Stirring and Mixing

Topology, Optimization, and those Pesky Walls

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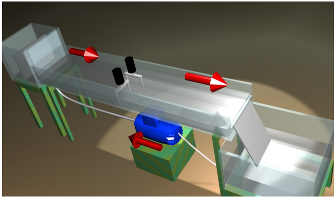
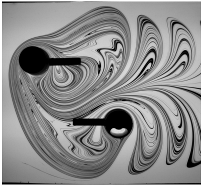
Stirring and Mixing of Viscous Fluids

- Viscous flows ⇒ no turbulence! (laminar)
- Open and closed systems
- Active (rods) and passive





Understand the mechanisms involved.
Characterize and optimize the efficiency of mixing.

Stirring and Mixing: What's the Difference?

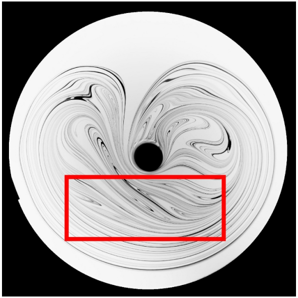
- Stirring is the mechanical motion of the fluid (cause);
- Mixing is the homogenisation of a substance (effect, or goal);
- Two extreme limits: Turbulent and laminar mixing, both relevant in applications;
- Even if turbulence is feasible, still care about energetic cost;
- For very viscous flows, use simple time-dependent flows to create chaotic mixing.
- Here we look at rod stirring and the impact of
  - the vessel walls on mixing rates;
  - the topology of the rod motions.

A Simple Example: Planetary Mixers

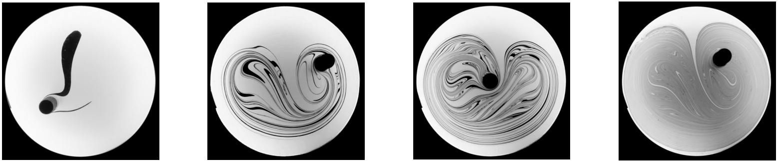
In food processing, rods are often used for stirring.

[movie 1] ©BLT Inc.

The Figure-Eight Stirring Protocol



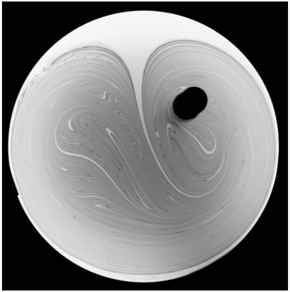
- Circular container of viscous fluid (sugar syrup);
- A rod is moved slowly in a ‘figure-eight’ pattern;
- Gradients are created by stretching and folding, the signature of chaos.

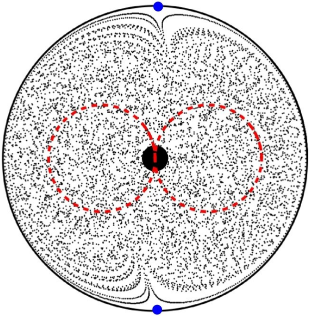


[movie 2] Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

The Mixing Pattern

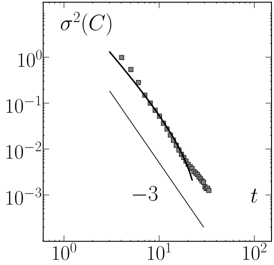
- Kidney-shaped mixed region extends to wall;
- Two parabolic points on the wall, one associated with injection of material;
- Asymptotically self-similar, so expect an exponential decay of the concentration ('strange eigenmode' regime).
  (Pierrehumbert, 1994; Rothstein et al., 1999; Voth et al., 2003)



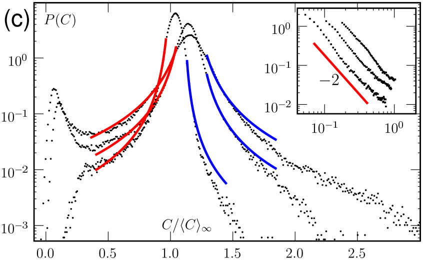


Mixing is Slower Than Expected

Concentration field in a well-mixed central region



Variance $= \int |\theta|^2 dV$

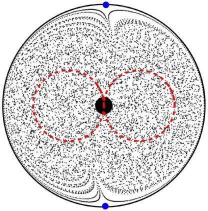


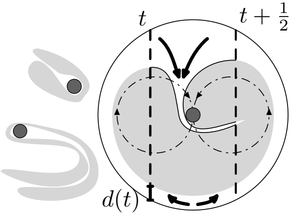
Concentration PDFs

$\Rightarrow$ Algebraic decay of variance $\neq$ Exponential

The 'stretching and folding' action induced by the rod is an exponentially rapid process (chaos!), so why aren't we seeing exponential decay?

Walls Slow Down Mixing

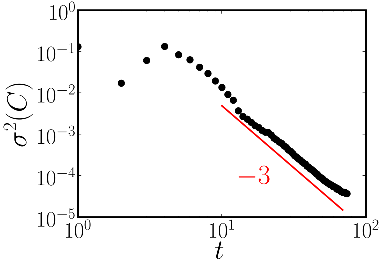




- Trajectories are (almost) everywhere chaotic
  $\Rightarrow$ but there is always poorly-mixed fluid near the walls;
- Re-inject unmixed (white) material along the unstable manifold of a parabolic point on the wall;
- No-slip at walls $\Rightarrow$ width of "white stripes" $\sim t^{-2}$ (algebraic (Chertkov & Lebedev, 2003; Salman & Haynes, 2007));
- Re-injected white strips contaminate the mixing pattern, in spite of the fact that stretching is exponential in the centre.

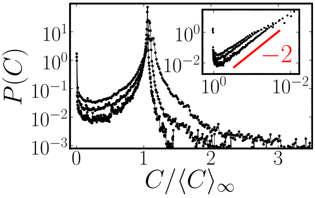
A Generic Scenario

- "Blinking vortex" (Aref, 1984) : numerical simulations



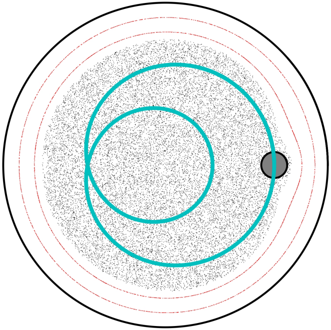
- 1-D Model: Baker's map + parabolic point

Reproduce statistical features of the concentration field;
Some analytical results possible.
(Gouillart et al., 2007, 2008)

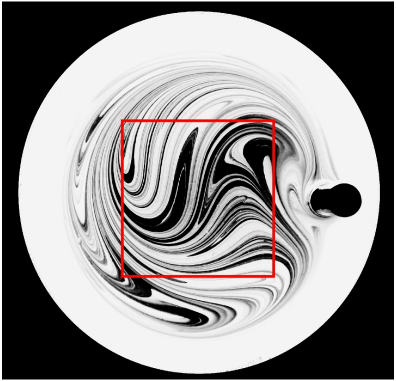


A Second Scenario

How do we mimic a slip boundary condition?

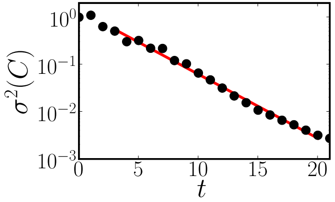


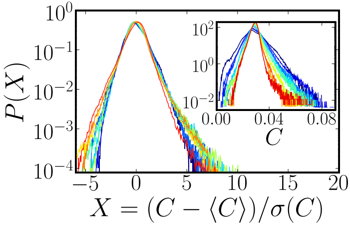
"Epitrochoid" protocol



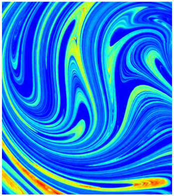
Central chaotic region + regular region near the walls.

Recover Exponential Decay

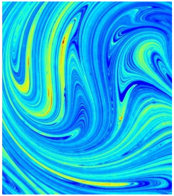




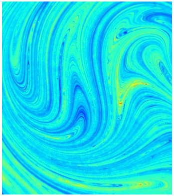
$t = 8$



$t = 12$



$t = 17$



. . . as well as 'true' self-similarity.

Another Approach: Rotate the Bowl!

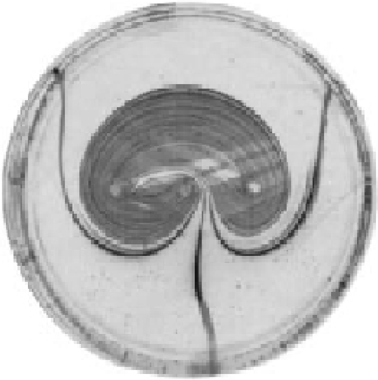
The Taffy Puller

This may not look like it has much to do with stirring, but notice how the taffy is stretched and folded exponentially.

Often the hydrodynamics are less important than the precise nature of the rod motion!

[movie 3]

Experiment of Boyland, Aref, & Stremler

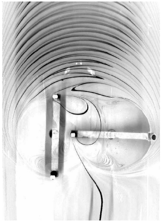


[movie 4] [movie 5]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Channel flow: Injection into mixing region

Injection against flow



Injection with flow

- Four-rod stirring device, used in industry;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimize stirring devices.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 6] [movie 7]

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \rightarrow \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: Categorize all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

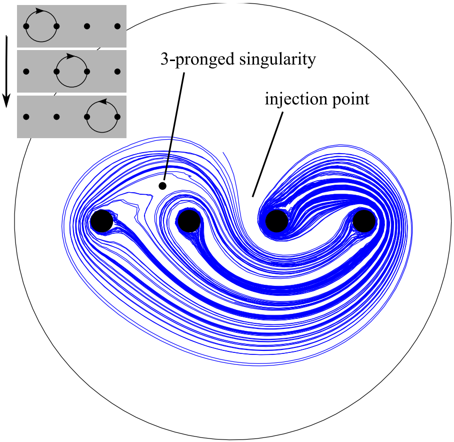
φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\, \mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1} \mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterize the isotopy class of φ.

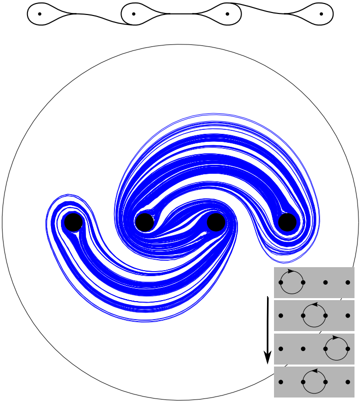
Number 3 is the one we want for good mixing

Visualizing a singular foliation

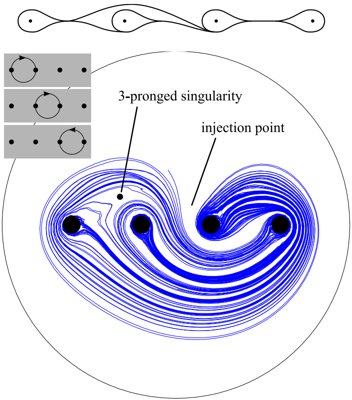


- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

Two types of stirring protocols for 4 rods



2 injection points

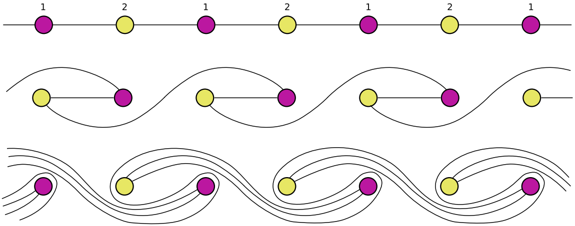


1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify foliations, and thus stirring protocols (Thiffeault et al., 2008).

Optimization

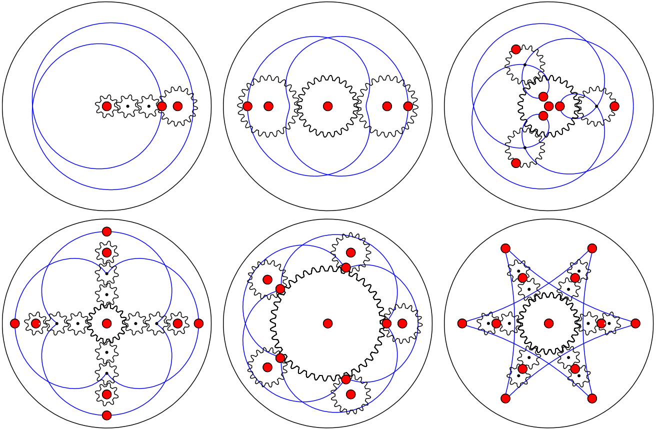
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1\,\sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).



- The dilatation per period is χ^2, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D'Alessandro et al. (1999)).
- Work with M. D. Finn (Adelaide).

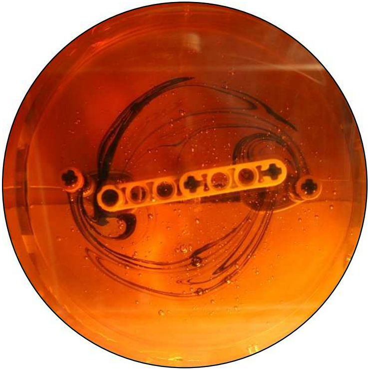
Silver Mixers!

- The designs with dilatation given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.



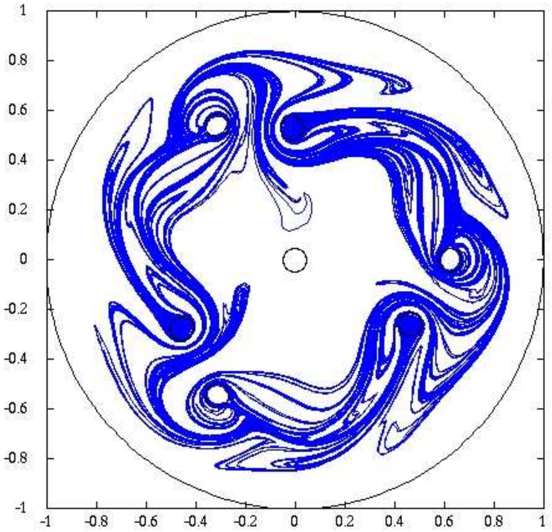
[movie 8]

Four Rods



[movie 9] [movie 10]

Six Rods



[movie 11]

Conclusions

- Walls can have a big impact and slow down mixing.
- It is sometimes possible to shield the walls from the mixing region, for instance by rotating the vessel.
- Having rods undergo 'braiding' motion guarantees a minimal amound of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions and periodic orbits according to their topological properties.
- We have an optimal design, the silver mixers.
- Need to also optimize other mixing measures, such as variance decay rate.

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