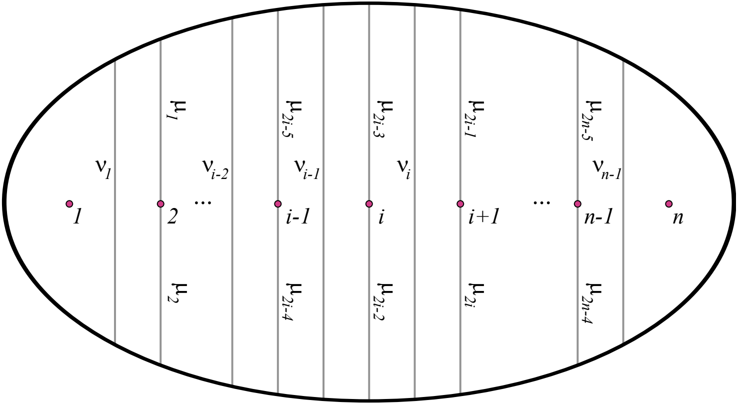
Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



Crossing numbers

Label the crossing numbers:



Dynnikov coordinates

Now take the difference of crossing numbers:

$$a_i = \tfrac{1}{2}\left(\mu_{2i} - \mu_{2i-1}\right),$$
$$b_i = \tfrac{1}{2}\left(\nu_i - \nu_{i+1}\right)$$

for $i = 1, \ldots, n-2$.

The vector of length $(2n-4)$,

$$\mathbf{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2})$$

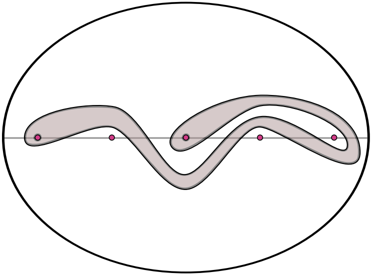
is called the Dynnikov coordinates of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can't do it with fewer than $2n-4$ numbers.

Intersection number

A useful formula gives the minimum intersection number with the 'horizontal axis':

$$L(\mathbf{u}) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i| \,,$$

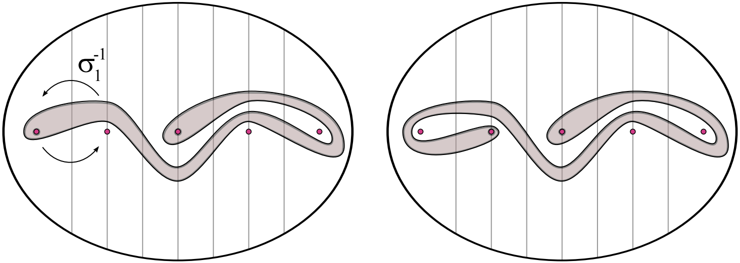


For example, the loop on the left has $L = 12$.

The crossing number grows proportionally to the the length.

Solution to problem 2: Action on coordinates

Moving the punctures according to a braid generator changes some crossing numbers:



There is an explicit formula for the change in the coordinates!

Action on loop coordinates

The update rules for σ_i acting on a loop with coordinates $(\mathbf{a}, \mathbf{b})$ can be written

$$\begin{aligned} a'_{i-1} &= a_{i-1} - b^+_{i-1} - \left(b^+_i + c_{i-1}\right)^+ , \\ b'_{i-1} &= b_i + c^-_{i-1} , \\ a'_i &= a_i - b^-_i - \left(b^-_{i-1} - c_{i-1}\right)^- , \\ b'_i &= b_{i-1} - c^-_{i-1} , \end{aligned}$$

where

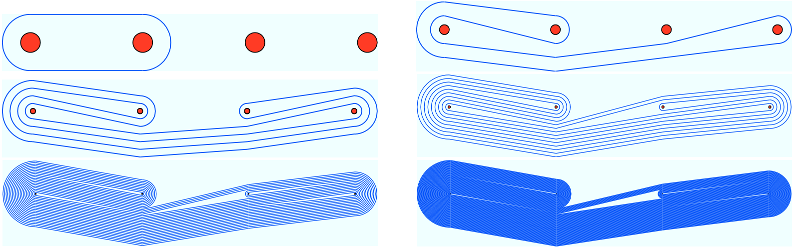
$$f^+ := \max(f, 0), \qquad f^- := \min(f, 0).$$

$$c_{i-1} := a_{i-1} - a_i - b^+_i + b^-_{i-1} .$$

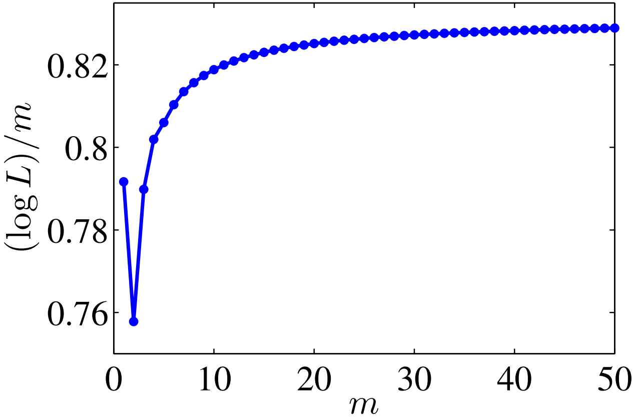
This is called a piecewise-linear action.
Easy to code up (see for example Thiffeault (2010)).

Growth of L

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:



Growth of L (2)

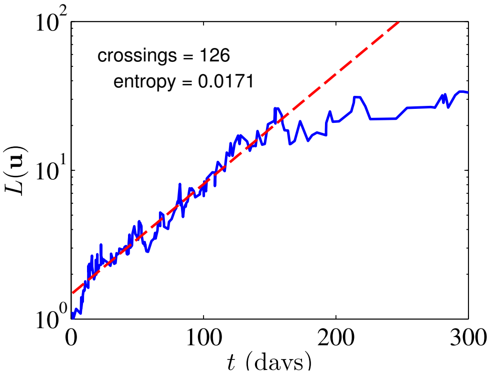


m is the number of times the braid acted on the initial loop.

[Moussafir (2006)]

Oceanic floats: Entropy

10 floats from Davis' Labrador sea data:



Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)

Some research directions

- The nature of the isotopy between the pA and real system.
- Which orbits dominate? (They live in folds — see for instance Cerbelli & Giona (2006); Thiffeault et al. (2009))
- Sharpness of the entropy bound (with Sarah Tumasz: arXiv.org/abs/1204.6730).
- Computational methods for isotopy class (random entanglements of trajectories – LCS method, see Allshouse & Thiffeault (2012) and Michael Allshouse's poster).
- 'Designing' for topological chaos (see Stremler & Chen (2007)).
- Combine with other measures, e.g., mix-norms (Mathew et al., 2005; Lin et al., 2011; Thiffeault, 2012).
- 3D?! (lots of missing theory)

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