Topology of Surfaces and the Stirring of Fluids

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

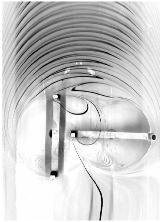
University of Michigan, 26 September 2008

Collaborators:

Matthew Finn	University of Adelaide
Emmanuelle Gouillart	CNRS / Saint-Gobain Recherche
Erwan Lanneau	CPT Marseille
Toby Hall	University of Liverpool

Channel flow: Injection into mixing region

Injection against flow



Injection with flow

- Four-rod stirring device, used in industry;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:

- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 1] [movie 2] [movie 3]

Mathematical description

Focus on closed systems.

Periodic stirring protocols in two dimensions can be described by a homeomorphism $\varphi : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- $\mathcal{S}$ is the disc with holes in it, corresponding to the stirring rods and distinguished periodic orbits.

Task: Categorise all possible φ.

φ and ψ are isotopic if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

Thurston–Nielsen classification theorem

φ is isotopic to a homeomorphism φ', where φ' is in one of the following three categories:

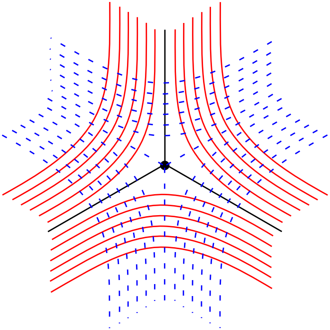
1. finite-order: for some integer $k > 0$, ${\varphi'}^k \simeq$ identity;
2. reducible: φ' leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. pseudo-Anosov: φ' leaves invariant a pair of transverse measured singular foliations, $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$, such that $\varphi'(\mathcal{F}^{\mathrm{u}}, \mu^{\mathrm{u}}) = (\mathcal{F}^{\mathrm{u}}, \lambda\,\mu^{\mathrm{u}})$ and $\varphi'(\mathcal{F}^{\mathrm{s}}, \mu^{\mathrm{s}}) = (\mathcal{F}^{\mathrm{s}}, \lambda^{-1}\mu^{\mathrm{s}})$, for dilatation $\lambda \in \mathbb{R}_+$, $\lambda > 1$.

The three categories characterise the isotopy class of φ.

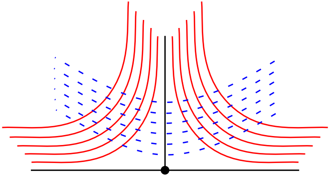
Number 3 is the one we want for good mixing

A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of pronged singularities.

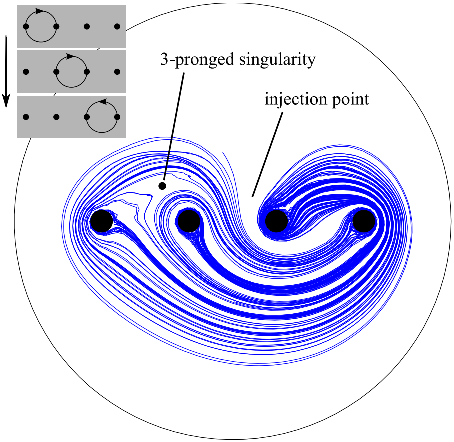


3-pronged singularity



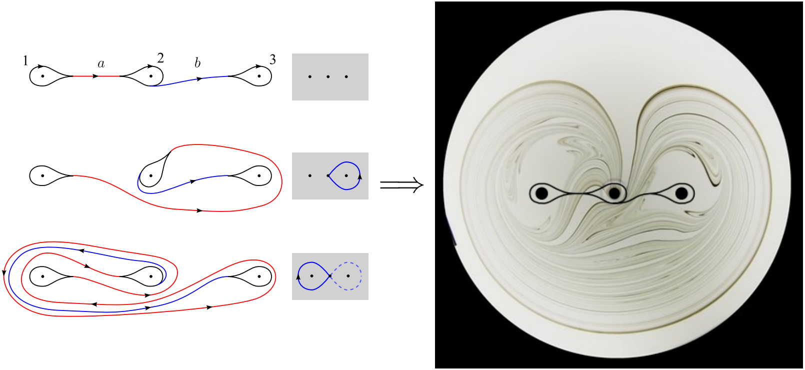
Boundary singularity

Visualising a singular foliation



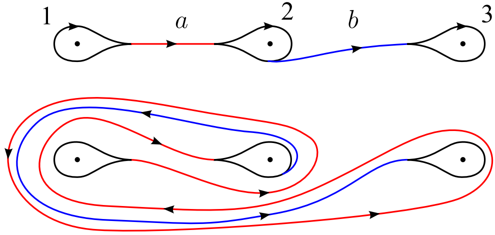
- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- Each rod has a 1-pronged singularity.
- One 3-pronged singularity in the bulk.
- One injection point (top): corresponds to boundary singularity;

Train tracks



Thurston introduced train tracks as a way of characterising the measured foliation. The name stems from the 'cusps' that look like train switches.

Train track map for figure-eight



$$a \mapsto a\,\bar{2}\,\bar{a}\,\bar{1}\,a\,b\,\bar{3}\,\bar{b}\,\bar{a}\,1\,a\,, \qquad b \mapsto \bar{2}\,\bar{a}\,\bar{1}\,a\,b$$

Easy to show that this map is efficient: under repeated iteration, cancellations of the type $a\,\bar{a}$ or $b\,\bar{b}$ never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++.)

Topological Entropy

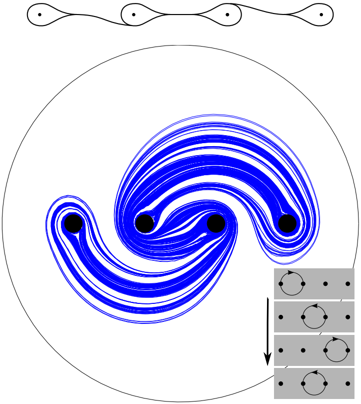
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, $\log \lambda$. This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurences of *a* and *b*, and write as matrix:

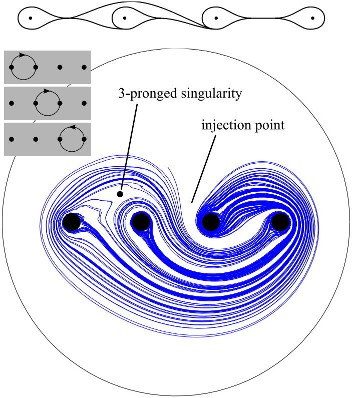
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = 1 + \sqrt{2} \simeq 2.41$. Hence, asymptotically, the length of the 'blob' is multiplied by 2.41 for each full stirring period.

Two types of stirring protocols for 4 rods



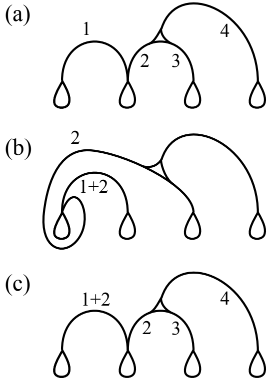
2 injection points



1 injection pt, 1 3-prong sing.

Topological index formulas allow us to classify train tracks, and thus stirring protocols.

Pseudo-Anosovs involve 'folding' the foliation



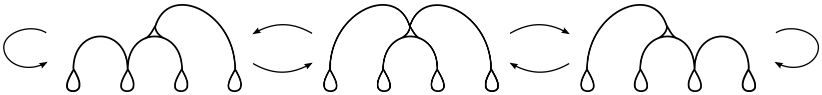
Build pA's 'in reverse,' by regarding them as a sequence of gluings or foldings of pieces of foliation.

Make a transition matrix showing how edges 1–4 are folded:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

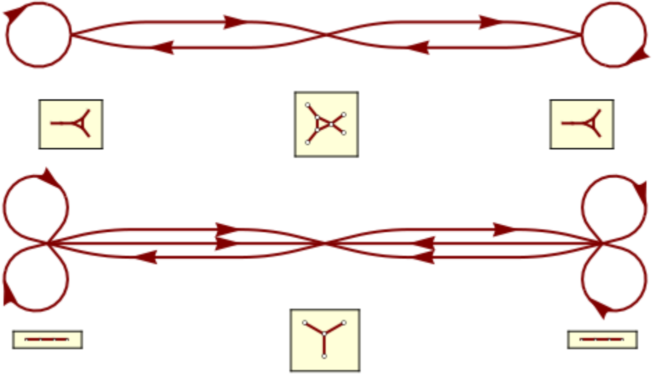
A train track folding automaton

The result is a folding automaton (a graph of train tracks):

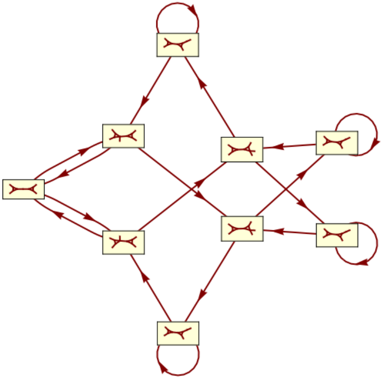


- Each arrow represents a folding of an edge onto another.
- A transition matrix is associated with each arrow.
- pA's are closed paths in this automaton, since they should leave the foliation invariant.
- All pA's are contained therein (up to conjugacy).

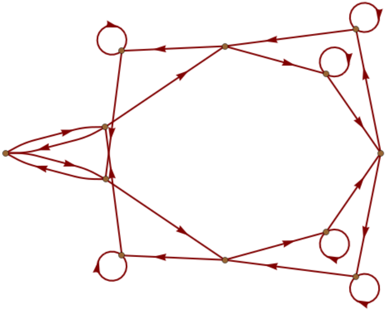
Automata can be simple. . .



Or elegant. . .

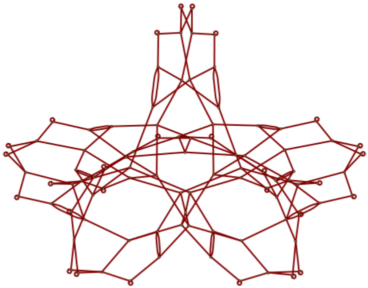


$n = 5, 2 \times$ 3-prong

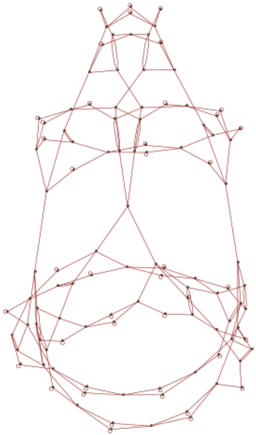


$n = 7, 2 \times$ 4-prong

Or pretty...

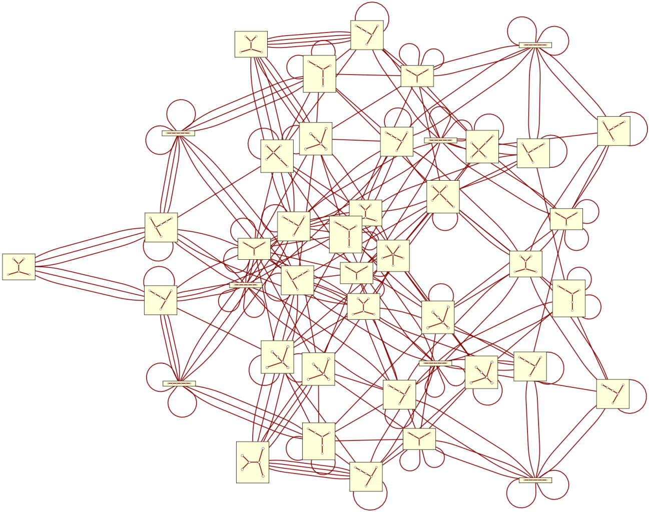


$n = 7, 4 \times 3$-prong
"The maple leaf"



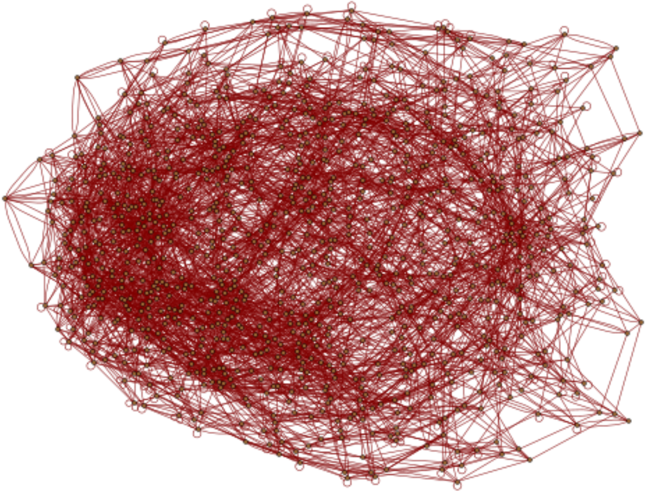
$n = 7, 2\times 3$-prongs, 1×4-prong
"The scarab"

Or rather large...



$n = 6$

Or just ridiculous. . .



$n = 7, 2 \times 3$-prongs (977 train tracks!)

The Minimiser problem

- On a given surface, which pA has the least λ?
- Known for $n = 3, 4, 5$ [Song, Ko & Los (2002); Ham & Song (2006)]
- Method: look at all closed paths until column or row norm exceeded.
- Combinatorics explode: on a computer,
    - $n = 3$: trivial;
    - $n = 4$: milliseconds;
    - $n = 5$: seconds;
    - $n = 7$: about 8 months? (still running!);
    - $n = 6$: decades??
- Minimiser is simple for n odd! New ideas are needed...
- Maximiser (completely diffferent question...)

Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amound of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions and periodic orbits according to their topological properties.
- Train track automata allow exploration of possible pseudo-Anosovs.
- Proof of minimiser for $n = 7$ (and other surfaces) ongoing.
- Maximiser? (Some results — the silver mixer)
- Holy grail: Three dimensions! (though current work applies to many 3D situations. . .)

References

Bestvina, M. & Handel, M. 1992 Train Tracks for ad Automorphisms of Free Groups. *Ann. Math.* **134**, 1–51.

Binder, B. J. & Cox, S. M. 2007 A Mixer Design for the Pigtail Braid. *Fluid Dyn. Res.* In press.

Boyland, P. L., Aref, H. & Stremler, M. A. 2000 Topological fluid mechanics of stirring. *J. Fluid Mech.* **403**, 277–304.

Boyland, P. L., Stremler, M. A. & Aref, H. 2003 Topological fluid mechanics of point vortex motions. *Physica D* **175**, 69–95.

D'Alessandro, D., Dahleh, M. & Mezić, I. 1999 Control of mixing in fluid flow: A maximum entropy approach. *IEEE Transactions on Automatic Control* **44**, 1852–1863.

Gouillart, E., Finn, M. D. & Thiffeault, J.-L. 2006 Topological Mixing with Ghost Rods. *Phys. Rev. E* **73**, 036311. arXiv:nlin/0510075.

Ham, J.-Y. & Song, W. T. 2006 The minimum dilatation of pseudo-Anosov 5-braids. arXiv:math.GT/0506295.

Kobayashi, T. & Umeda, S. 2006 Realizing pseudo-Anosov egg beaters with simple mecanisms. Preprint.

Moussafir, J.-O. 2006 On the Entropy of Braids. In submission, arXiv:math.DS/0603355.

Song, W. T., Ko, K. H., & Los, J. E. 2002 Entropies of braids. *J. Knot Th. Ramifications* **11**, 647–666.

Thiffeault, J.-L. 2005 Measuring topological chaos. *Phys. Rev. Lett.* **94**, 084502. arXiv:nlin/0409041.

Thiffeault, J.-L. & Finn, M. D. 2006 Topology, Braids, and Mixing in Fluids. *Phil. Trans. R. Soc. Lond. A* **364**, 3251–3266. arXiv:nlin/0603003.

Thiffeault, J.-L., Finn, M. D., Gouillart, E., Hall, T. 2008 Topology of Chaotic Mixing Patterns. *Chaos* **18**, 033123. arXiv:0804.2520.

Thurston, W. P. 1988 On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc.* **19**, 417–431.