
Topological Kinematics of Mixing

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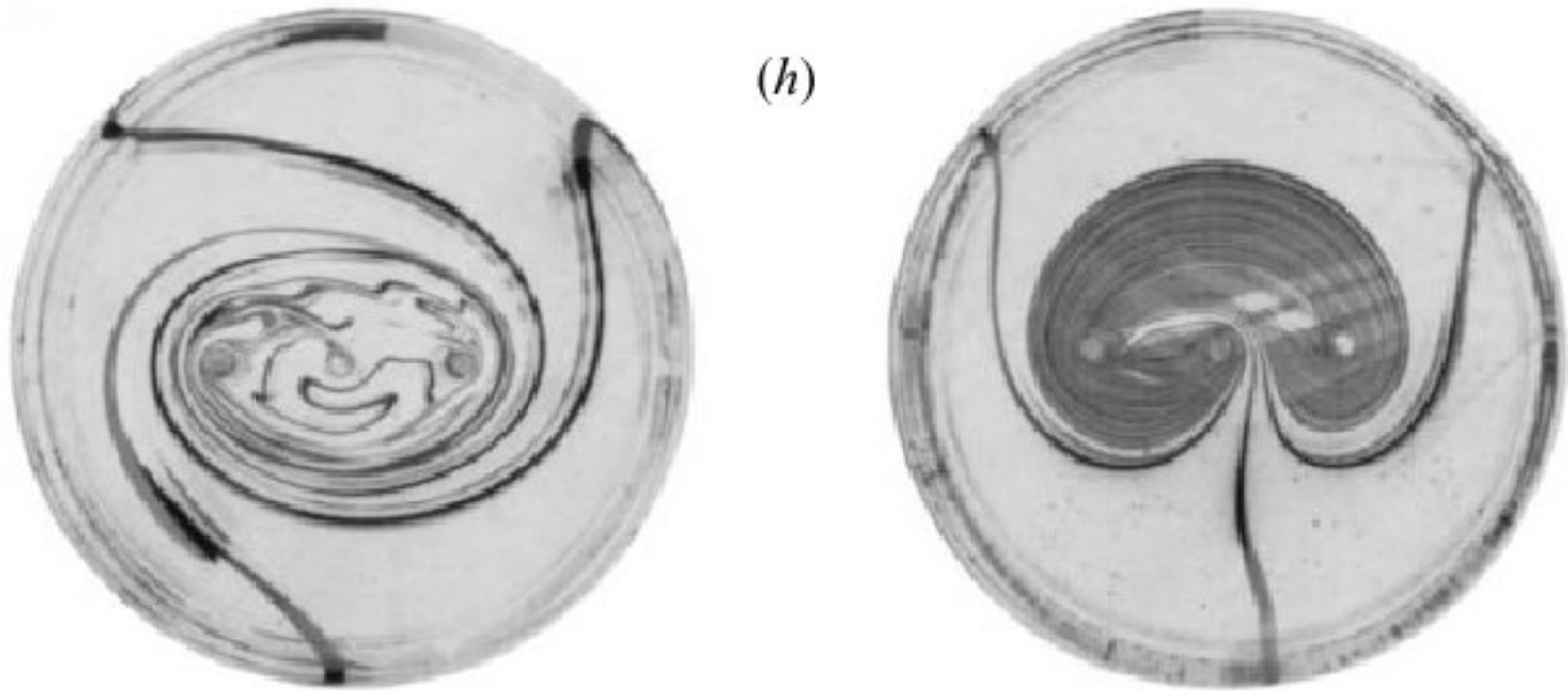
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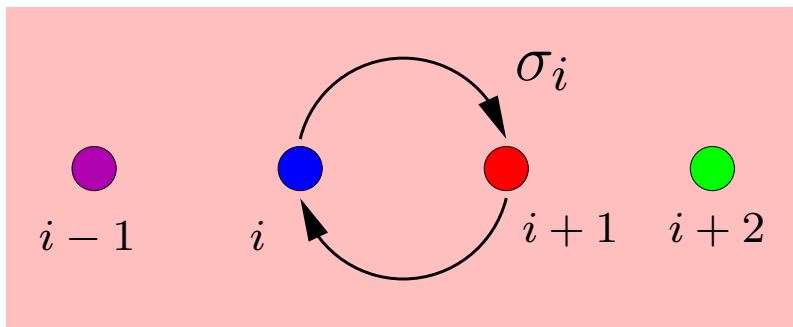
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Experiment of Boyland *et al.*



[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)] [Movies by Matt Finn: [boyland1](#) [boyland2](#)]

Generators of the n -Braid Group



A generator

$$\sigma_i, \quad i = 1, \dots, n-1$$

is the clockwise interchange of the i th and $(i+1)$ th rod.

The generators obey the **presentation**

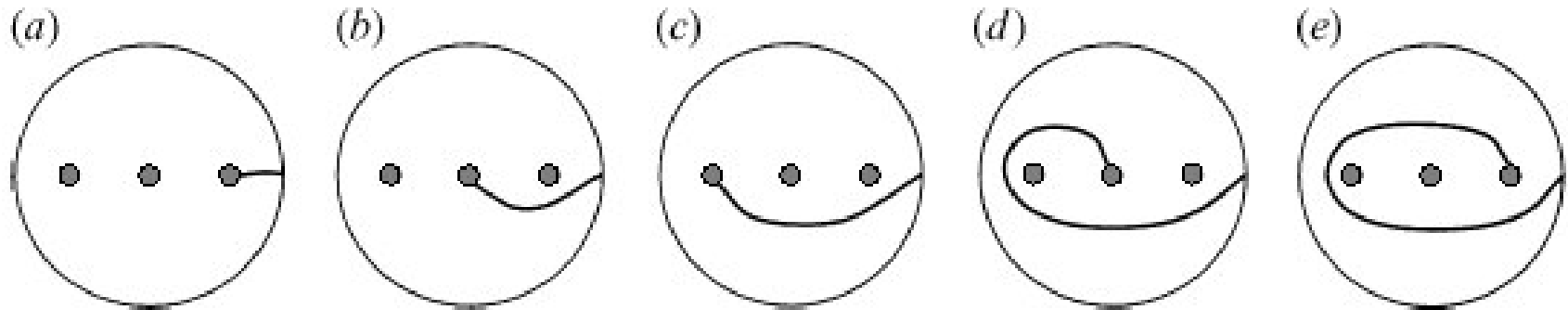
$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1$$

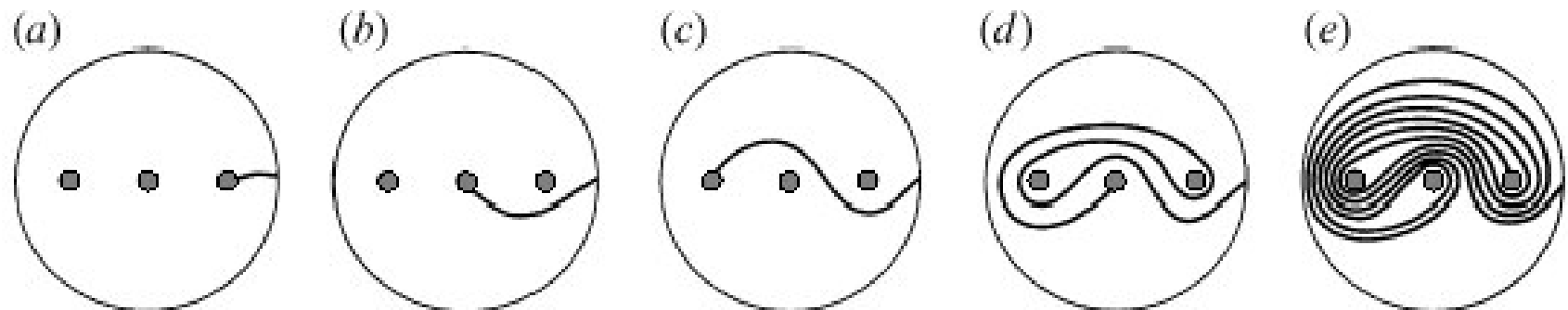
These generators are used to characterise the motion of the rods.

The Two BAS Stirring Protocols

$\sigma_1\sigma_2$ protocol



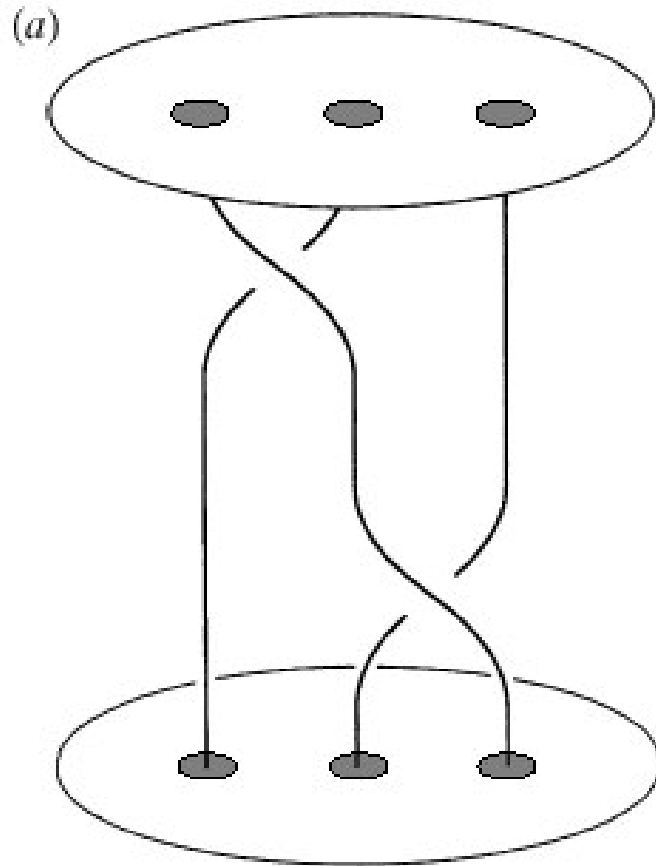
$\sigma_1^{-1}\sigma_2$ protocol



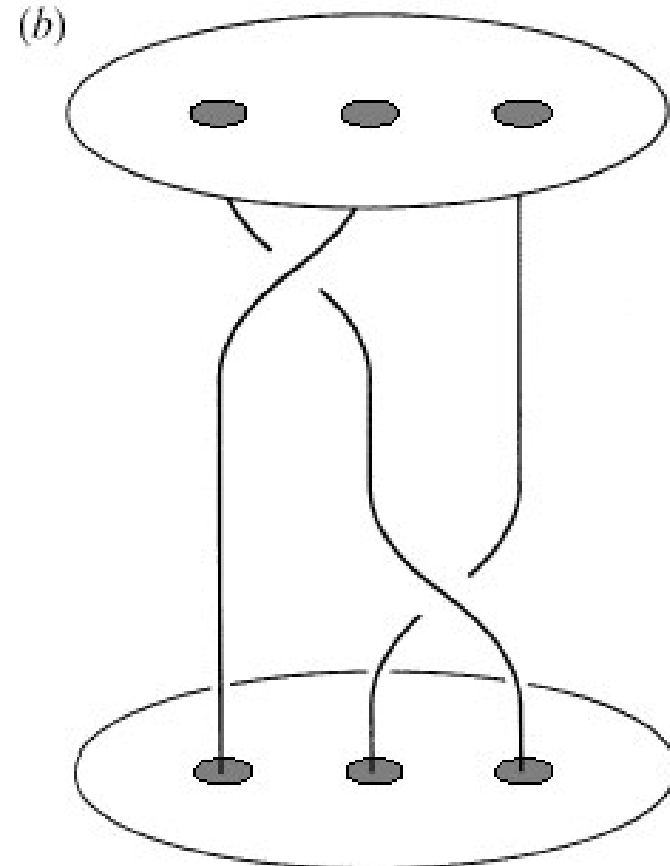
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Rod Trajectories as Braids

$\sigma_1\sigma_2$ protocol



$\sigma_1^{-1}\sigma_2$ protocol



Time ↑

[P. L. Boyland, H. Aref, and M. A. Stremler, J. Fluid Mech. **403**, 277 (2000)]

Matrix Representation of the σ_i

We use the convenient **Burau** representation

$$\sigma_i = I_{i-2} \oplus \begin{pmatrix} 1 & -\tau & 0 \\ 0 & -\tau & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

where $\tau \in \mathbb{C}$. The matrices are $(n-1) \times (n-1)$.

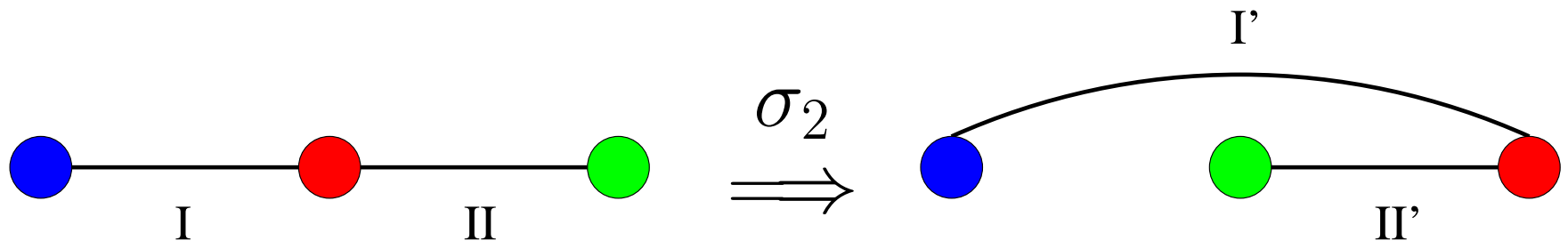
The Burau matrices satisfy the presentation (of course), but for $n > 4$ they do **not** provide a faithful representation.

This is of no great consequence for our purposes.

Topological Entropy of a Braid

Practically speaking, the topological entropy of a braid is a **lower bound** on the **line-stretching exponent** of the flow!

This is Eminently Reasonable™:



The Burau representation has an awesome property: if Σ is the Burau representation of a braid word,

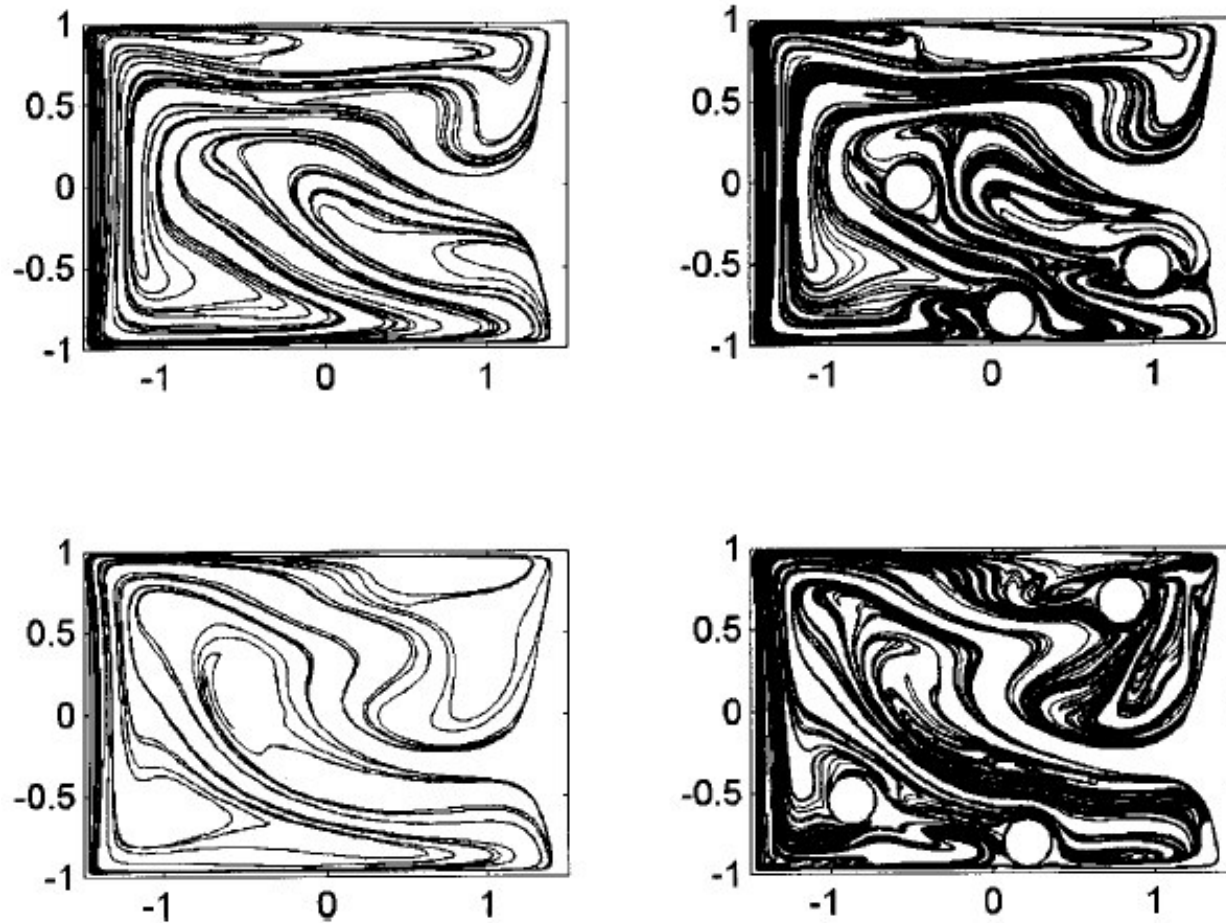
$$\text{topological entropy of braid} \geq \sup_{|\tau|=1} \text{spr } \Sigma \quad \text{[Kolev]}$$

where **spr** denotes the spectral radius (the magnitude of the largest eigenvalue).

The Difference between BAS's Two Protocols

- The matrices associated the generators have eigenvalues on the unit-circle.
- The first (bad) stirring protocol has eigenvalues on the unit circle
- The second (good) protocol has largest eigenvalue $(3 + \sqrt{5})/2 = 2.6180$.
- So for the second protocol the length of a line joining the rods grows exponentially!
- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (**Thurston–Nielsen theorem**).

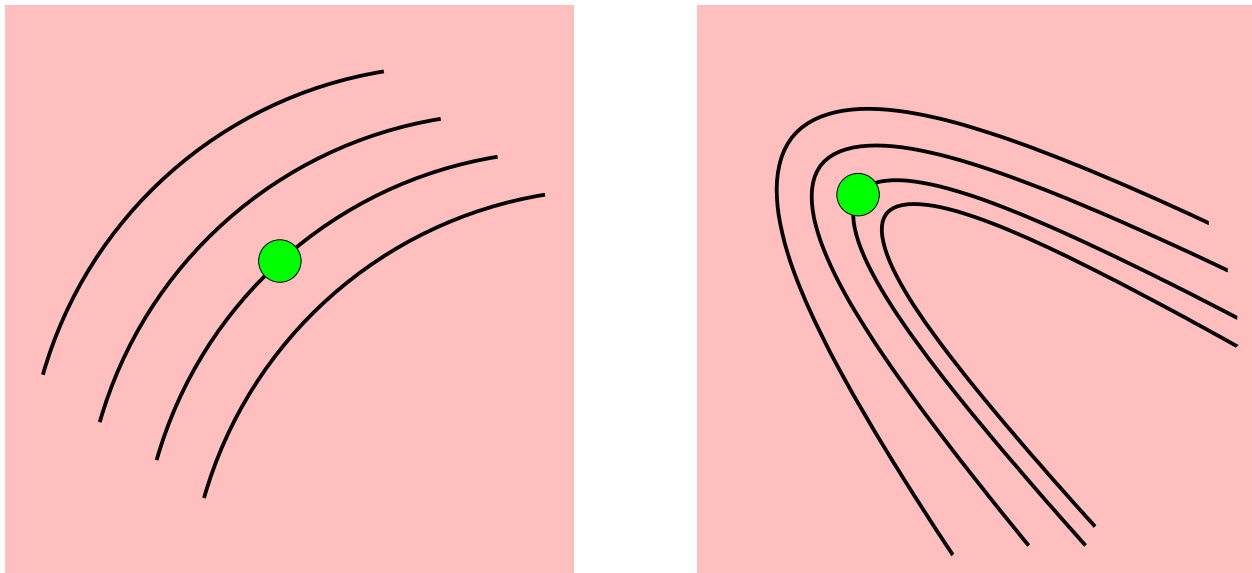
Freely-moving Rods in a Cavity Flow



[A. Vikhansky, *Physics of Fluids* **15**, 1830 (2003)]

Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).

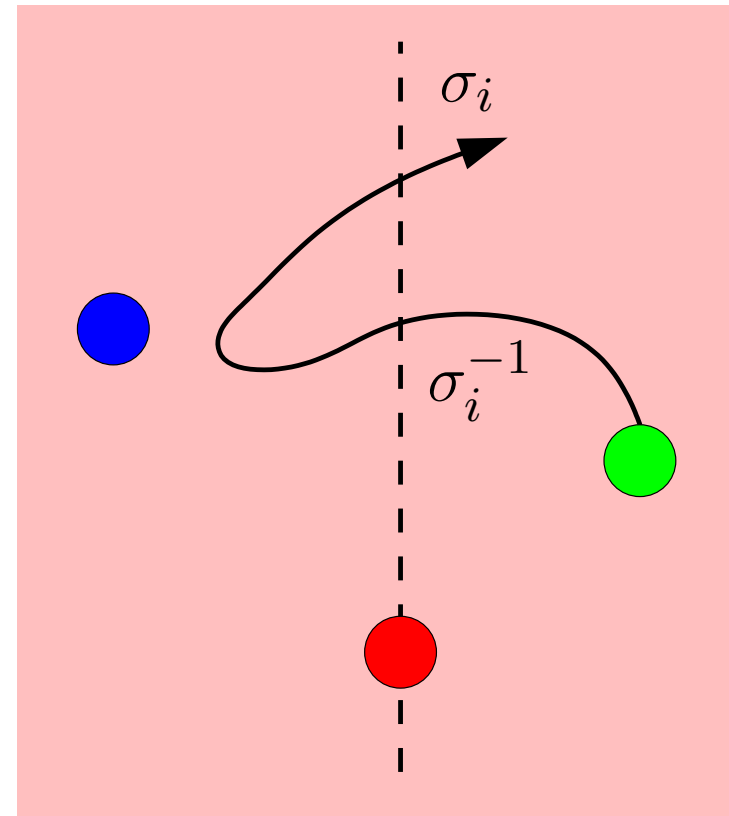
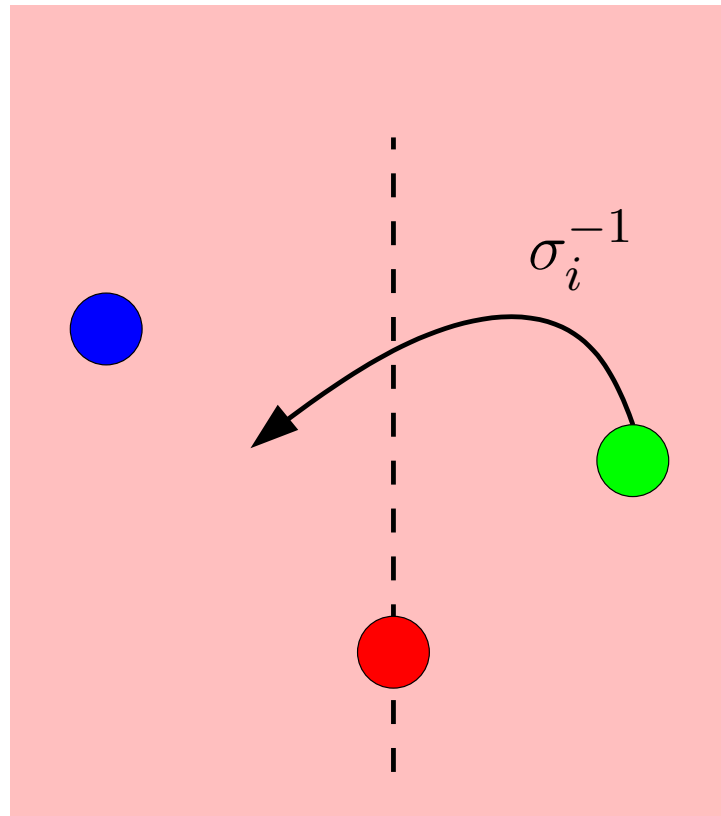


Material lines must bend around the orbit: **it acts just like a “rod”!**

The idea: pick any n fluid particles and follow them.

How do they braid around each other?

Detecting Braiding Events



In the second case there is no net braid: the two elements cancel each other.

Random Sequence of Braids

We end up with a sequence of braids, with matrix representation

$$\Sigma^{(N)} = \sigma^{(N)} \dots \sigma^{(2)} \sigma^{(1)}$$

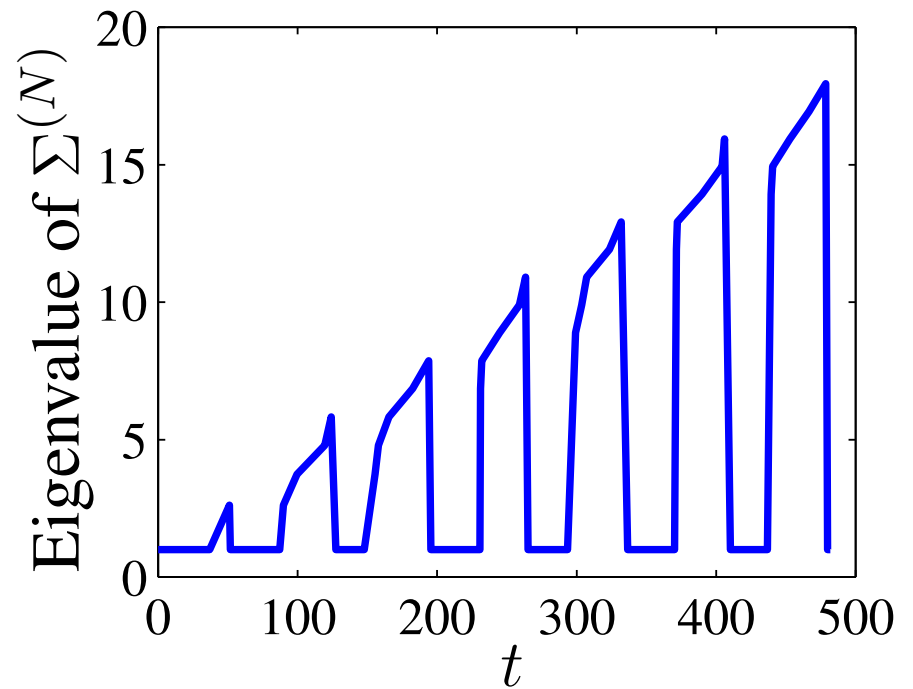
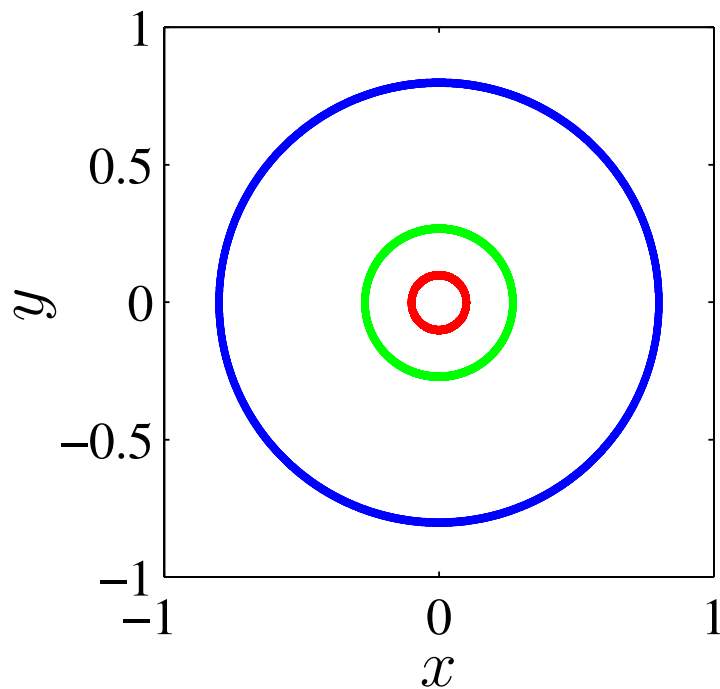
where $\sigma^{(\mu)} \in \{\sigma_i, \sigma_i^{-1}\}$ and N is the number of braiding events detected after a time t .

The largest eigenvalue of $\Sigma^{(N)}$ is a measure of the **complexity of the braiding motion**, called the **braiding factor**.

Random matrix theory says that the braiding factor can **grow exponentially!** We call the rate of exponential growth the **braiding Lyapunov exponent** or just **braiding exponent**.

Non-braiding Motion

First consider the motion of of three points in concentric circles with irrationally-related frequencies.

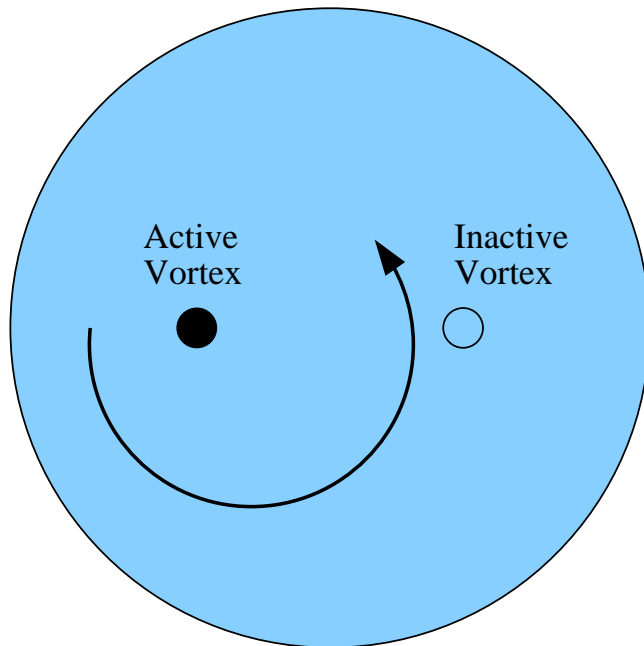


The braiding factor grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.

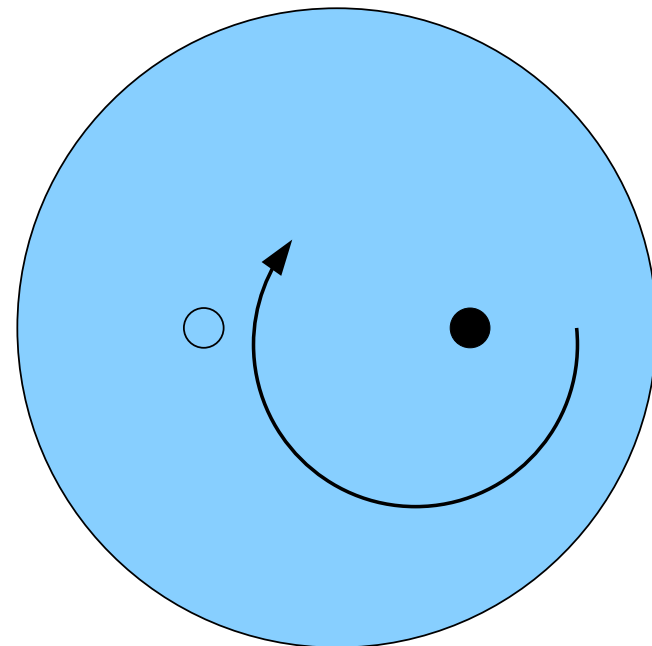
Blinking-vortex Flow

To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won't do).

Aref's **blinking-vortex flow** is ideal.



First half of period

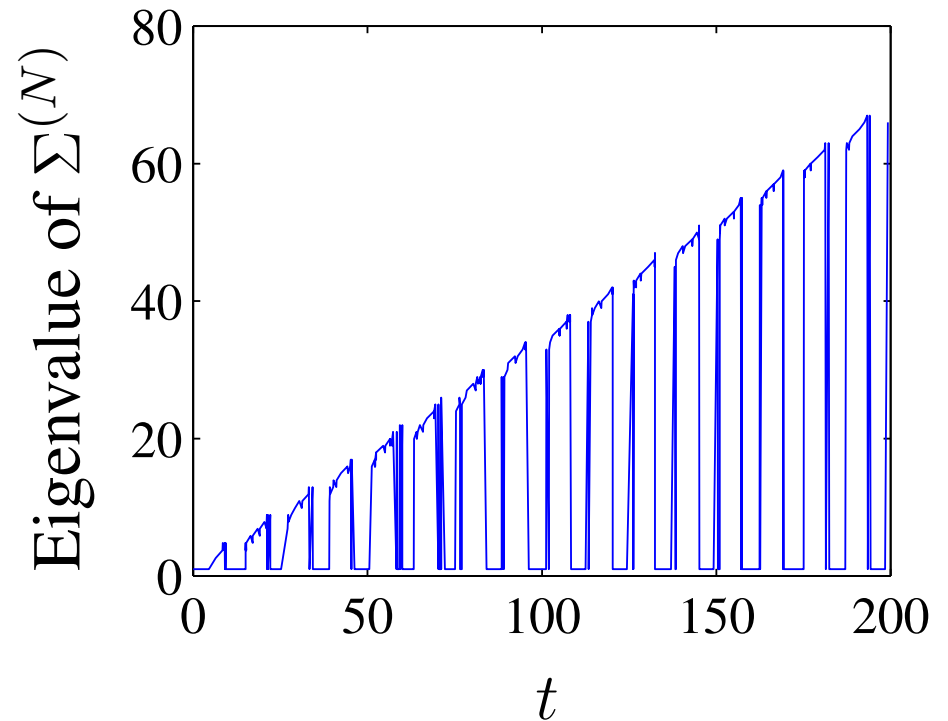
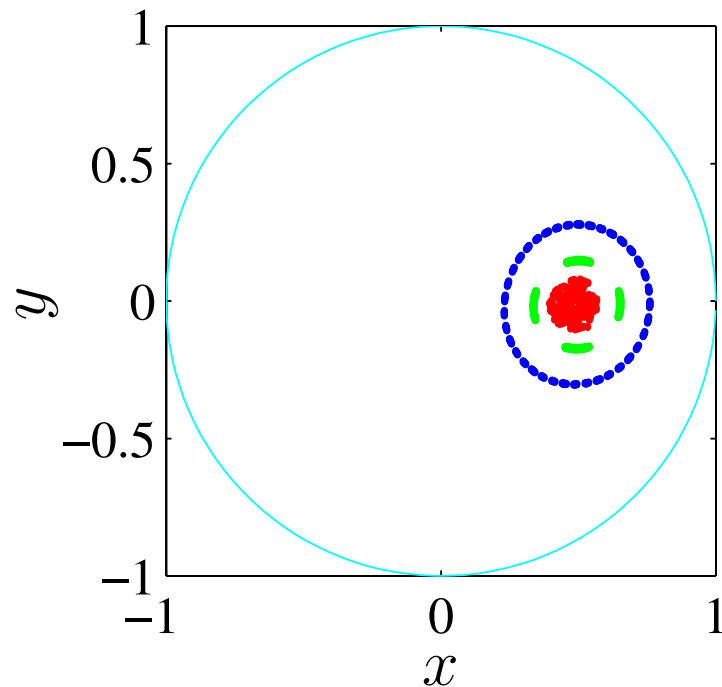


Second half of period

The only parameter is the circulation Γ of the vortices.

Blinking Vortex: Non-braiding Motion

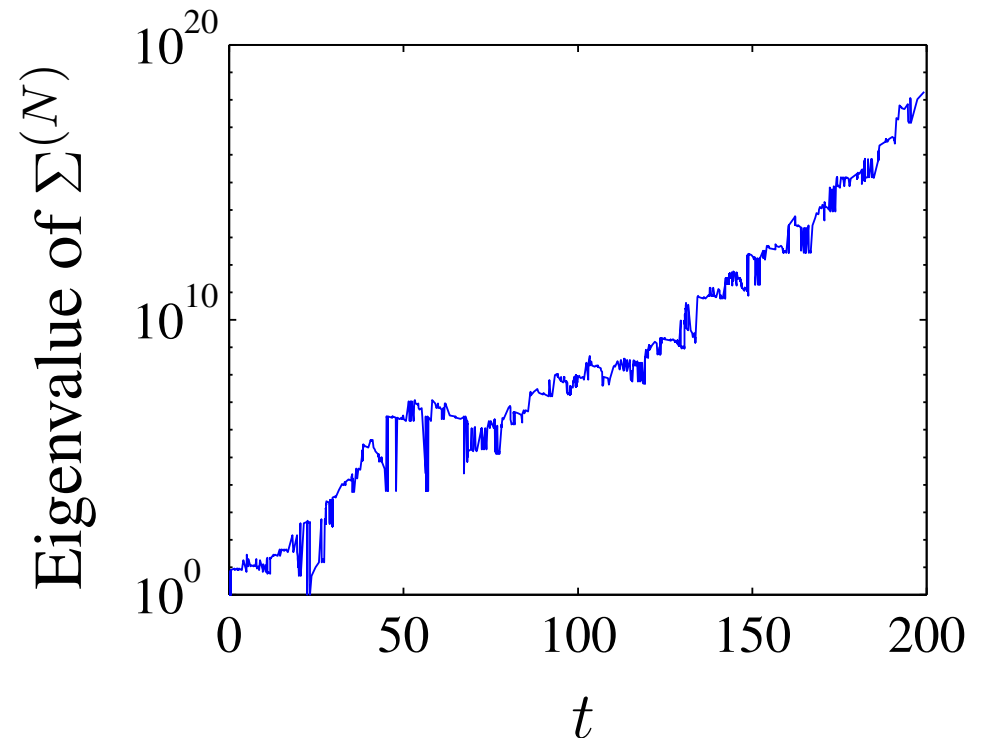
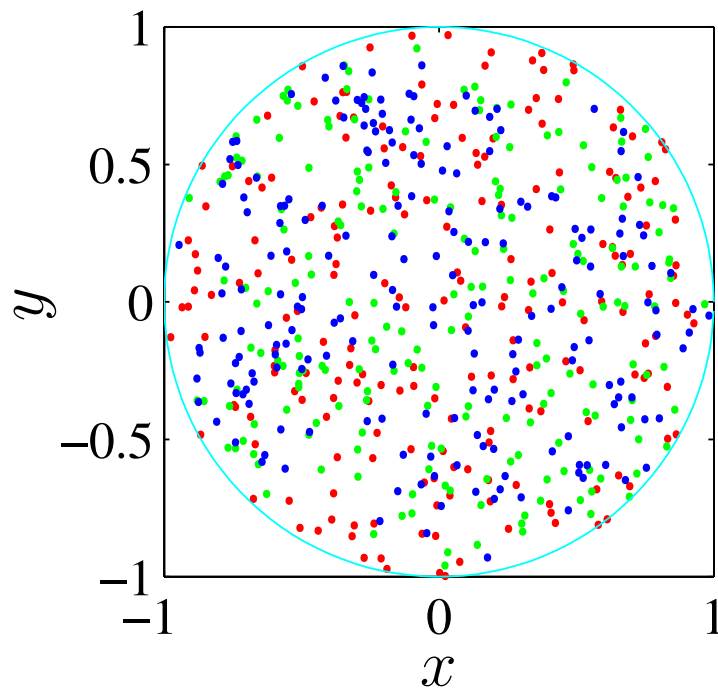
For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.



One of the orbits is chaotic, the other two are closed.

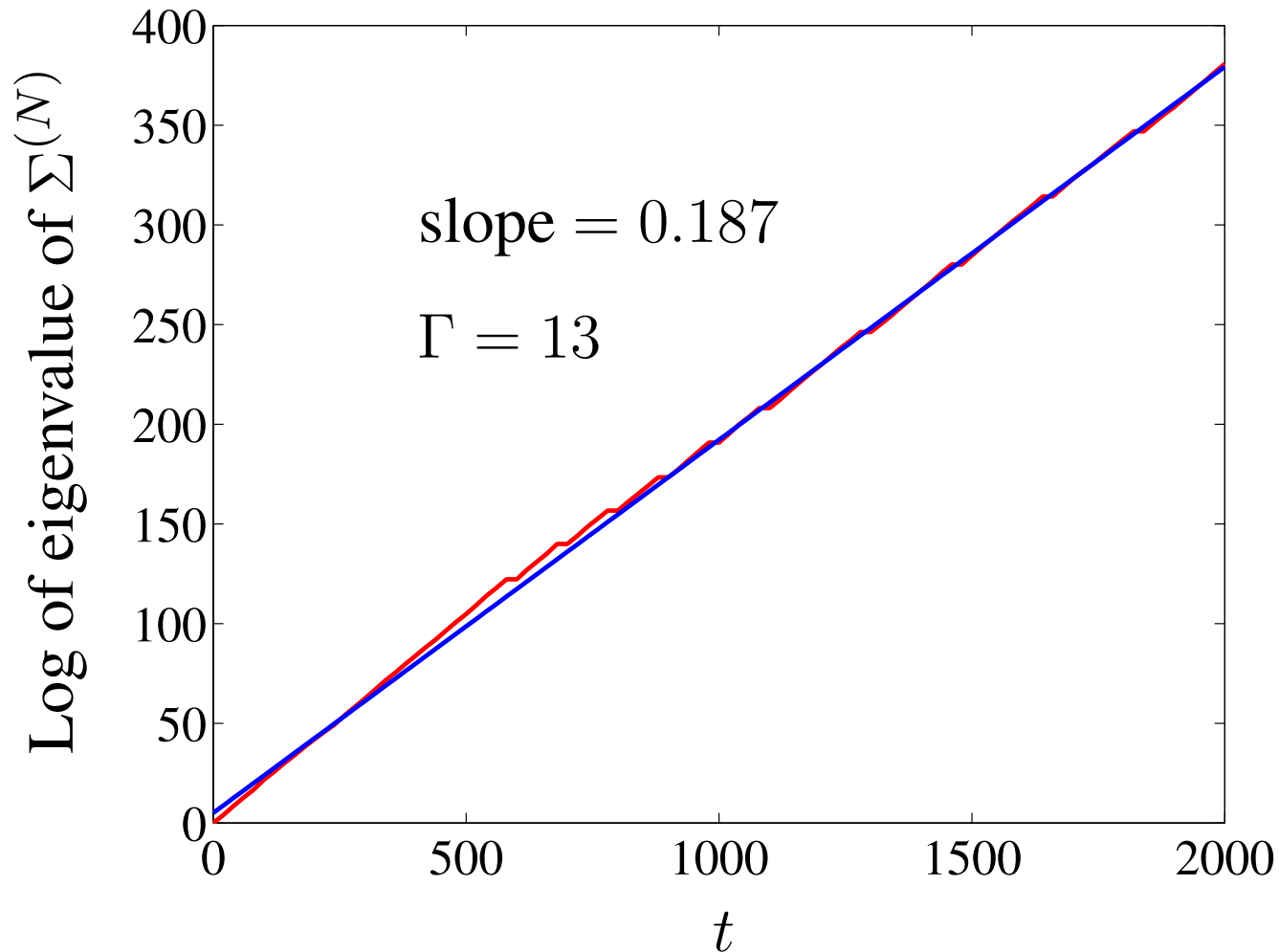
Blinking Vortex: Braiding Motion

For $\Gamma = 13$, the blinking vortex is globally chaotic.



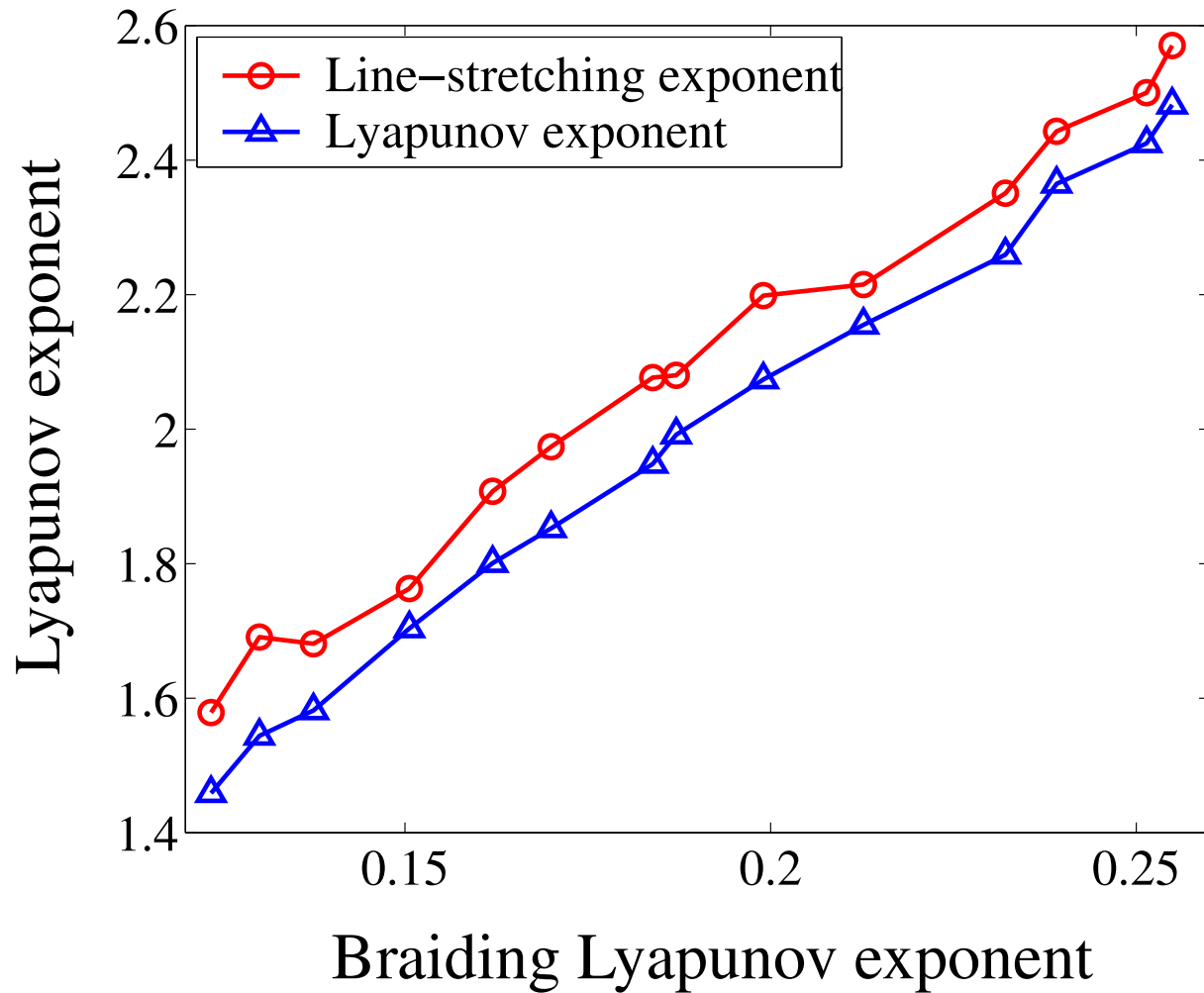
The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order 10^{20} rather than 80!

Averaging over many Triplets



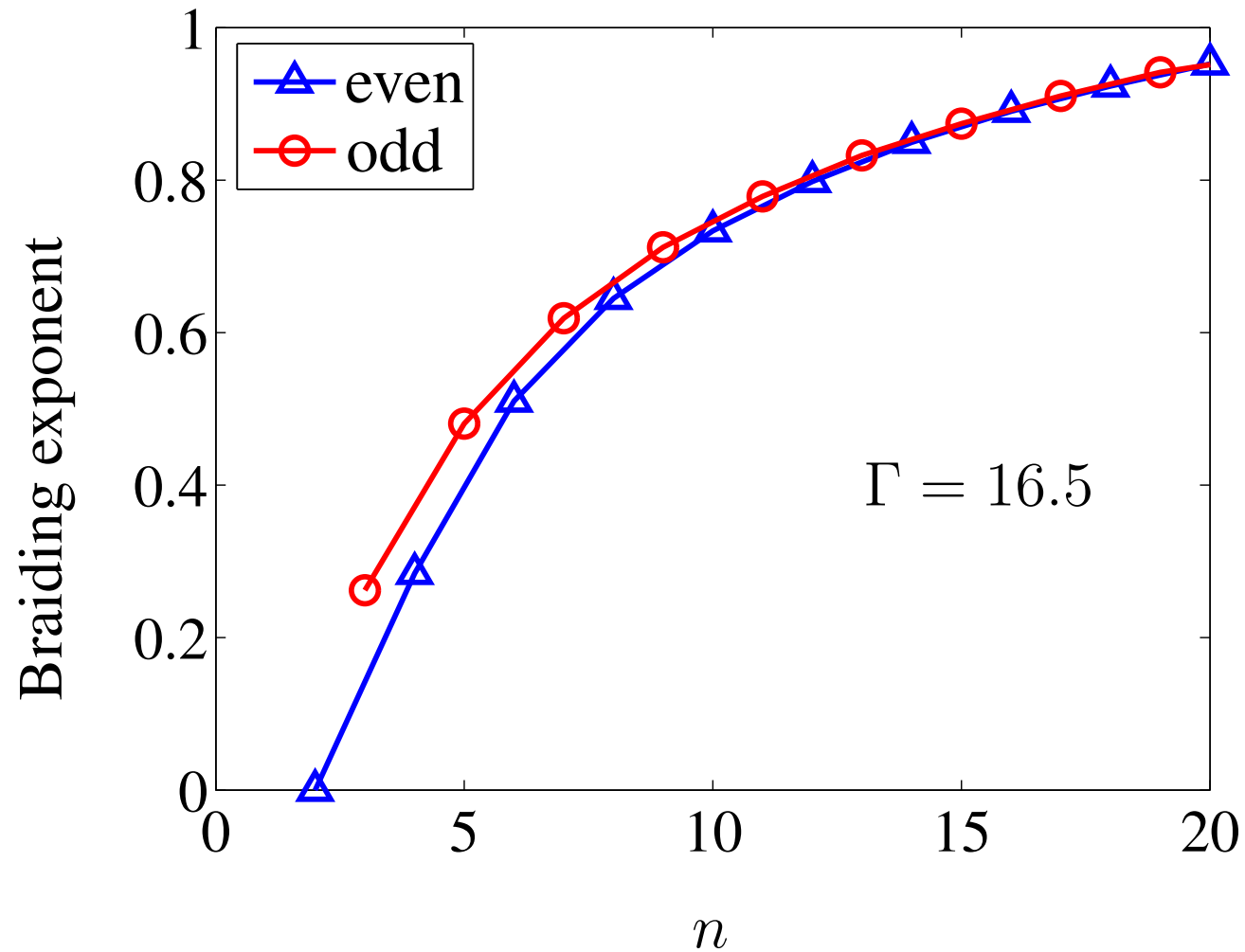
Averaged over 100 random triplets.

Comparison with Lyapunov Exponents

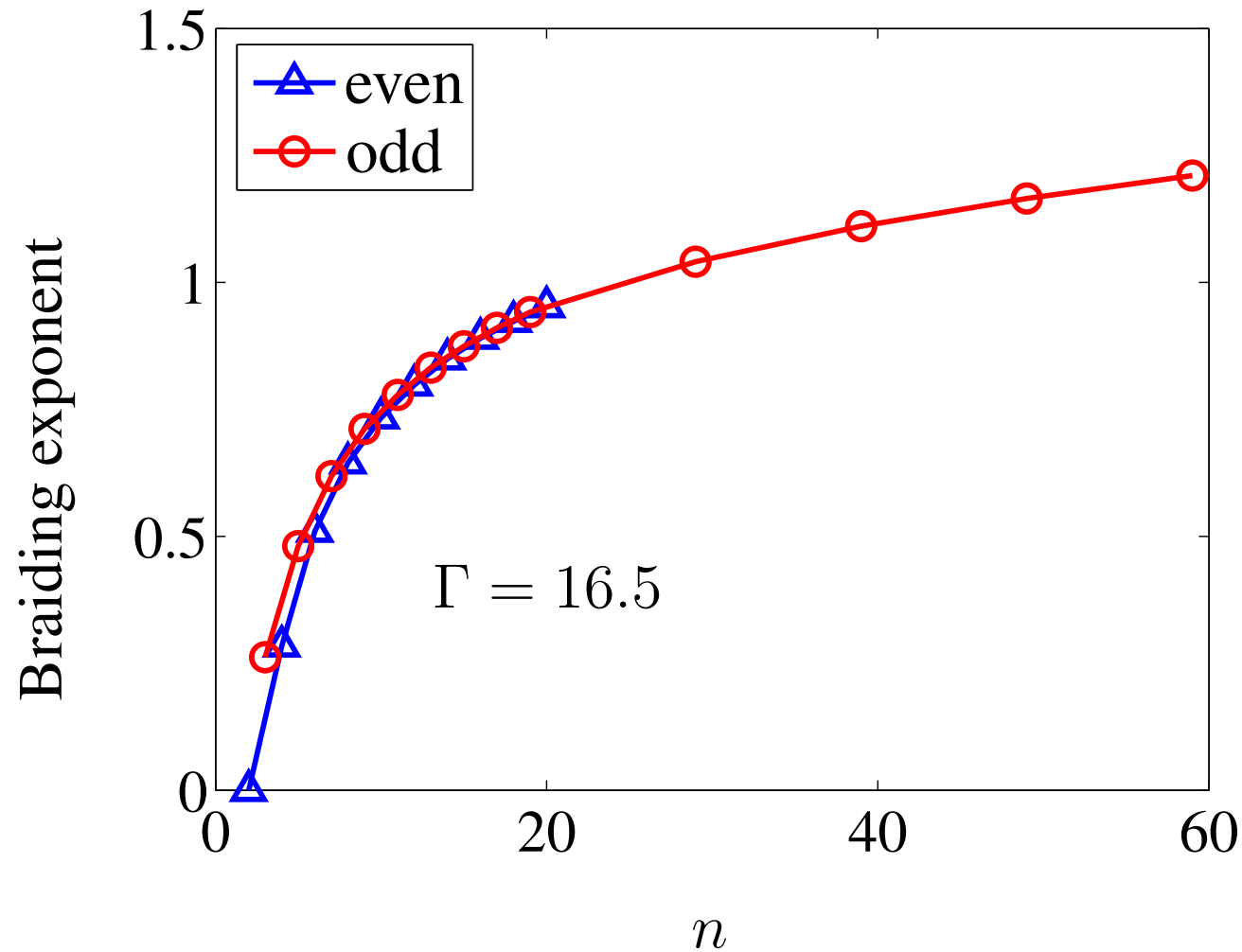


Γ varies from 8 to 20.

Beyond Three Particles



But does it Saturate?



Well, it really should...

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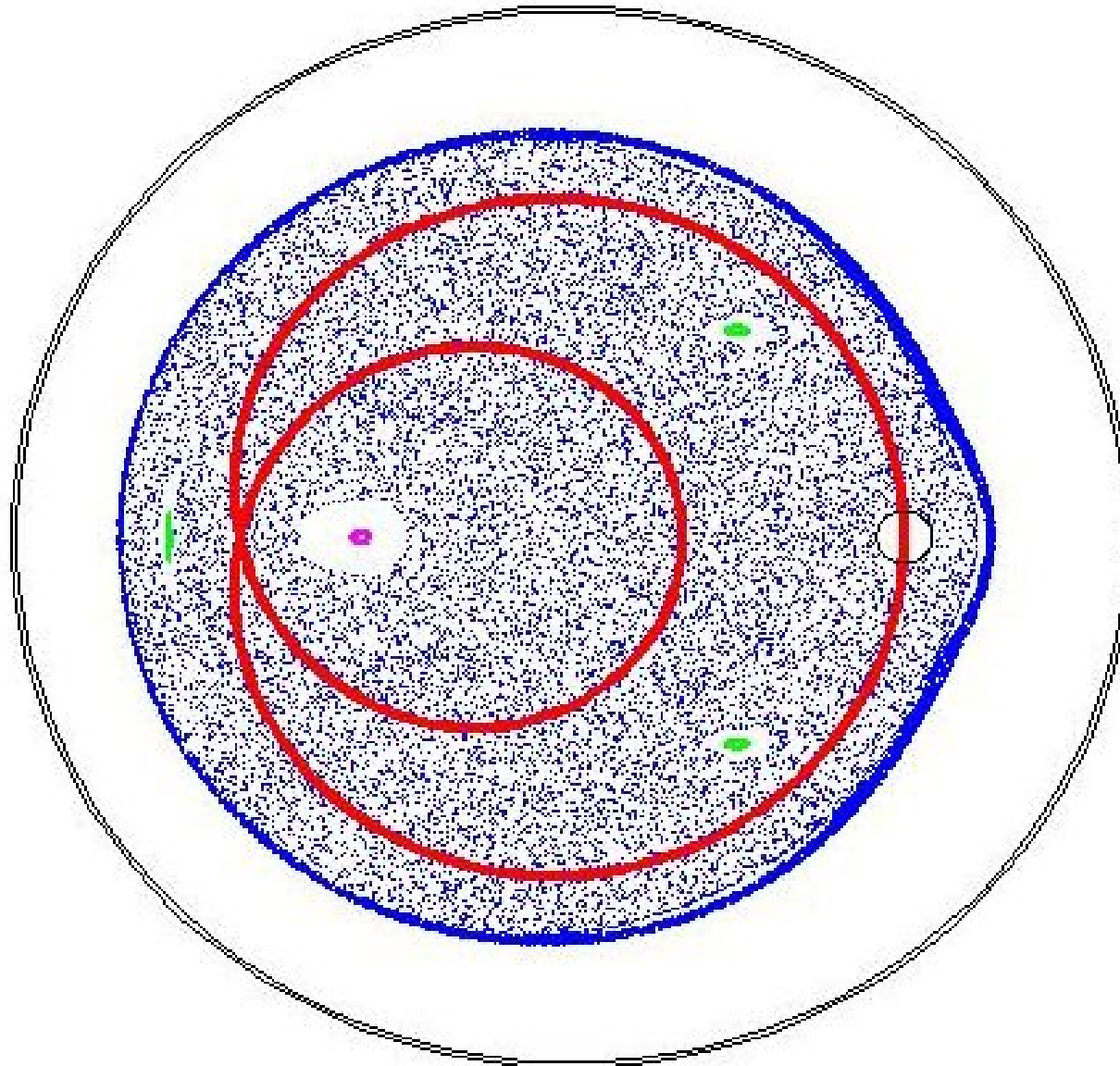
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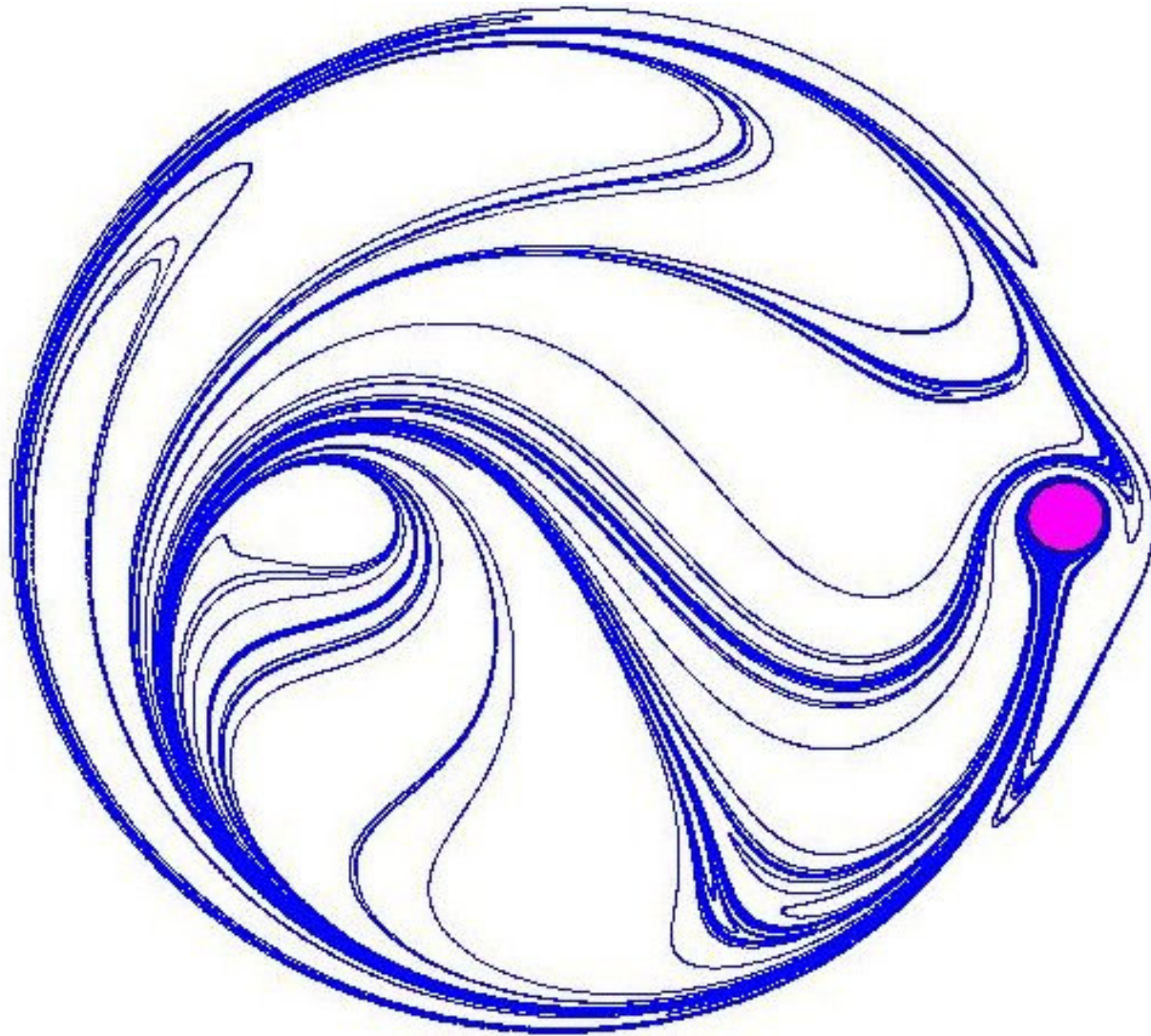
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Poincaré Section



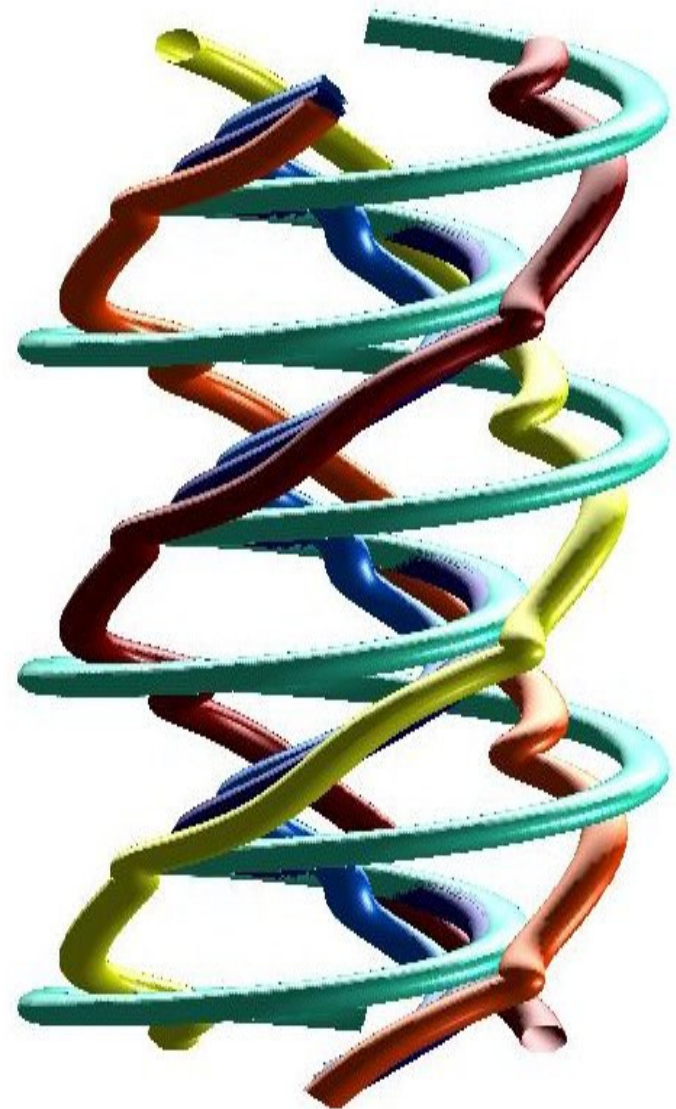
Stretching of Lines



Motion of Islands

Make a braid from the motion of the rod and the **periodic islands**.

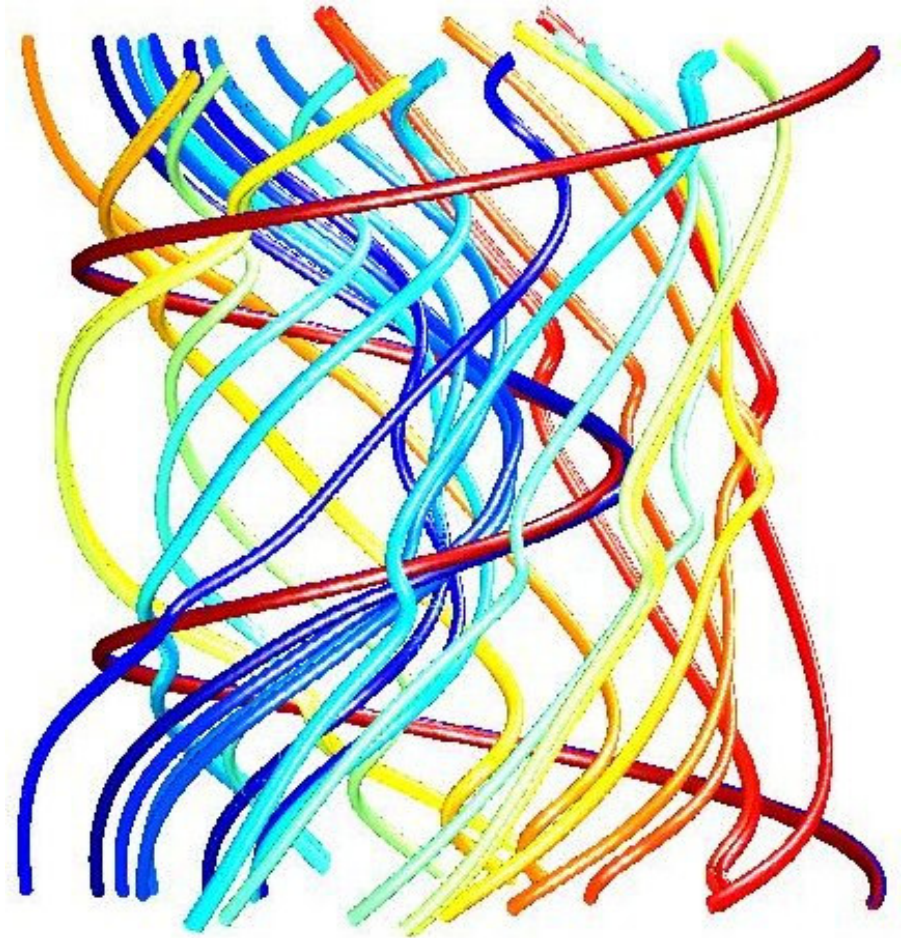
Most (74%) of the topological entropy is accounted for by this braid.



Motion of Islands and Unstable Periodic Orbits

Now we also include **unstable** periodic orbits as well as the stable ones (islands).

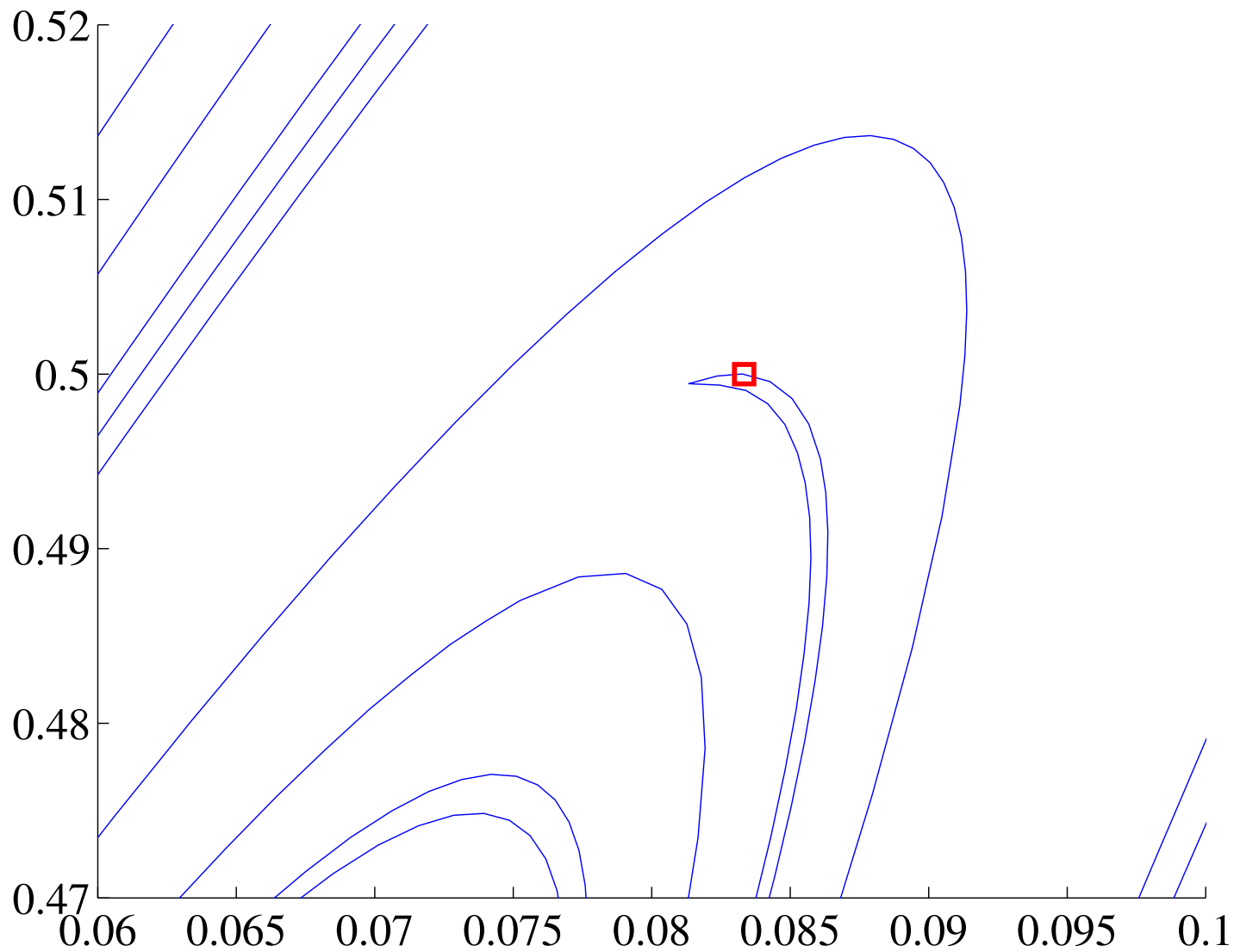
Almost all (99%) of the topological entropy is accounted for by this braid.



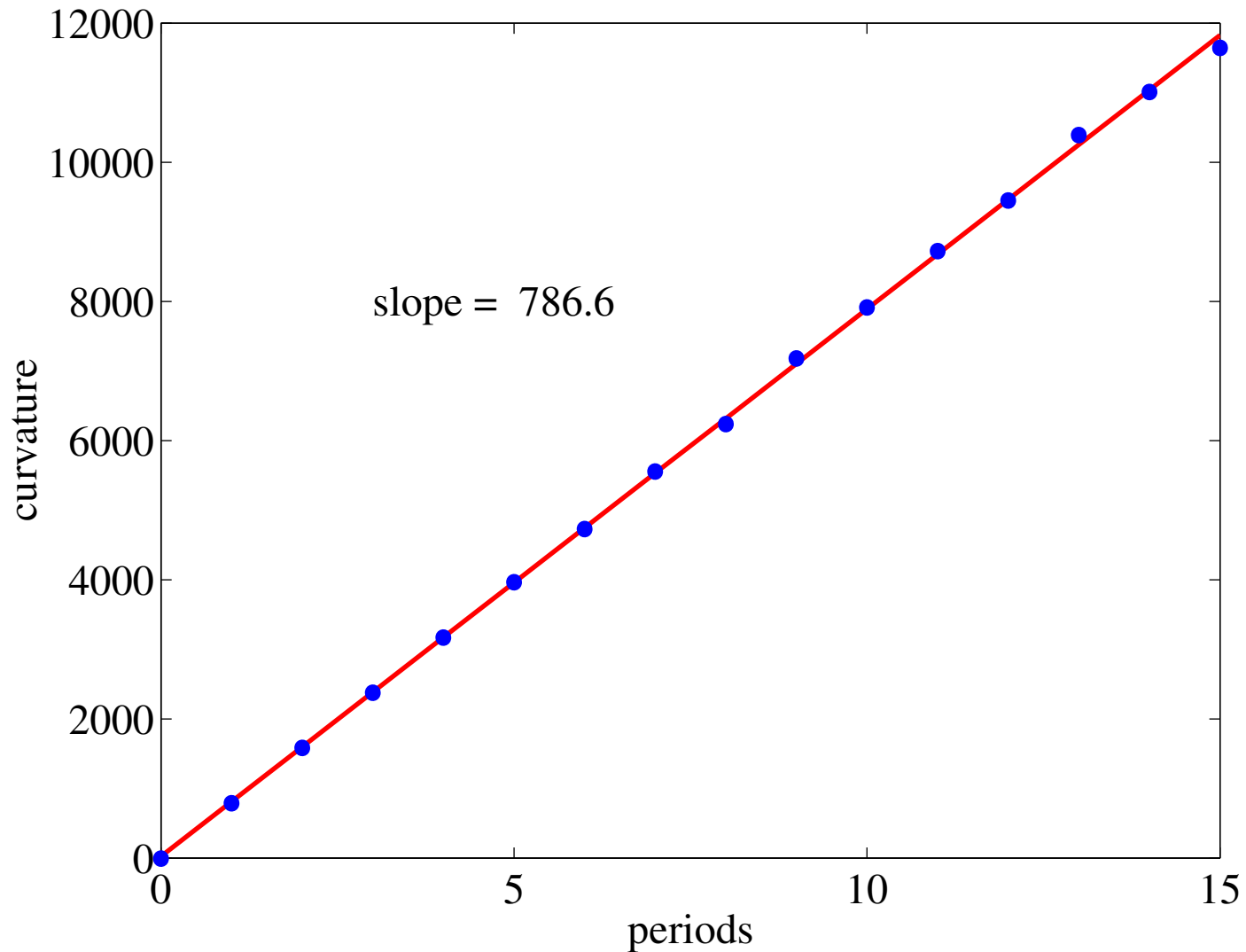
Periodic Orbits as Rods

[movie: sf_periodic.avi]

Blowup of the Tip



Curvature of the Tip



[Preliminary result: not sure this is generic.]

Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- The complexity of a braid can be represented by the largest eigenvalue of a product of matrices—the braiding factor.
- Any collection of n particles can potentially braid.
- The complexity of the braid is a good measure of chaos.
- No need for infinitesimal separation of trajectories or derivatives of the velocity field.
- Many issues to investigate: faithfulness of representation, improved lower bound for topological entropy, reducibility of braids, size of “ghost rod” for periodic orbits...
- See [J-LT, *Phys. Rev. Lett.* **94**, 084502 \(2005\)](#).