Chaotic Mixing in a Diffeomorphism of the Torus

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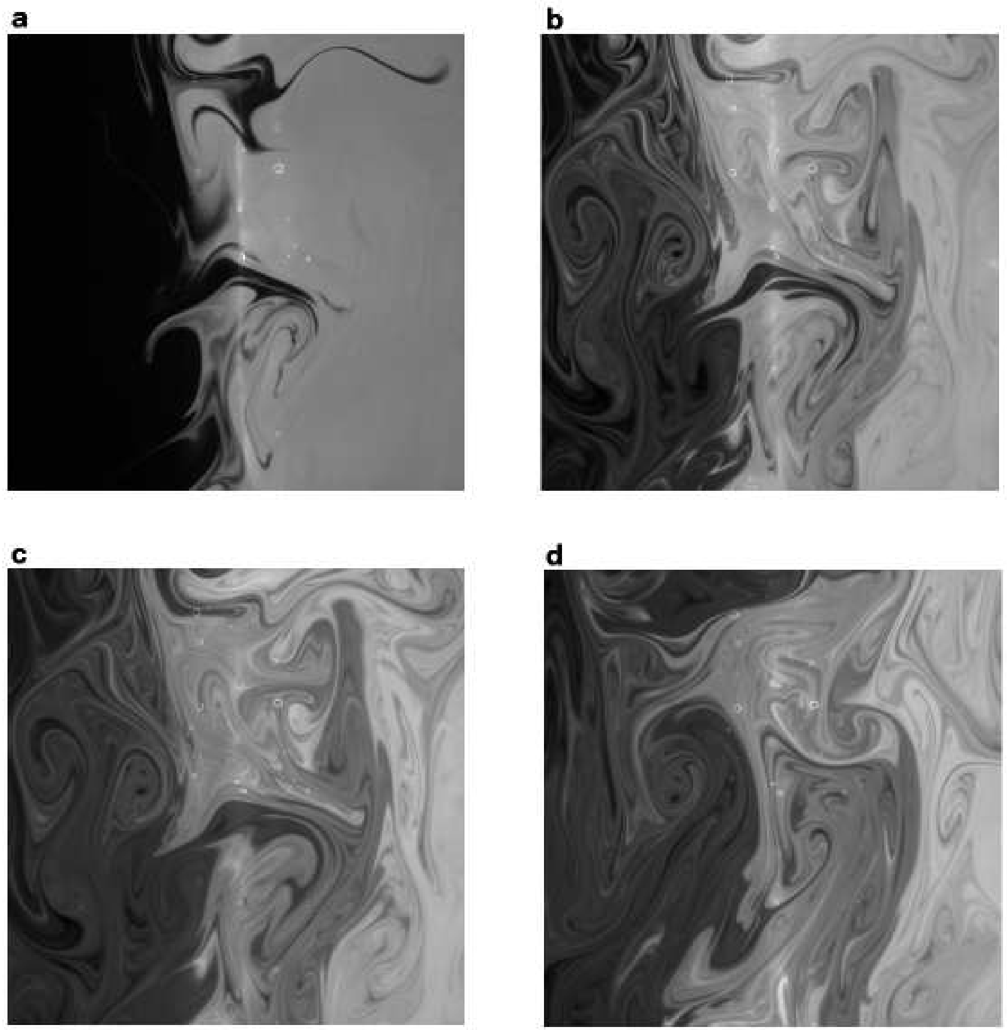
Experiment of Rothstein et al. (1999)

Regular array of magnets

[Rothstein, Henry, and Gollub, Nature **401**, 770 (1999)]

Persistent Pattern

Disordered array ($i = 2, 20, 50, 50.5$)



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Global theory:

- Eigenfunction of advection–diffusion operator.
- Applies if initial scale large.
- [Pierrehumbert, Chaos Sol. Frac. (1994)] Strange eigenmode
  [Fereday et al., PRE (2002)] Baker's map
  [Sukhatme and Pierrehumbert, PRE (2002)] Unified description

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- Eigenfunction of advection–diffusion operator.
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- Today: Focus on Global theory.
- Map allows analytical results.

A Diffeomorphism of the Torus

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0,1]^2$,

$$\mathcal{M}(\boldsymbol{x}) = \mathbb{M} \cdot \boldsymbol{x} + \phi(\boldsymbol{x}),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \phi(\boldsymbol{x}) = \frac{K}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

so that $\mathbb{M} \cdot \boldsymbol{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $K = 0$ the stretching of phase-space elements is uniform in space (homogeneous).

Advection and Diffusion

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution $\theta^{(i-1)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \mathcal{H}_\epsilon\,\theta^{(i-1)}(\mathcal{M}^{-1}(\boldsymbol{x}))$$

where ϵ is the diffusivity, and the heat operator $\mathcal{H}_\epsilon$ and kernel h_ϵ are

$$\mathcal{H}_\epsilon\theta(\boldsymbol{x}) := \int_{\mathbb{T}^2} h_\epsilon(\boldsymbol{x}-\boldsymbol{y})\theta(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y};$$

$$h_\epsilon(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}^2\epsilon).$$

Transfer Matrix

Fourier expand $\theta^{(i)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{\theta}^{(i)}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \,.$$

The effect of advection and diffusion becomes

$$\hat{\theta}^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{q}} \mathbb{T}_{\boldsymbol{k}\boldsymbol{q}}\, \hat{\theta}^{(i-1)}_{\boldsymbol{q}},$$

with the transfer matrix,

$$\mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} := \int_{\mathbb{T}^2} \exp\left(2\pi\mathrm{i}\left(\boldsymbol{q}\cdot\boldsymbol{x} - \boldsymbol{k}\cdot\mathcal{M}(\boldsymbol{x})\right) - \epsilon\,\boldsymbol{q}^2\right)\,\mathrm{d}\boldsymbol{x},$$

$$= \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{0,Q_2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left(\left(k_1+k_2\right)K\right), \qquad \boldsymbol{Q} := \boldsymbol{k}\cdot\mathbb{M} - \boldsymbol{q},$$

where the J_Q are the Bessel functions of the first kind.

Variance: A measure of mixing

In the absence of diffusion ($\epsilon = 0$), the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} \left|\theta^{(i)}(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}}^{(i)}, \qquad \sigma_{\boldsymbol{k}}^{(i)} := \left|\hat{\theta}_{\boldsymbol{k}}^{(i)}\right|^2$$

is preserved. (We assume the spatial mean of θ is zero.)

For $\epsilon > 0$ the variance decays.

We consider the case $\epsilon \ll 1$, of greatest practical interest.

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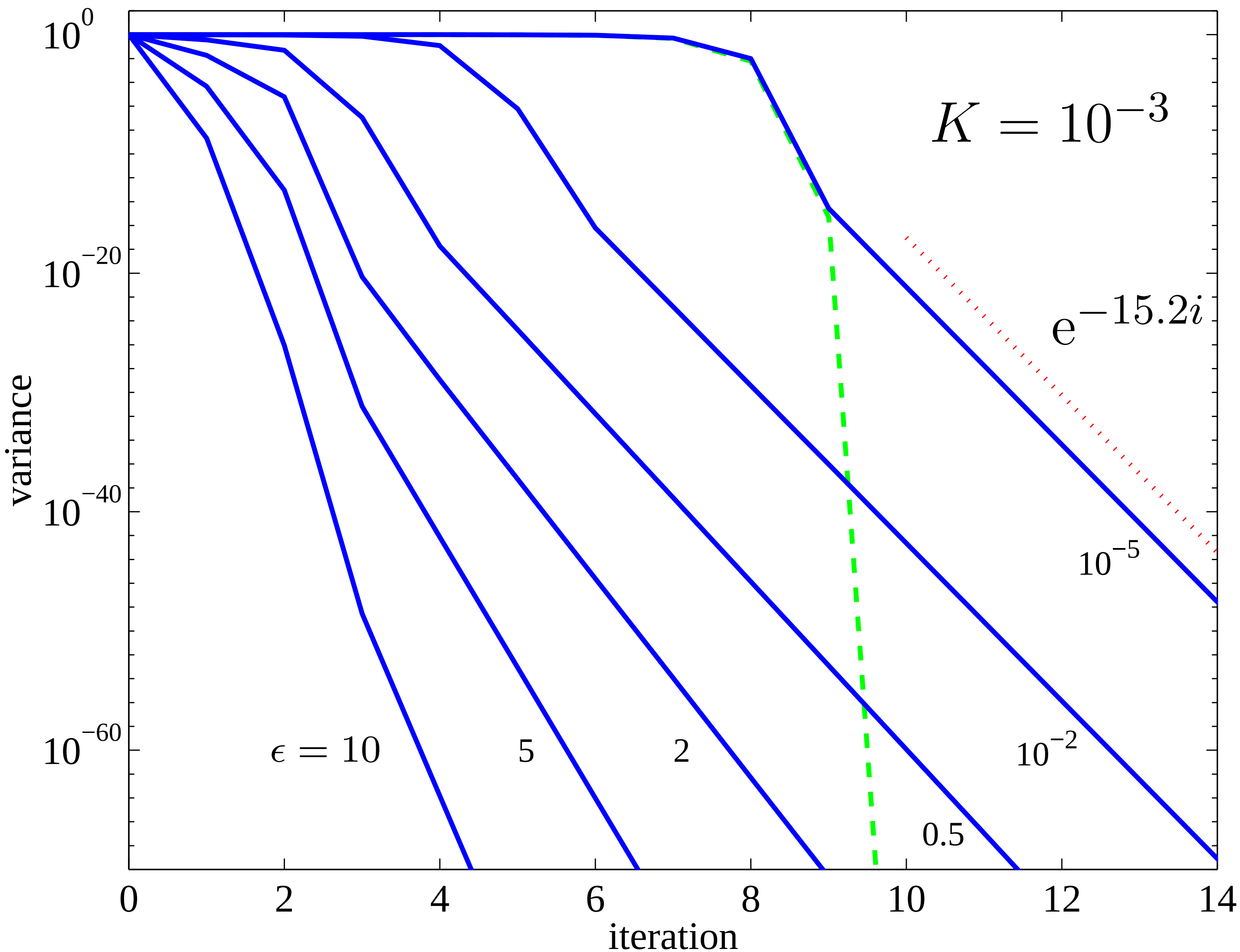
- The variance is initially constant;
- It then undergoes a rapid superexponential decay;
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.

Decay of Variance



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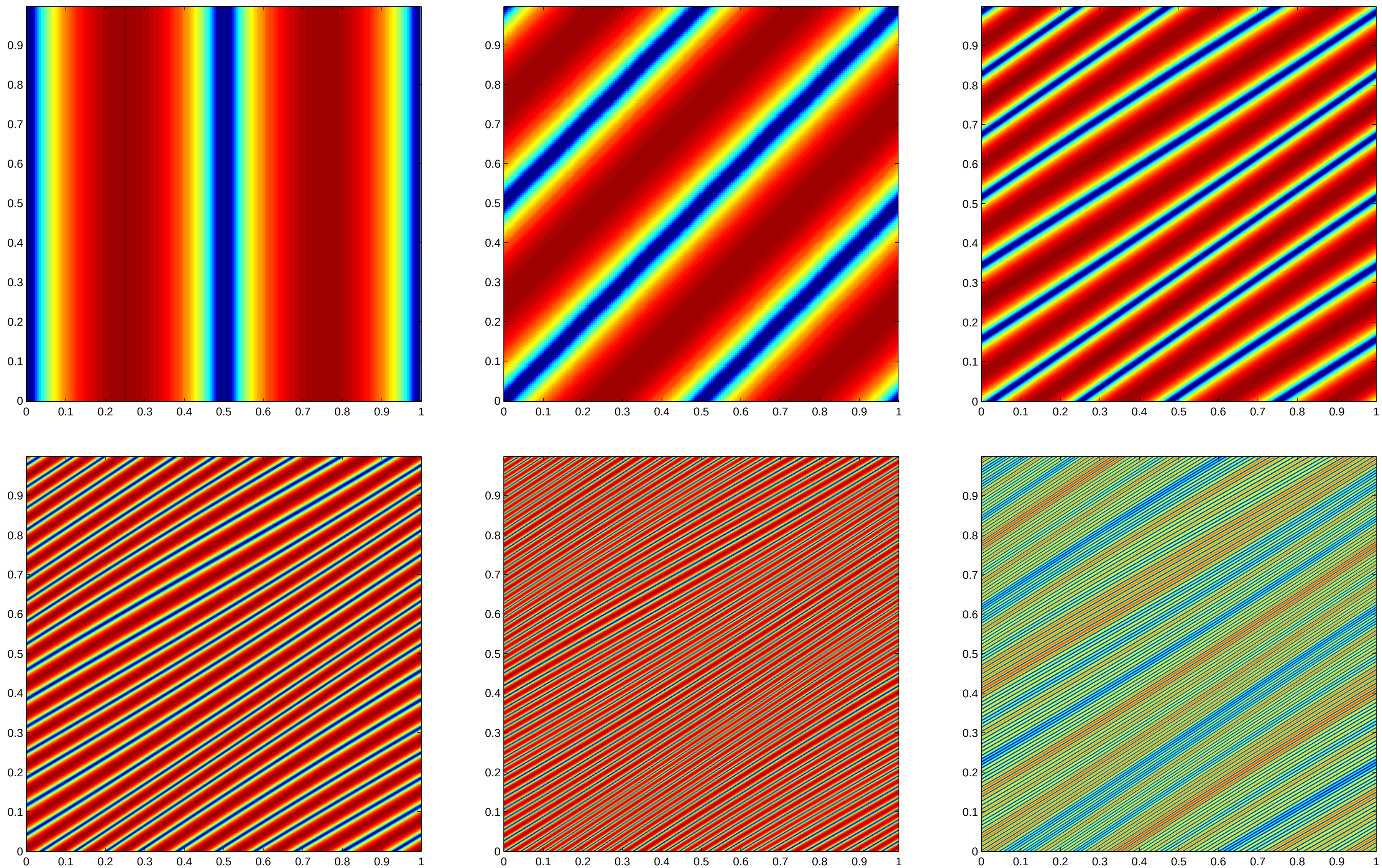
- This is the well-known "filamentation" effect in chaotic flows: the stretching and folding action of the flow causes rapid variation of the temperature across the folds.
- Can no longer neglect diffusion after a number of iterations

$$i_1 \simeq 1 + (\log \epsilon^{-1} / \log \Lambda^2) \simeq 6 \quad \text{for } \epsilon = 10^{-5},$$

where $\Lambda = (3 + \sqrt{5})/2$ is the largest eigenvalue of $\mathbb{M}^{-1}$.

Variance: 5 iterations for $K = 0.3$ and $\epsilon = 10^{-3}$



Superexponential Phase

For small K and $\boldsymbol{k}$, we have $J_0\left((k_1+k_2)K\right) \gg J_1\left((k_1+k_2)K\right)$, so we set $K=0$ and retain only the $Q_1=0$ term in the transfer matrix,

$$\mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} = \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{\boldsymbol{0},\boldsymbol{Q}} + \mathcal{O}\big((k_1+k_2)^2K^2\big)\,;$$

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If initially the variance is concentrated in a single wavenumber $\boldsymbol{q}_0$, then after one iteration it will all be in $\boldsymbol{q}_0\cdot\mathbb{M}^{-1}$, after two in $\boldsymbol{q}_0\cdot\mathbb{M}^{-2}$, etc.

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But each time the variance is multiplied by the diffusive decay factor $\exp(-\epsilon\,\boldsymbol{q}^2)$, with $\boldsymbol{q}$ getting exponentially larger; the net decay is thus superexponential.

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- This direct cascade process dominates at first, but it is so efficient that eventually we must examine the effect of the the wave term, which is felt through the Bessel functions in the transfer matrix.
- Can the wave term lead to the formation of an eigenfunction of the advection–diffusion operator, which would imply exponential decay?

An Eigenfunction?

Recall:

$$\mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} = \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{0,Q_2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left(\left(k_1+k_2\right)K\right), \qquad \boldsymbol{Q} := \boldsymbol{k}\cdot\mathbb{M} - \boldsymbol{q},$$

Consider a matrix element for which $Q_1 \neq 0$. This means that the initial ($\boldsymbol{q}$) and final ($\boldsymbol{k}$) wavenumbers connected by that matrix element can differ from $\boldsymbol{k}\cdot\mathbb{M} = \boldsymbol{q}$ by Q_1 in their first component.

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Is it possible for a wavenumber to be mapped back onto itself by such a coupling? Seek solutions to

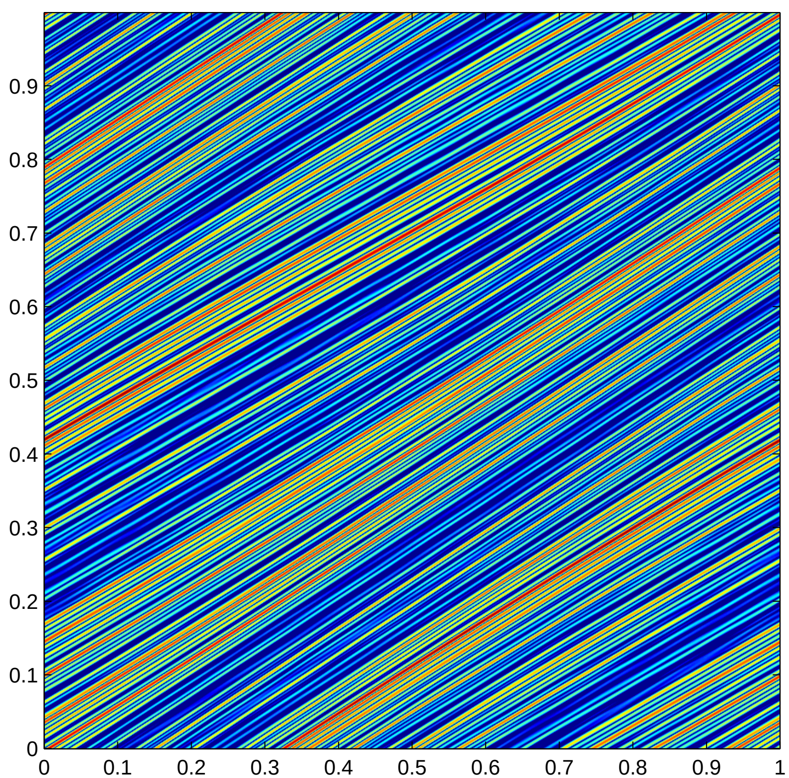
$$(q_1 \;\; q_2)\cdot\mathbb{M} = (q_1 + Q_1 \;\; q_2) \qquad \Longrightarrow \qquad (q_1 \;\; q_2) = (0 \;\; Q_1)\,.$$

The matrix element connecting the $(0 \;\; Q_1)$ mode to itself is

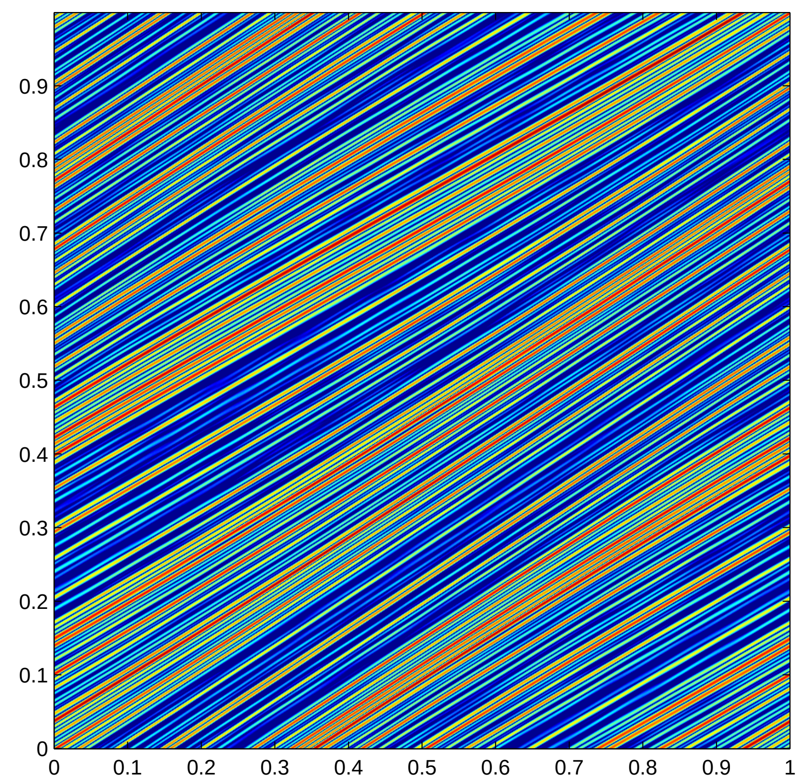
$$\mathbb{T}_{(0\;Q_1),(0\;Q_1)} = \mathrm{e}^{-\epsilon\,Q_1^2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left(Q_1\,K\right).$$

Eigenfunction for $K = 0.3$ and $\epsilon = 10^{-3}$

(Renormalized by decay rate)



$i = 25$



$i = 30$

Decay Rate

For small K, the dominant Bessel function is J_1, so the decay factor μ^2 for the variance is

$$\mu = \left|\mathbb{T}_{(0\ 1),(0\ 1)}\right| = \mathrm{e}^{-\epsilon}\, J_1\,(K) = \tfrac{1}{2}K + \mathcal{O}\big(\epsilon\, K, K^2\big)\,.$$

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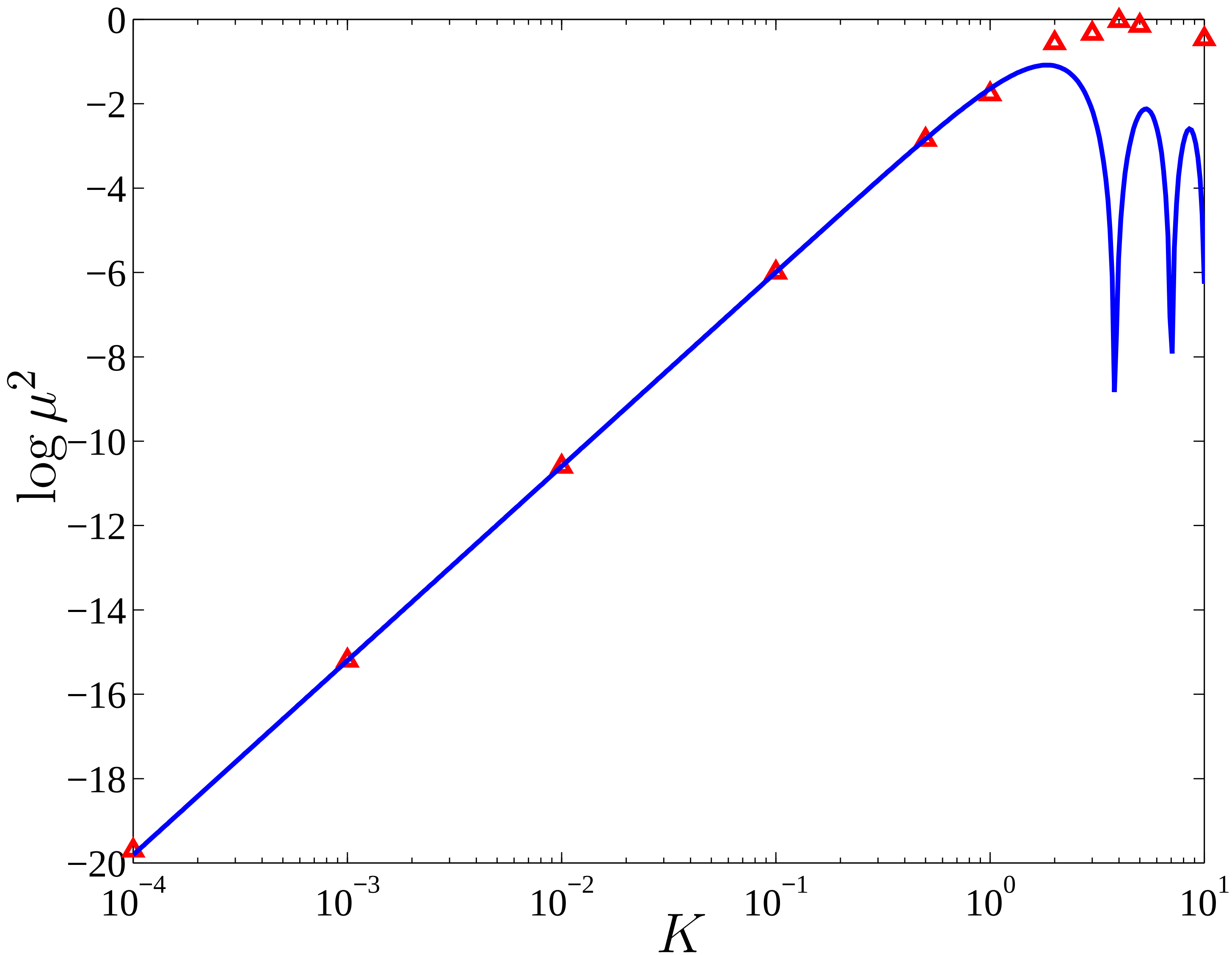
This is an analogous result to the baker's map [Fereday et al., PRE (2002)], except that here the agreement with numerical results is good for K quite close to unity.

This is because in the baker's map the discontinuity generates many slowly-decaying harmonics at each step.

Decay Rate as $\epsilon \to 0$



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- The superexponential decay depletes the variance very rapidly until all that is left is variance in the exponentially decaying mode $(0\ \ 1)$.
- The superexponential phase thus ends when the variance at large wavenumbers equals that in mode $(0\ \ 1)$.
- Assuming that the variance resides entirely in the $\boldsymbol{k}_0 = (0\ \ 1)$ mode initially, the condition for breakdown is

$$\mu^{i_2} = \exp\left(-\epsilon \left\| \boldsymbol{k}_0 \cdot \mathbb{M}^{-(i_2-1)} \right\|^2\right),$$

where μ^2 is the decay factor of the variance in $\boldsymbol{k}_0$.

Transition (continued)

After substituting $\left\|\boldsymbol{k}_0 \cdot \mathbb{M}^{-(i_2-1)}\right\| \simeq \Lambda^{i_2-1}$, solve numerically for i_2.

For $K = 10^{-3}$ and $\epsilon = 10^{-5}$, we have $i_2 \simeq 9.2$, numerical results.

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For $\epsilon \ll 1$, approximate solution given by

$$i_2 \simeq 1 + \log\left(\epsilon^{-1} \log \mu^{-1}\right) / \log \Lambda^2,$$

which gives $i_2 \simeq 8$ for $K = 10^{-3}$, $\epsilon = 10^{-5}$.

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Subtracting $i_1 = 1 + \log \epsilon^{-1} / \log \Lambda^2$, the onset of the superexponential phase, we find the duration of the phase is

$$i_2 - i_1 \simeq \frac{\log \log \mu^{-1}}{\log \Lambda^2}.$$

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- Observed in experiments? There μ tends to be closer to unity, so unlikely. But…

Variance Spectrum of the Eigenfunction

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- The long-wavelength mode $(0\ \ 1)$ is the bottleneck that determines the decay rate, for small K.
- But this dominant mode does not determine the structure of the eigenfunction.
- In fact, a very small amount of the total variance actually resides in that bottleneck mode: the variance is concentrated at small scales.

Cascade

The variance is taken out of the $(0 \;\; 1)$ mode by the map: there is a cascade to larger wavenumber through the action of $\mathbb{M}^{-1}$:

$$(0 \;\; 1) \to (-1 \;\; 2) \to (-3 \;\; 5) \to (-8 \;\; 13) \to \ldots.$$

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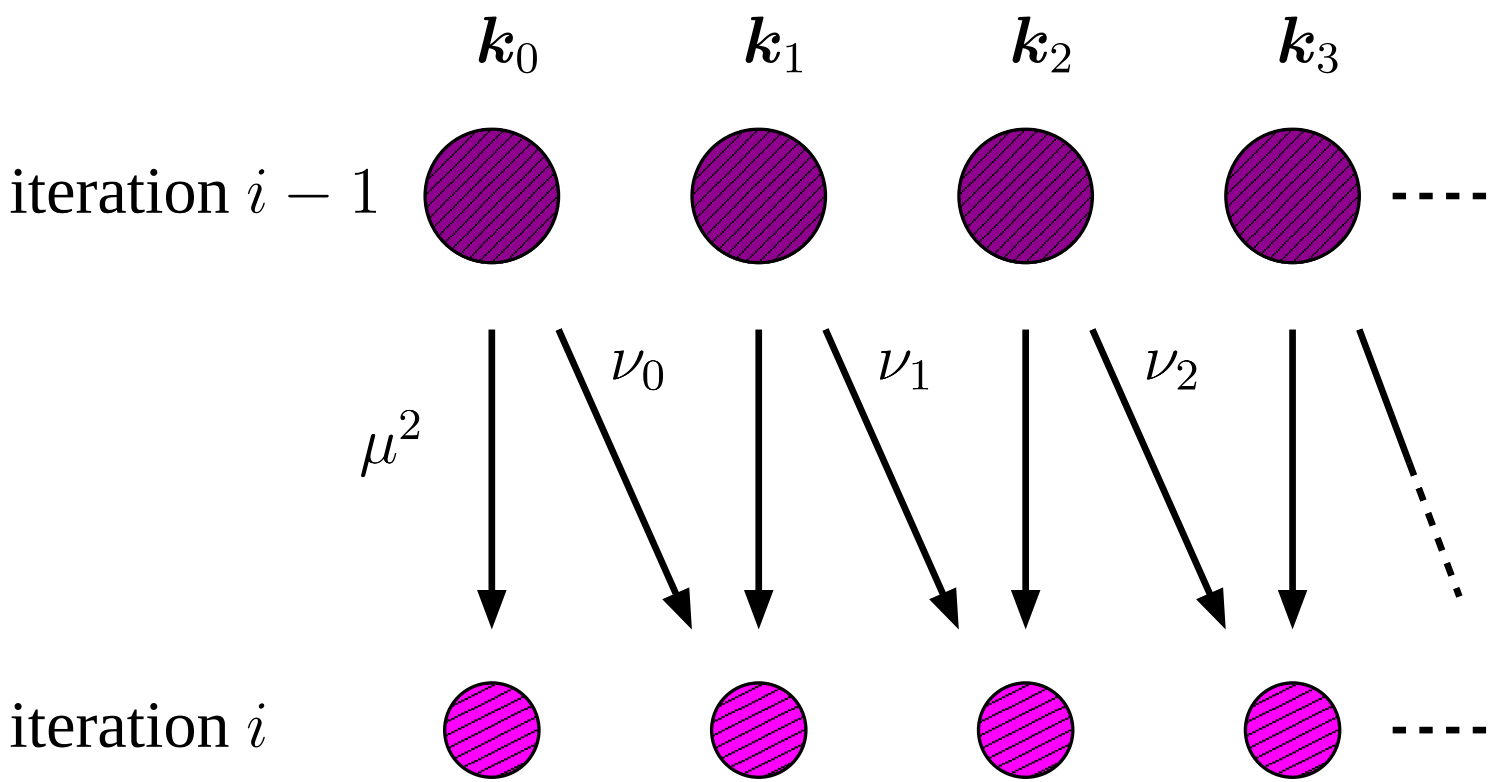
But we have seen that the exponential decay rate suggests that the scalar concentration is in an eigenfunction of the advection–diffusion operator.

What is going on?

Eigenfunction: One Iteration

The wavenumbers are mapped back to themselves, with their variance decreased by a uniform factor $\mu^2 < 1$ (vertical arrows). But at the same time the modes are mapped to next one down the cascade following the diagonal arrows.



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If we denote by $\sigma_n^{(i)} := |\hat{\theta}_{\boldsymbol{k}_n}|^2$ the variance in mode $\boldsymbol{k}_n$ at the ith iteration, we have

$$\sigma_n^{(i)} = \mu^2\,\sigma_n^{(i-1)}, \qquad n = 0, 1, \ldots,$$

$$\sigma_n^{(i)} = \nu_{n-1}\,\sigma_{n-1}^{(i-1)}, \qquad n = 1, 2, \ldots.$$

Eigenfunction and Cascade

The decrease in variance for each of the diagonal arrows is diffusive and is given by the factor $\nu_n = \exp(-2\epsilon\, \boldsymbol{k}_n^2)$.

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These two recurrences can be combined to give

$$\Sigma_n^{(i)} := \frac{\sigma_n^{(i)}}{\sigma_0^{(i)}} = \frac{\nu_{n-1}\,\nu_{n-2}\,\cdots\,\nu_0}{\mu^{2n}} = \mu^{-2n}\,\exp\left(-2\epsilon \sum_{m=0}^{n-1} \boldsymbol{k}_m^2\right),$$

where $\Sigma_n^{(i)}$ is the relative variance in the nth mode.

Eigenfunction and Cascade (cont'd)

The wavenumber is given by the exponential recursion,

$$\|\boldsymbol{k}_n\| \simeq \Lambda\,\|\boldsymbol{k}_{n-1}\| \quad \Longrightarrow \quad \|\boldsymbol{k}_n\| \simeq \Lambda^n\,\|\boldsymbol{k}_0\| = \Lambda^n\,.$$

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Solve for $n = \log \|\boldsymbol{k}_n\| \, / \, \log \Lambda$ and rewrite the relative variance as

$$\Sigma_n^{(i)} \simeq \|\boldsymbol{k}_n\|^{-2 \log \mu / \log \Lambda} \exp\left(-2\epsilon \boldsymbol{k}_n^2 / \Lambda^2\right) ,$$

where we retained only the $\boldsymbol{k}_{n-1}^2$ (last) term of the sum.

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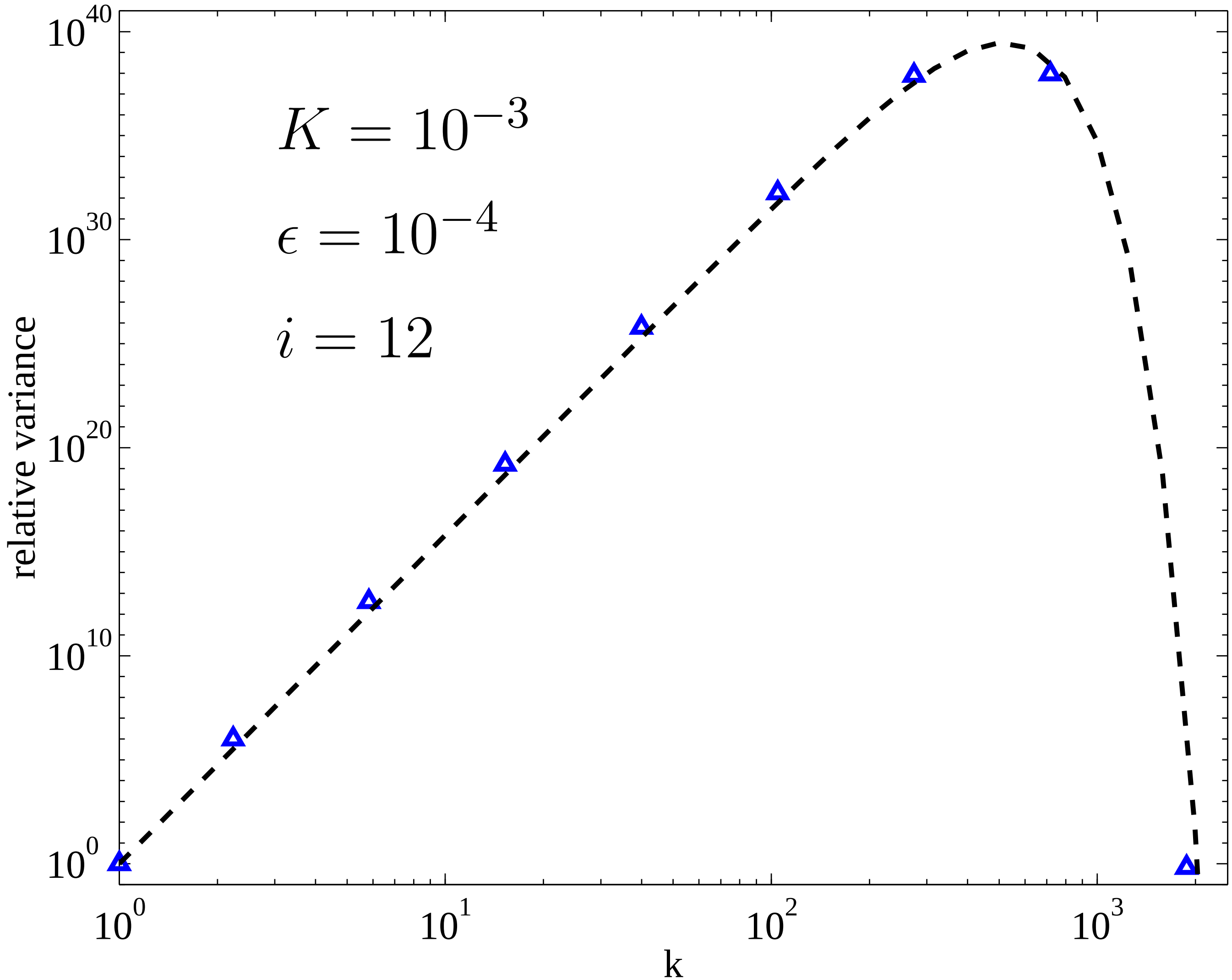
where we retained only the $\boldsymbol{k}_{n-1}^2$ (last) term of the sum.
Does not (and should not) depend on the iteration number, i, and depends only on n through $\boldsymbol{k}_n$. Find

$$\boxed{\Sigma(k) = k^{2\zeta} \exp\left(-2\epsilon k^2/\Lambda^2\right) , \qquad \zeta := -\log\mu/\log\Lambda,}$$

the spectrum of relative variance.

Spectrum of Variance



Spectrum of Variance (cont'd)

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Find the maximum of $\Sigma(k)$,

$$k_{\mathrm{m}} = \Lambda\,(\zeta/2\epsilon)^{1/2}, \qquad \Sigma(k_{\mathrm{m}}) = k_{\mathrm{m}}^{2\zeta}\,\mathrm{e}^{-\zeta} = k_{\mathrm{m}}^{2\zeta}\,\mu^{\log\Lambda}.$$

The peak wavenumber thus scales as $\epsilon^{-1/2}$, the same scaling as the dissipation scale.
The relative variance in that peak wavenumber scales as $\epsilon^{-\zeta}$.

k_{m} largest wavenumber that must be included in a numerical calculation to capture the decay of variance correctly (fewer?).

Conclusions

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