Topological Kinematics of Mixing

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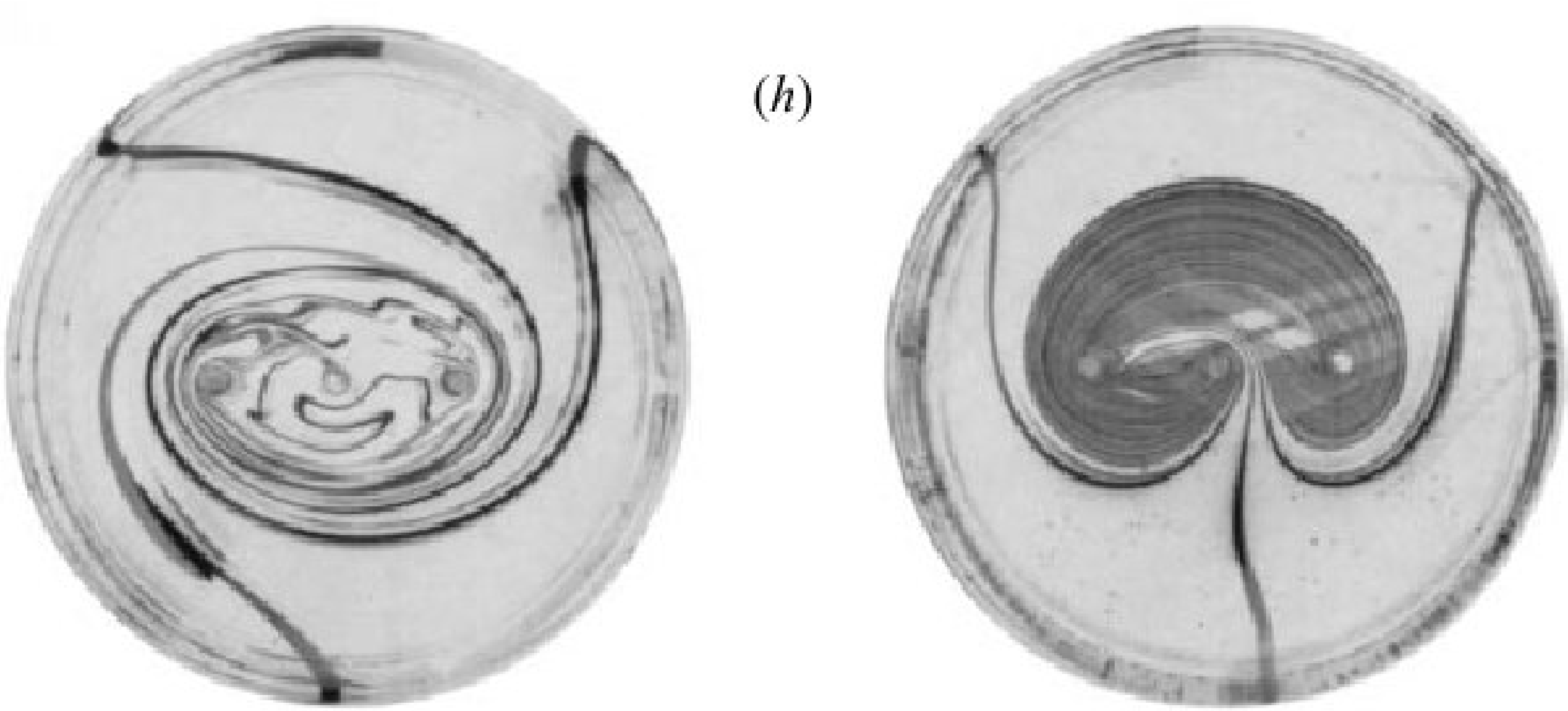
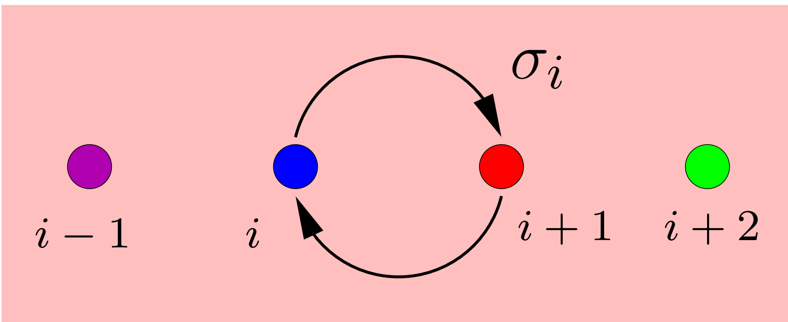
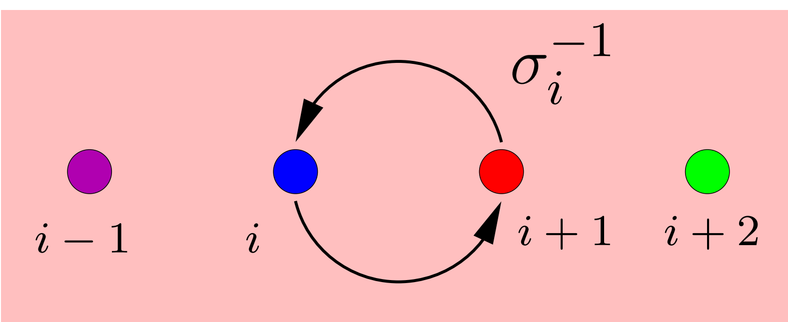
Experiment of Boyland *et al.*



[P. L. Boyland, H. Aref, and M. A. Stremler, J. Fluid Mech. **403**, 277 (2000)] [Movies by Matt Finn: boyland1 boyland2]

Generators of the n-Braid Group





A generator

$$\sigma_i\,, \quad i = 1, \ldots, n-1$$

is the clockwise interchange of the i th and $(i+1)$th rod.

The generators obey the presentation

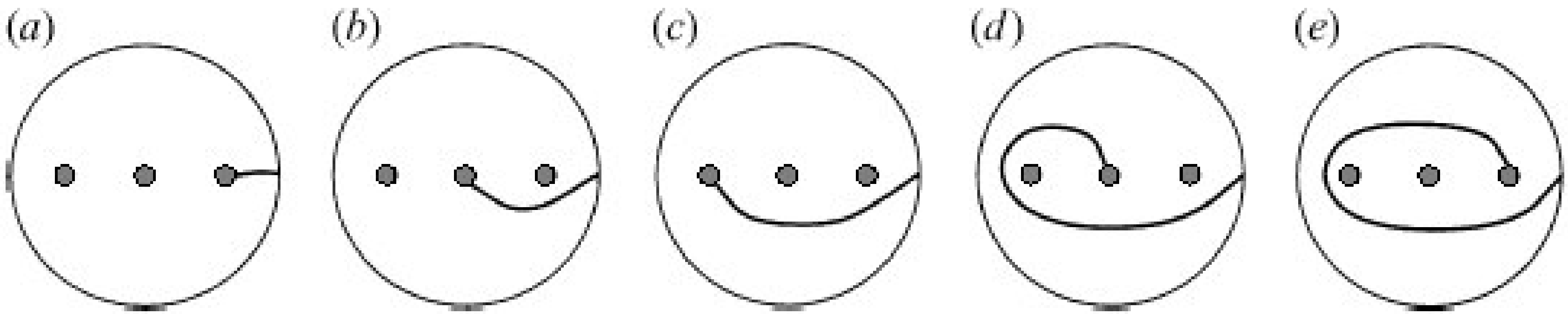
$$\sigma_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\sigma_i$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| > 1$$

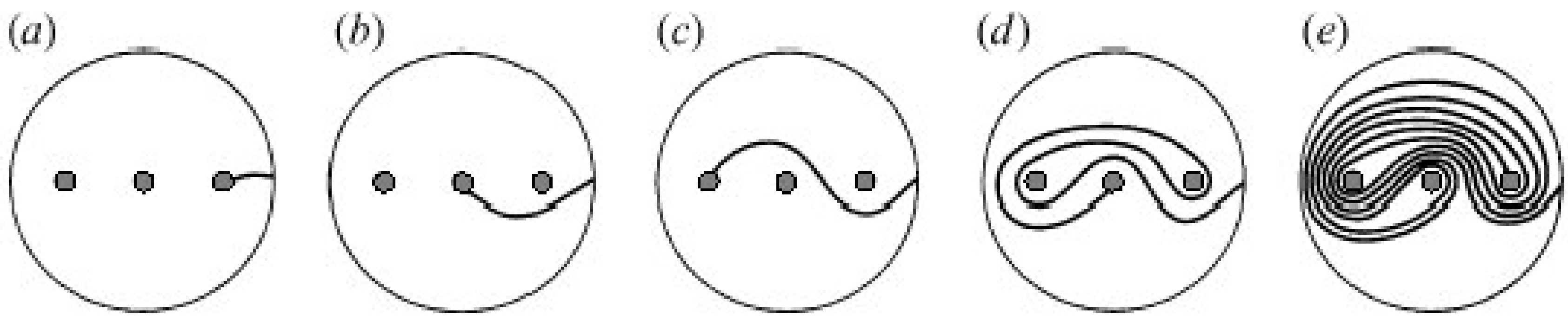
These generators are used to characterise the motion of the rods.

The Two BAS Stirring Protocols

$\sigma_1\sigma_2$ protocol



$\sigma_1^{-1}\sigma_2$ protocol

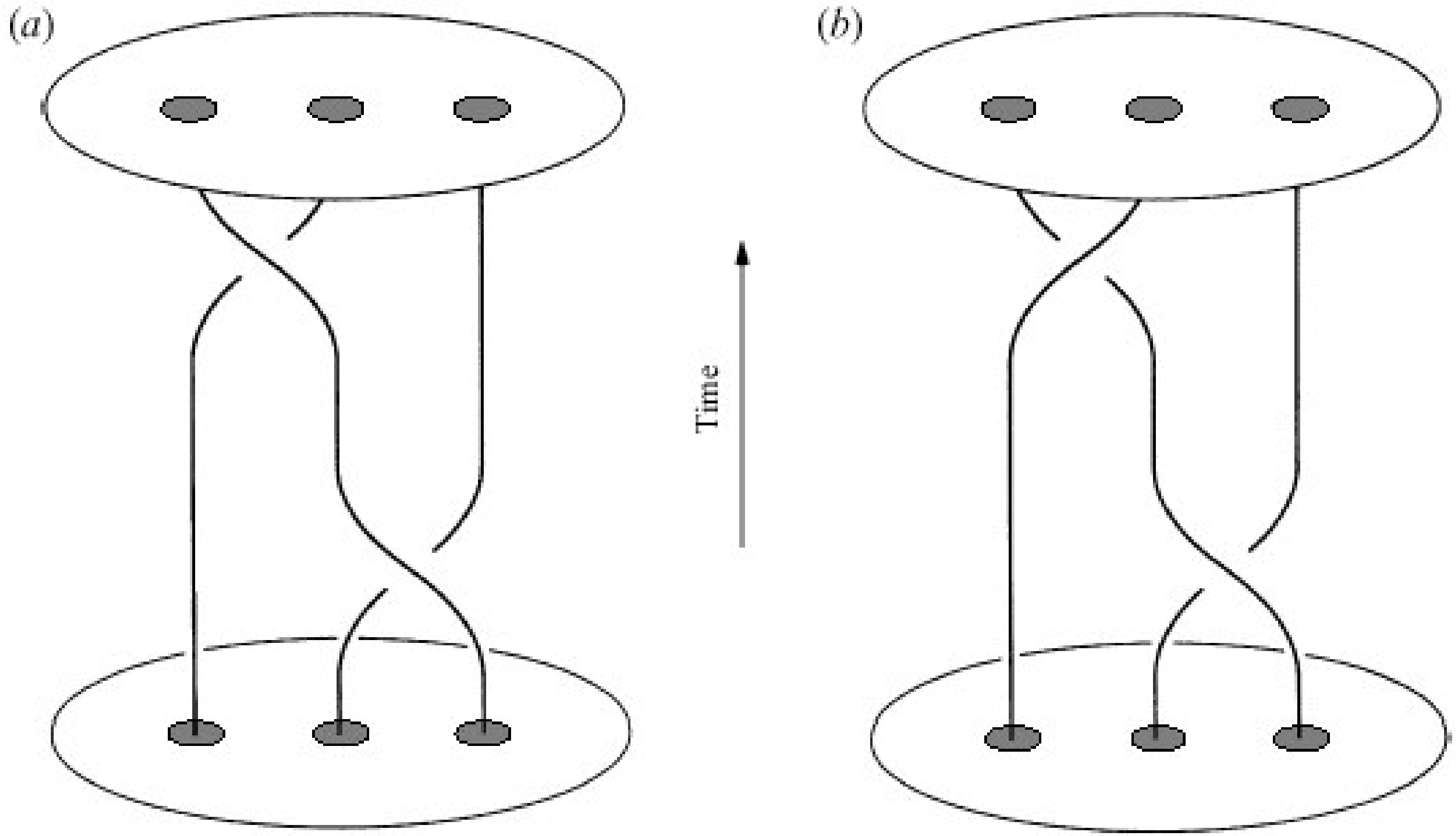


[P. L. Boyland, H. Aref, and M. A. Stremler, J. Fluid Mech. **403**, 277 (2000)]

Rod Trajectories as Braids

$\sigma_1 \sigma_2$ protocol

$\sigma_1^{-1} \sigma_2$ protocol



[P. L. Boyland, H. Aref, and M. A. Stremler, J. Fluid Mech. **403**, 277 (2000)]

Matrix Representation of the σ_i

We use the convenient Burau representation

$$\sigma_i = I_{i-2} \oplus \begin{pmatrix} 1 & -\tau & 0 \\ 0 & -\tau & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

where $\tau \in \mathbb{C}$. The matrices are $(n-1) \times (n-1)$.

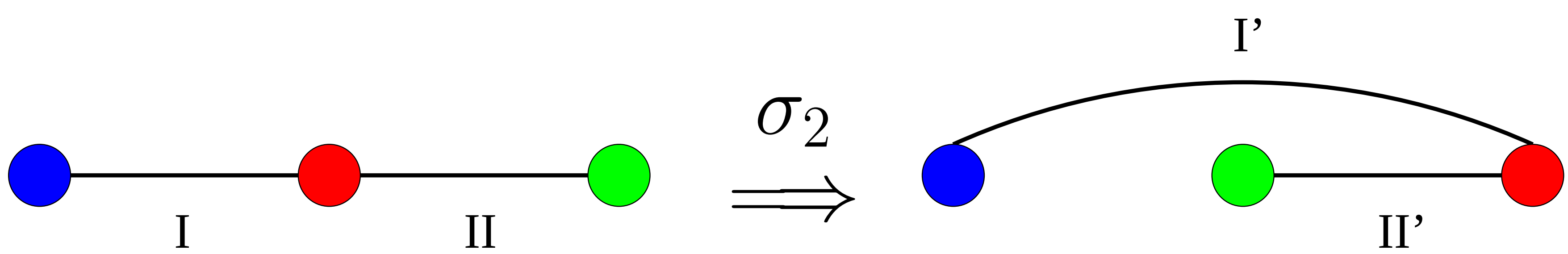
The Burau matrices satisfy the presentation (of course), but for $n > 4$ they do not provide a faithful representation.

This is of no great consequence for our purposes.

Topological Entropy of a Braid

Practically speaking, the topological entropy of a braid is a lower bound on the line-stretching exponent of the flow!

This is Eminently Reasonable™:



The Burau representation has an awesome property: if Σ is the Burau representation of a braid word,

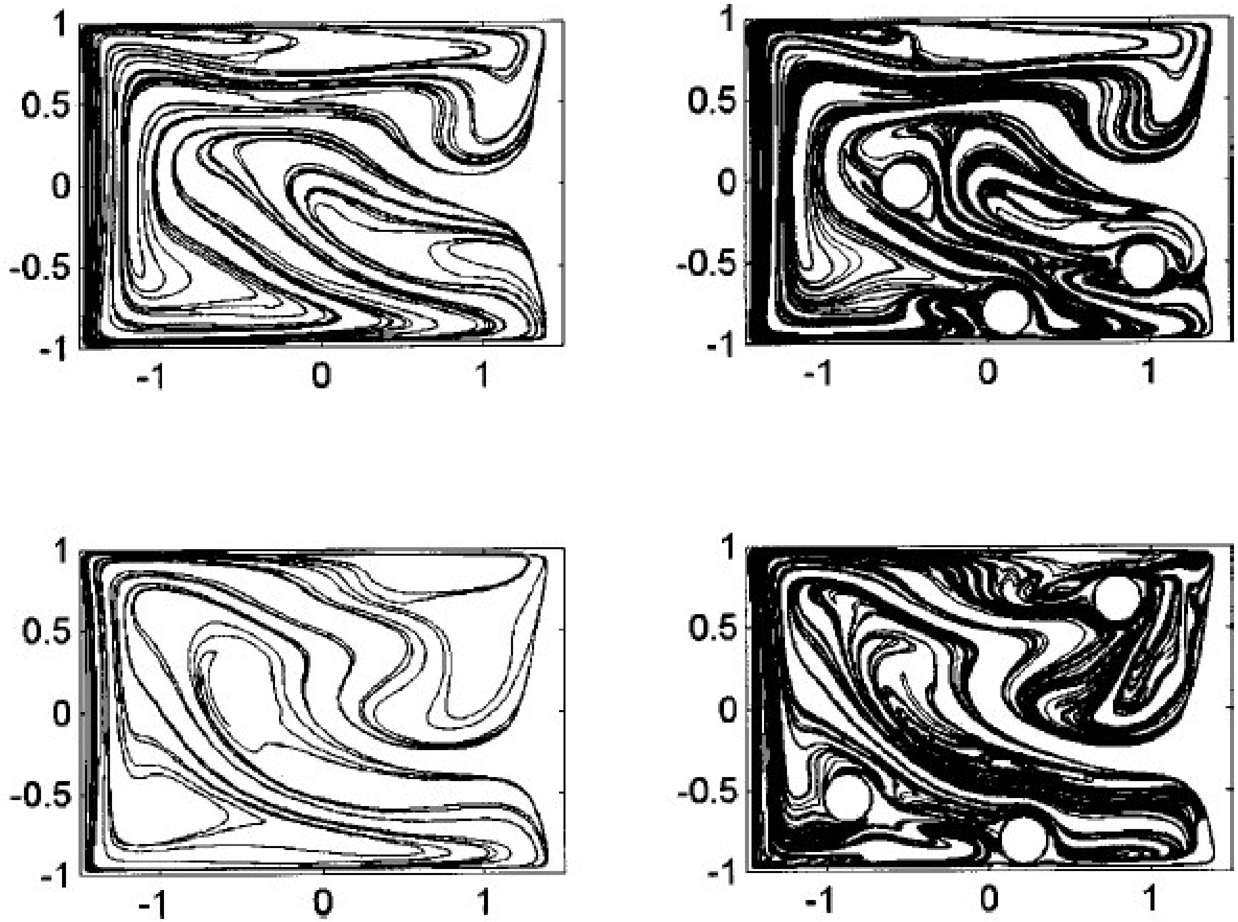
$$\text{topological entropy of braid} \geq \sup_{|\tau|=1} \operatorname{spr} \Sigma \qquad \text{[Kolev]}$$

where spr denotes the spectral radius (the magnitude of the largest eigenvalue).

The Difference between BAS's Two Protocols

- The matrices associated the generators have eigenvalues on the unit-circle.
- The first (bad) stirring protocol has eigenvalues on the unit circle
- The second (good) protocol has largest eigenvalue $(3+\sqrt{5})/2 = 2.6180$.
- So for the second protocol the length of a line joining the rods grows exponentially!
- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).

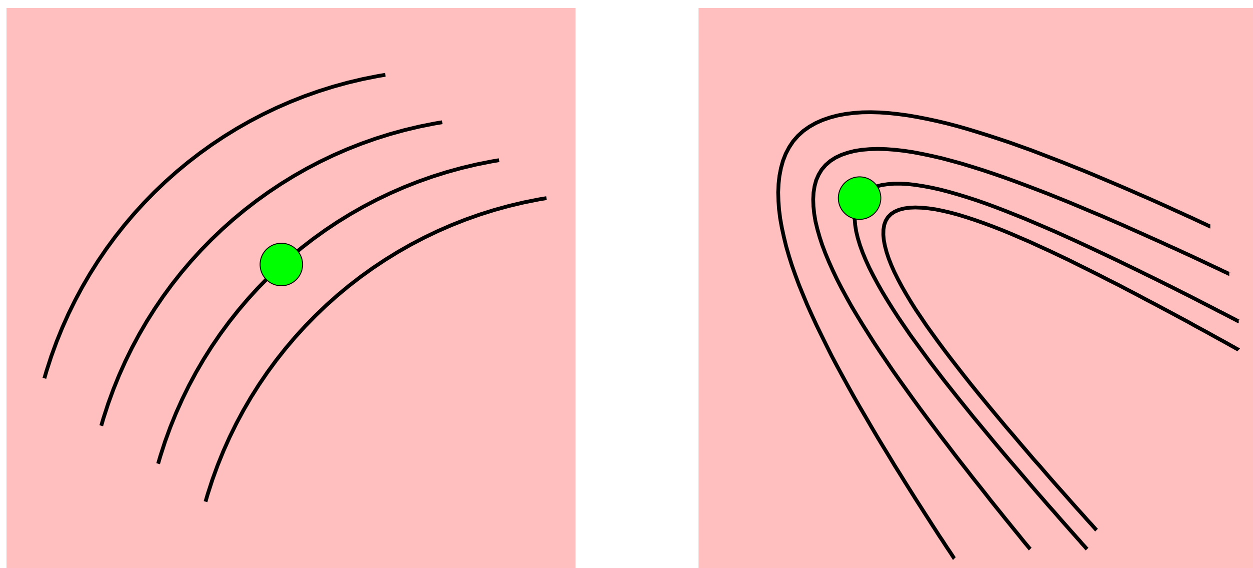
Freely-moving Rods in a Cavity Flow



[A. Vikhansky, Physics of Fluids **15**, 1830 (2003)]

Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).

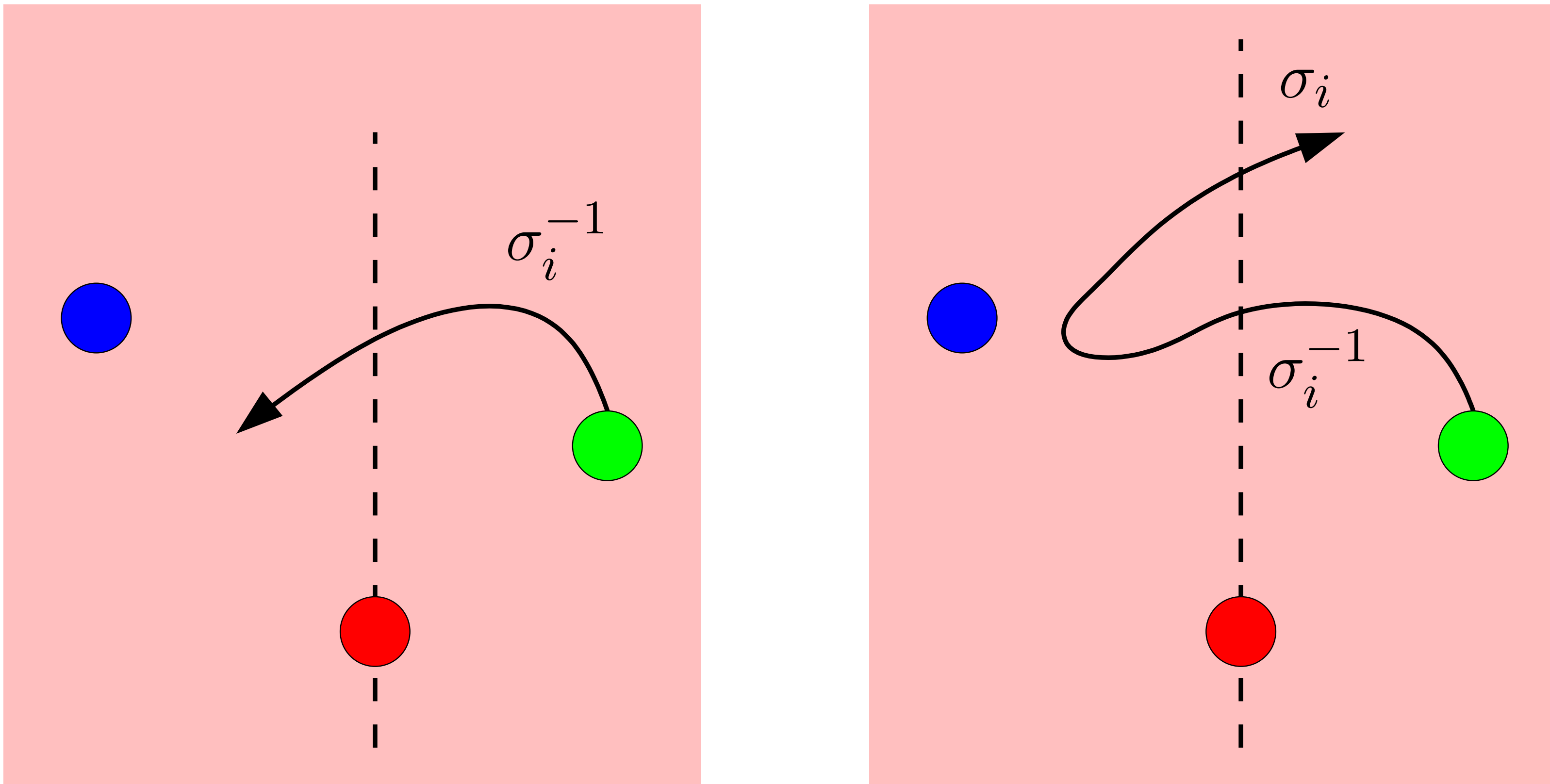


Material lines must bend around the orbit: it acts just like a “rod”!

The idea: pick any n fluid particles and follow them.

How do they braid around each other?

Detecting Braiding Events



In the second case there is no net braid: the two elements cancel each other.

Random Sequence of Braids

We end up with a sequence of braids, with matrix representation

$$\Sigma^{(N)} = \sigma^{(N)} \cdots \sigma^{(2)} \sigma^{(1)}$$

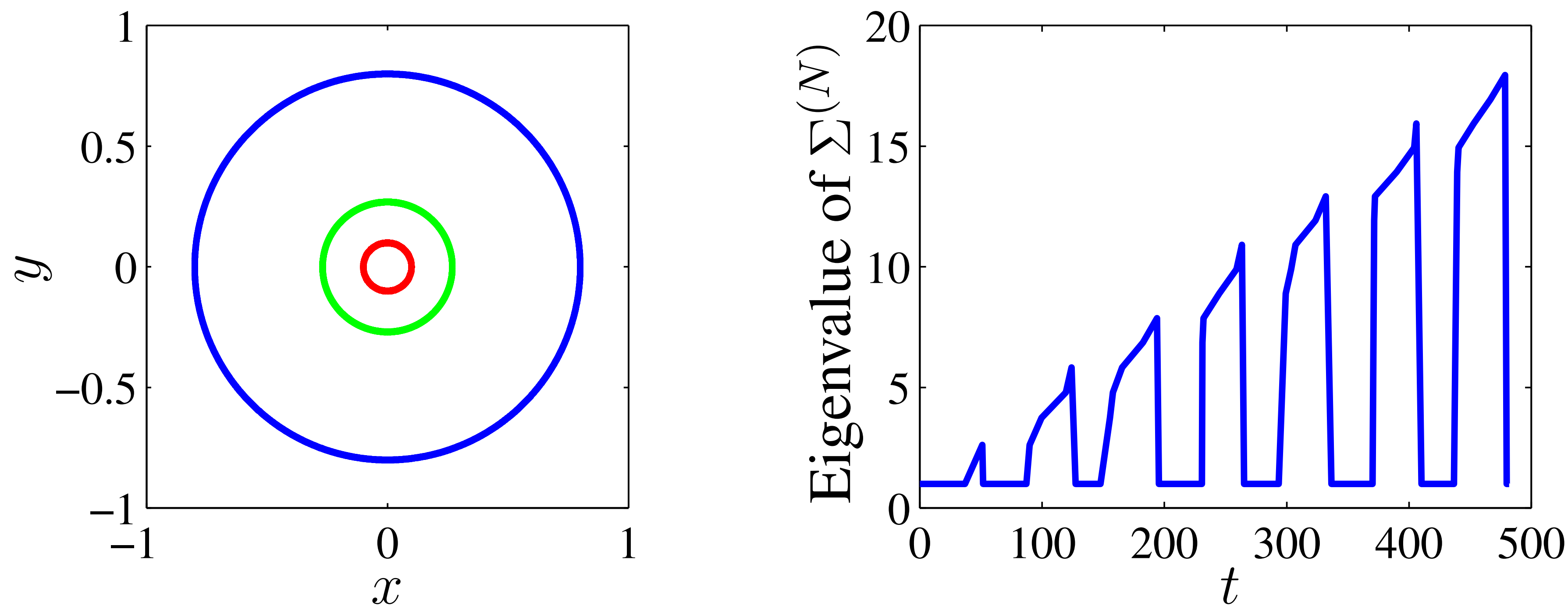
where $\sigma^{(\mu)} \in \{\sigma_i, \sigma_i^{-1}\}$ and N is the number of braiding events detected after a time t.

The largest eigenvalue of $\Sigma^{(N)}$ is a measure of the complexity of the braiding motion, called the braiding factor.

Random matrix theory says that the braiding factor can grow exponentially! We call the rate of exponential growth the braiding Lyapunov exponent or just braiding exponent.

Non-braiding Motion

First consider the motion of of three points in concentric circles with irrationally-related frequencies.

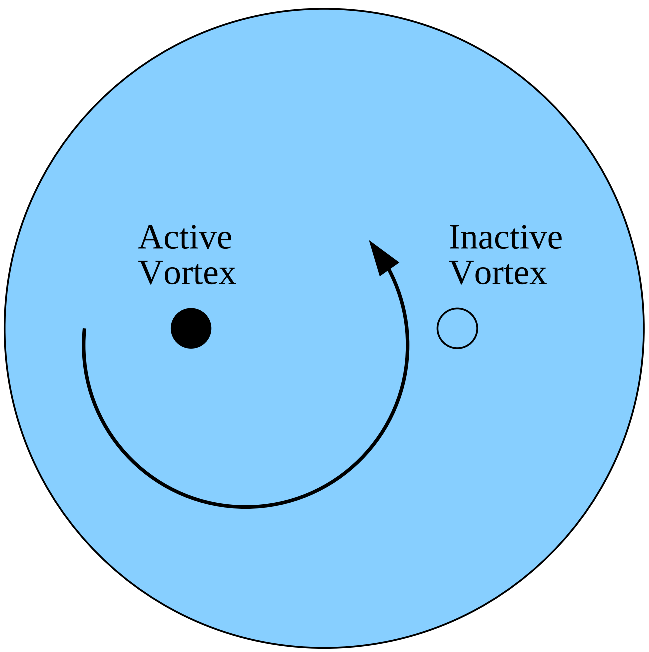


The braiding factor grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.

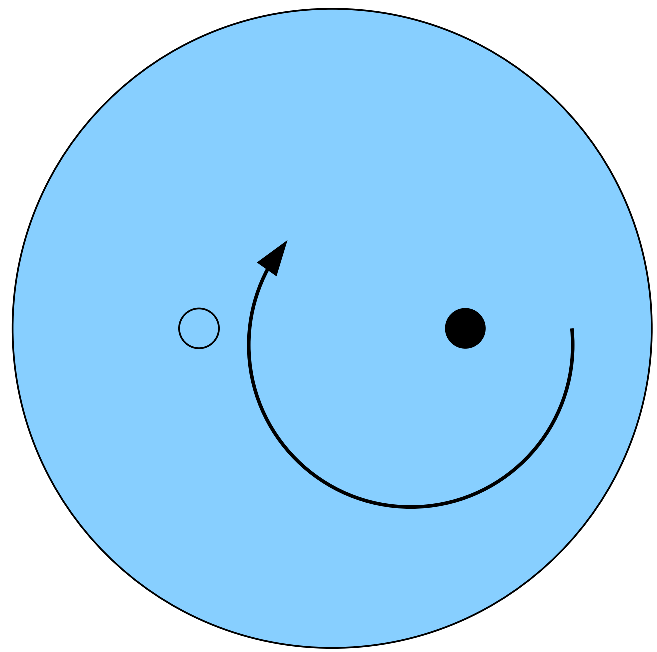
Blinking-vortex Flow

To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won't do).

Aref's blinking-vortex flow is ideal.



First half of period

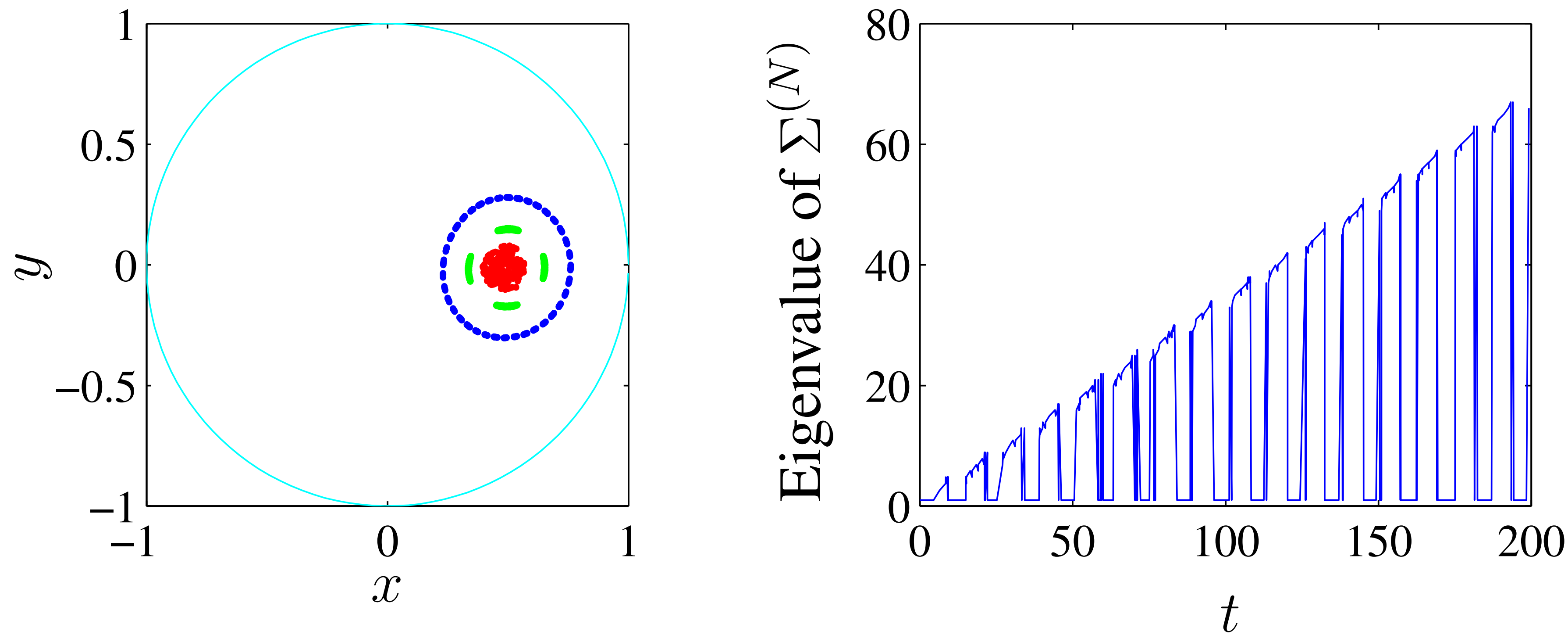


Second half of period

The only parameter is the circulation Γ of the vortices.

Blinking Vortex: Non-braiding Motion

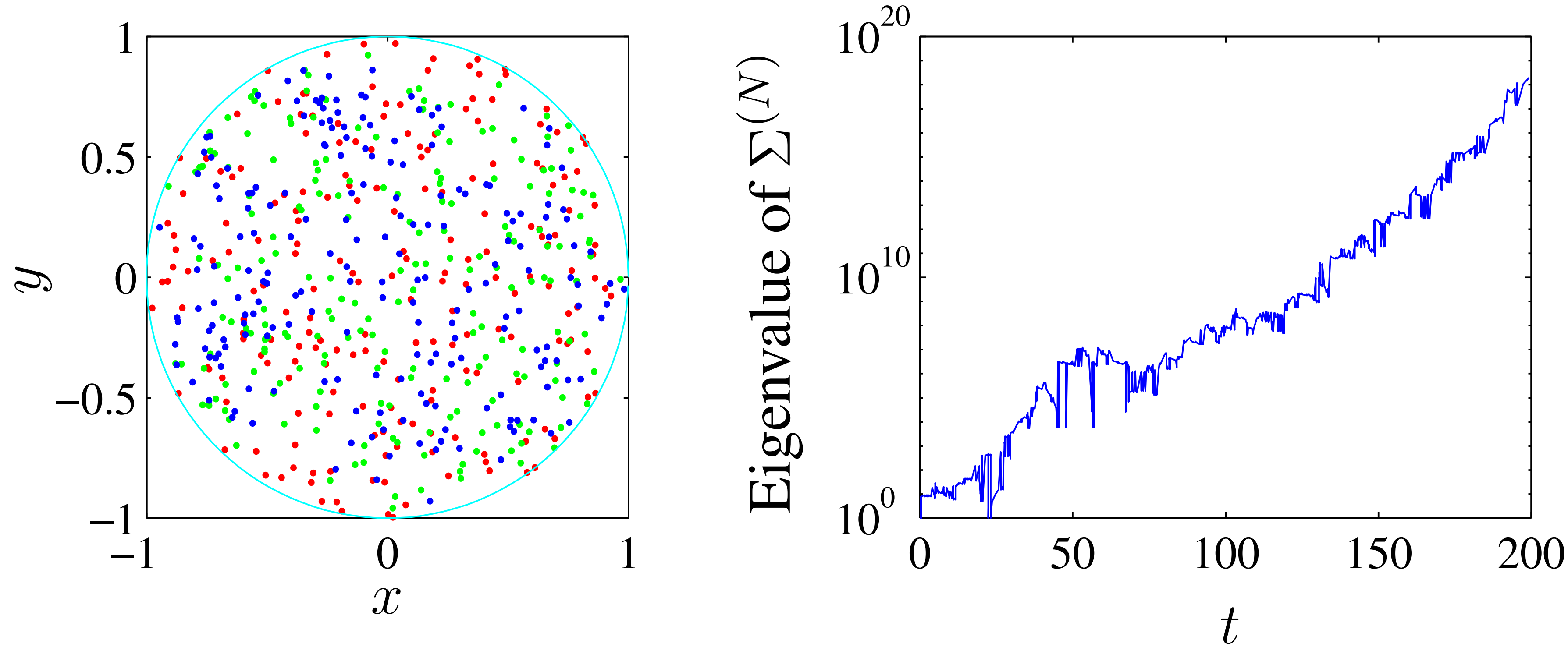
For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.



One of the orbits is chaotic, the other two are closed.

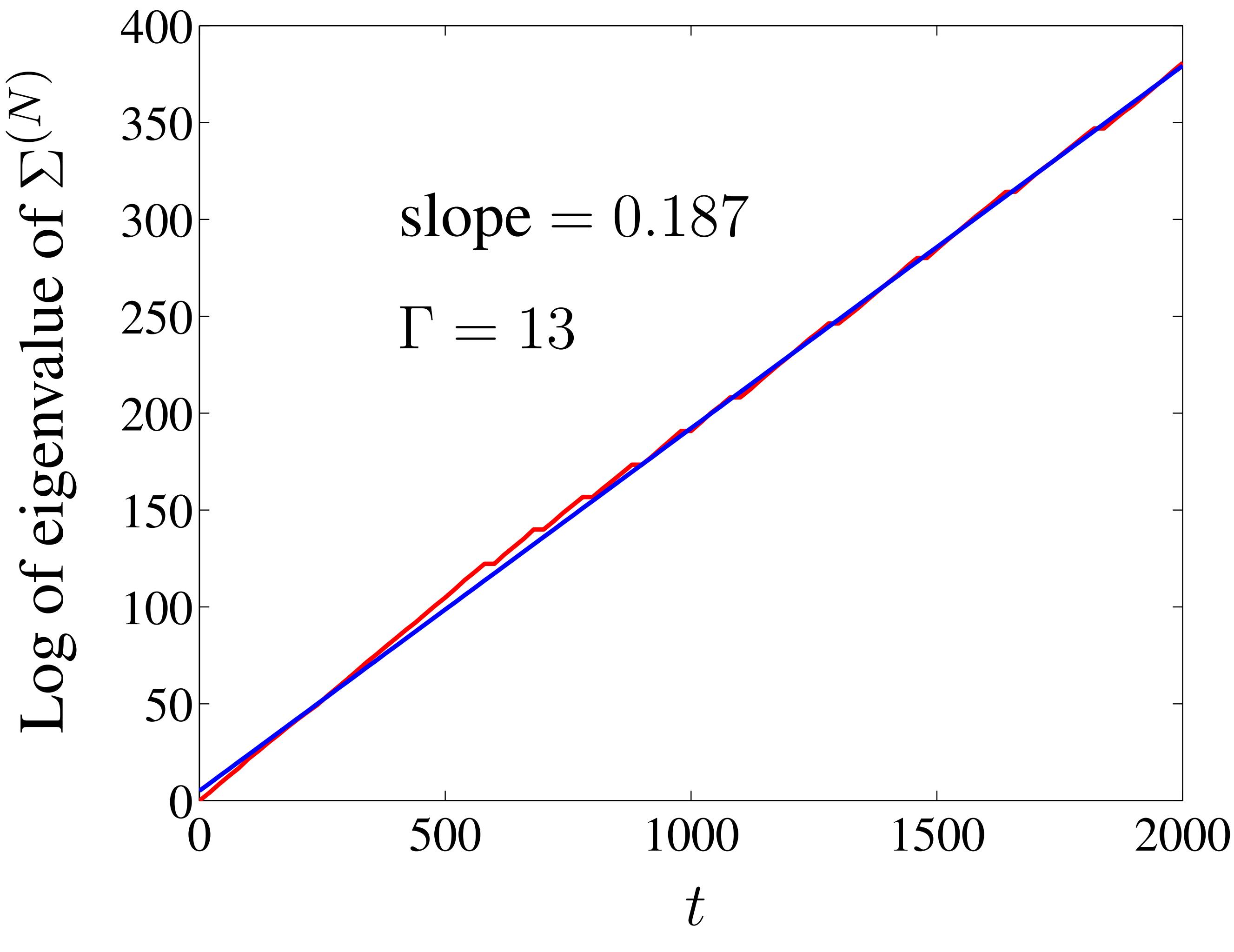
Blinking Vortex: Braiding Motion

For $\Gamma = 13$, the blinking vortex is globally chaotic.



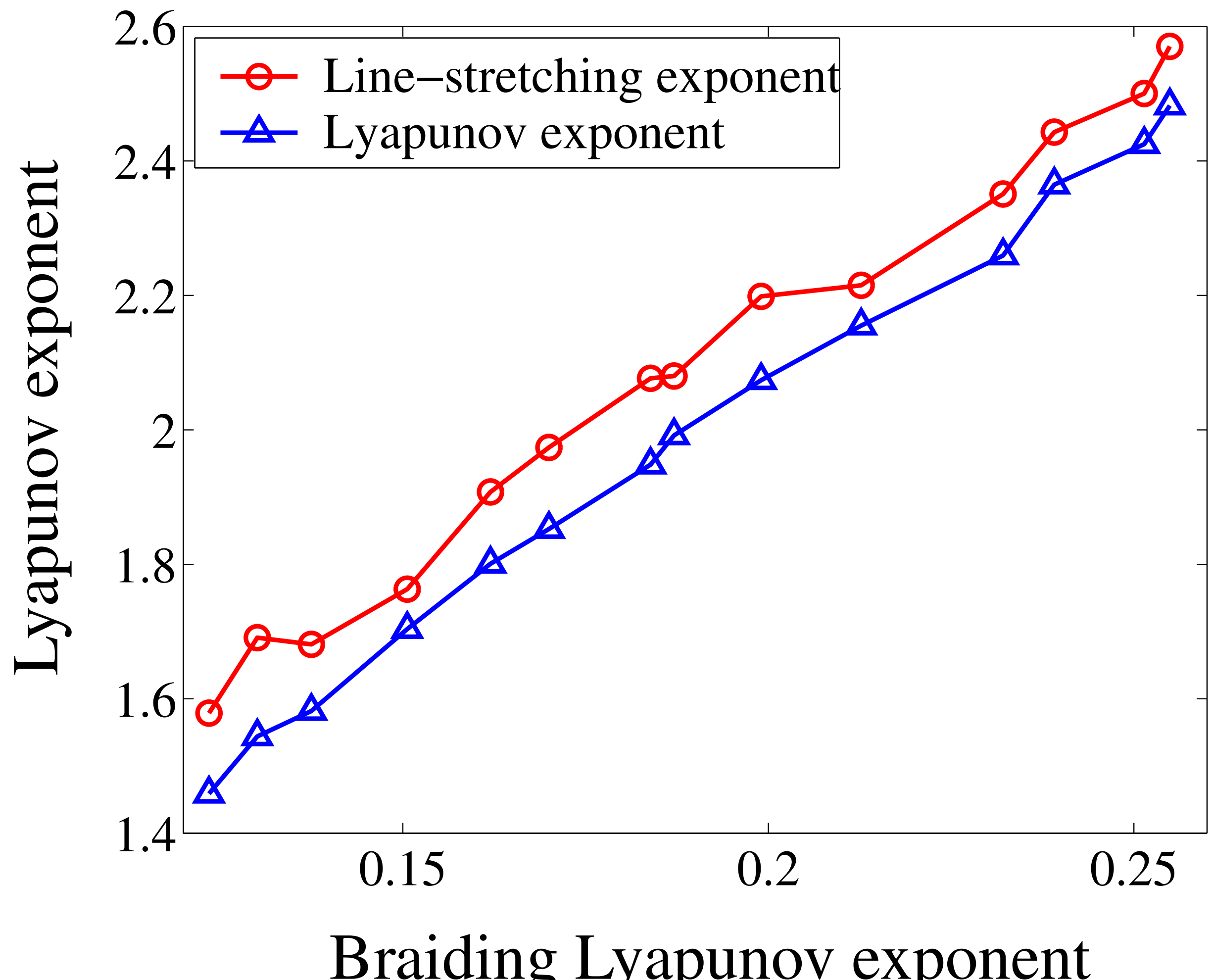
The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order 10^{20} rather than 80!

Averaging over many Triplets



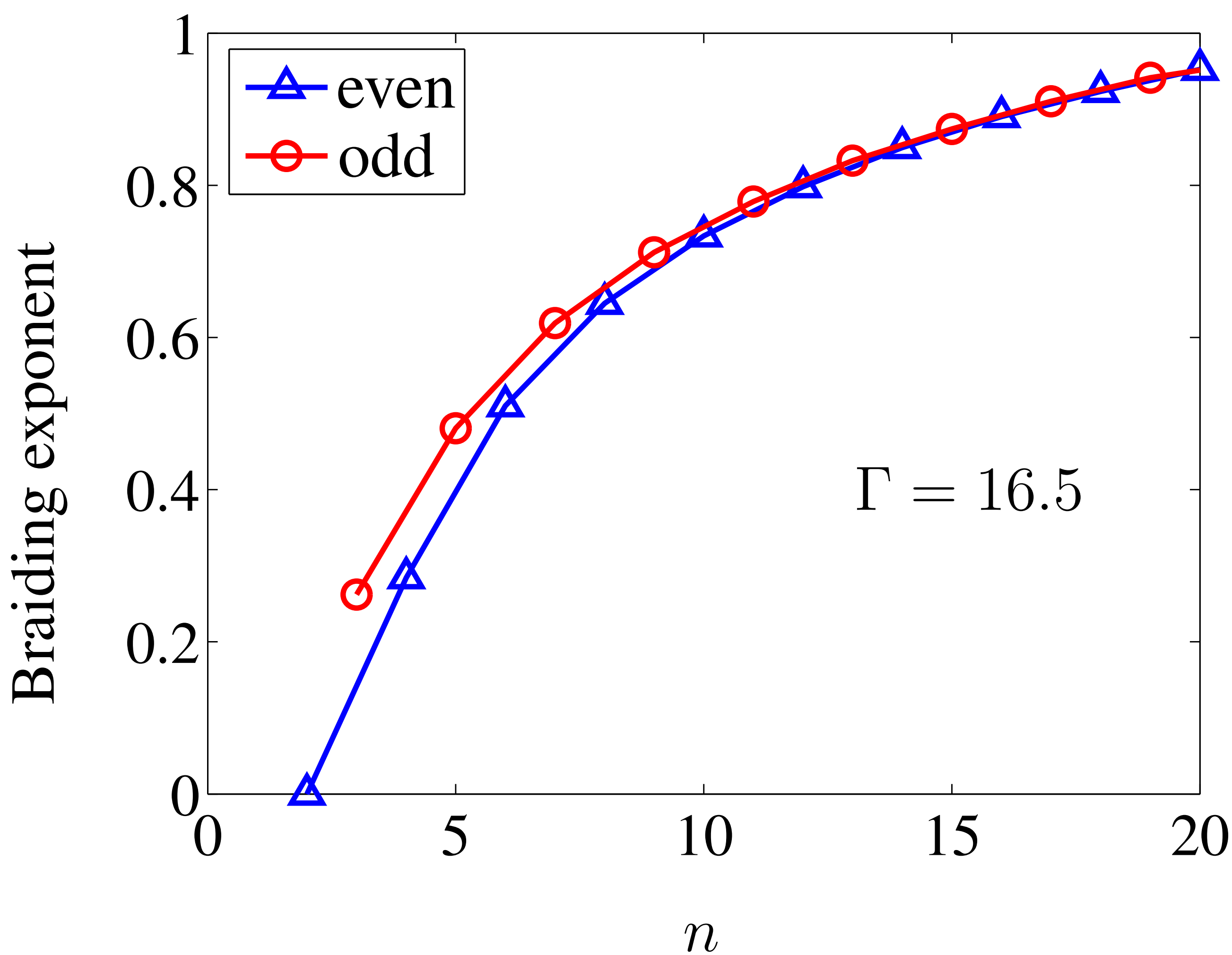
Averaged over 100 random triplets.

Comparison with Lyapunov Exponents

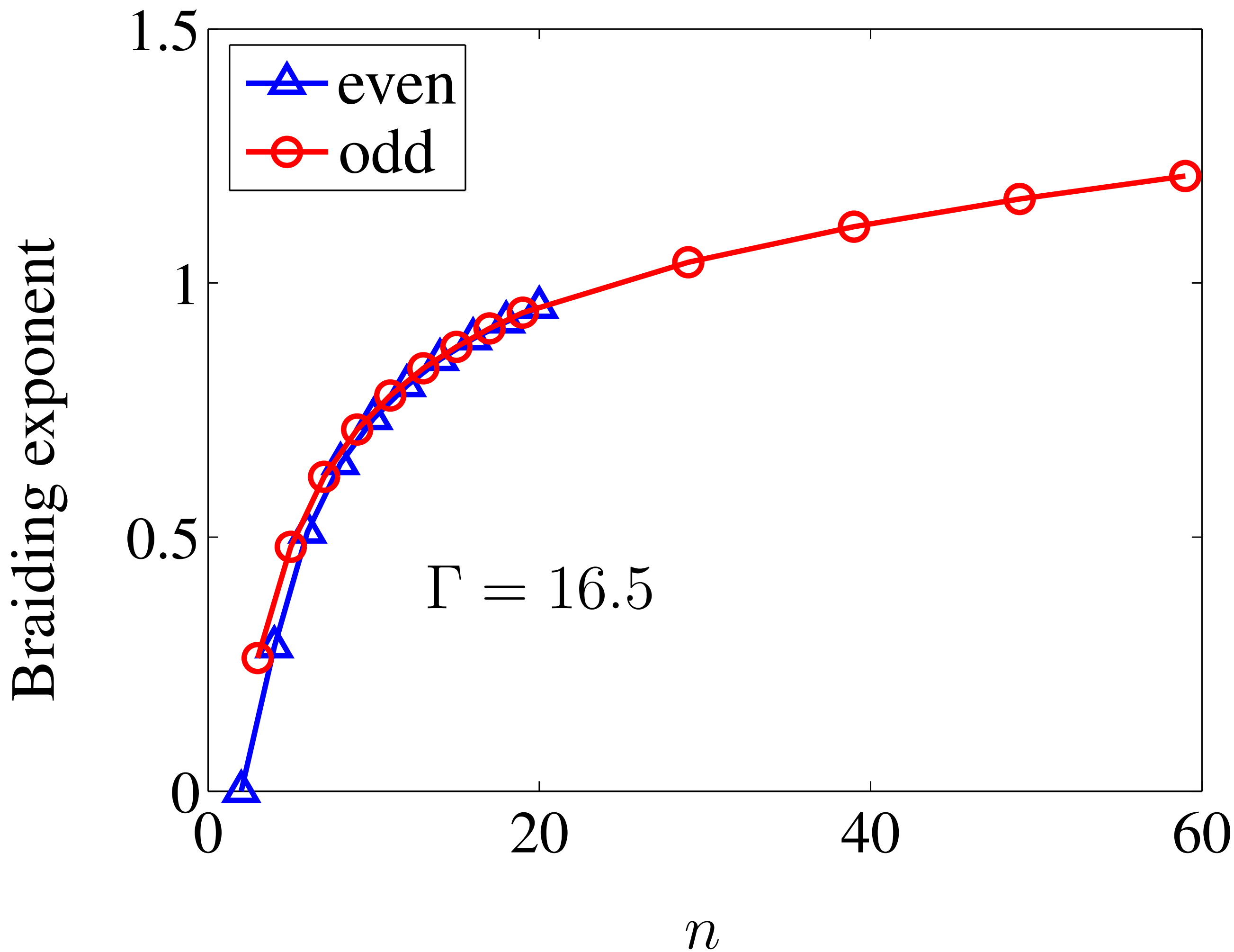


Γ varies from 8 to 20.

Beyond Three Particles



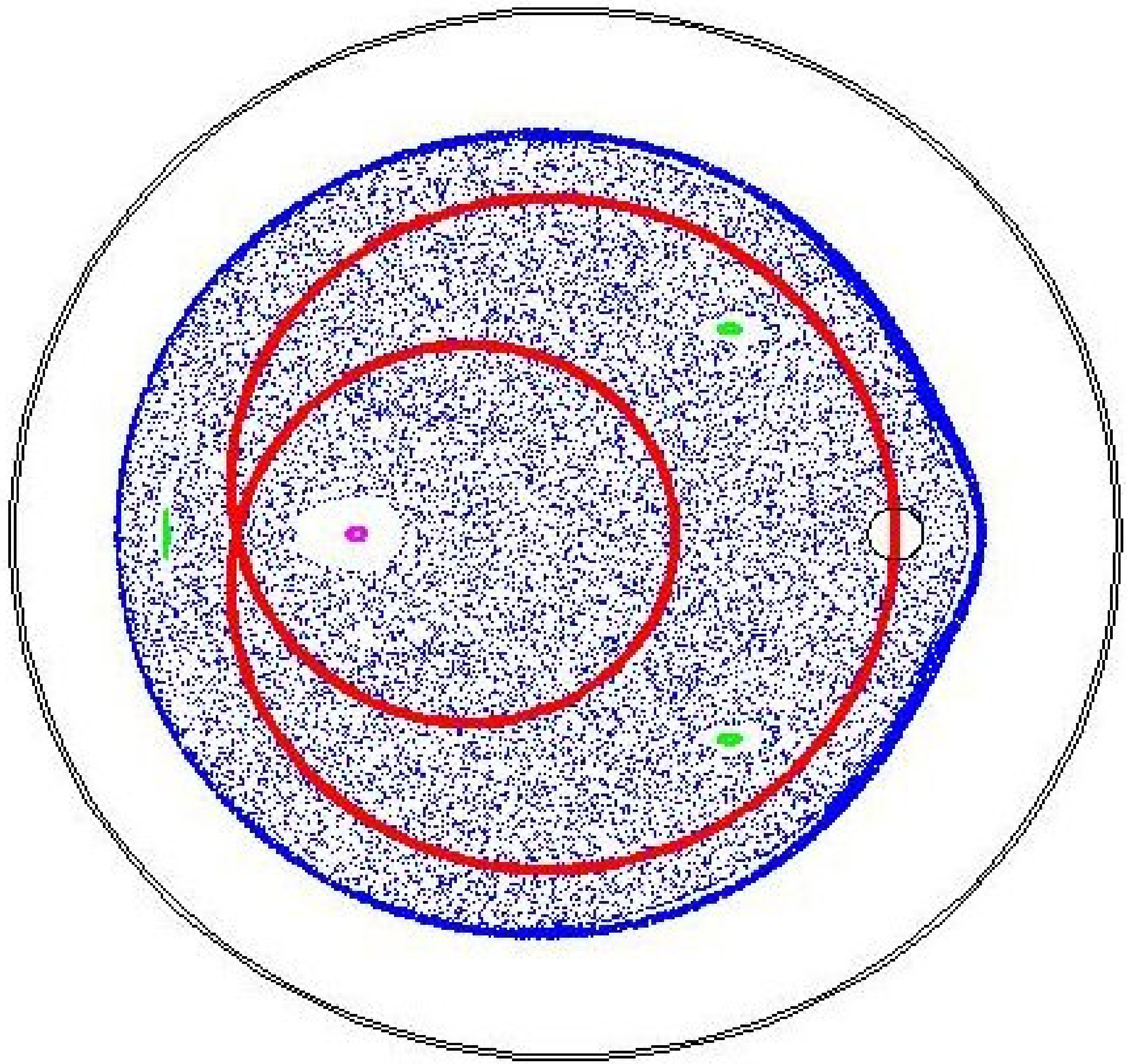
But does it Saturate?



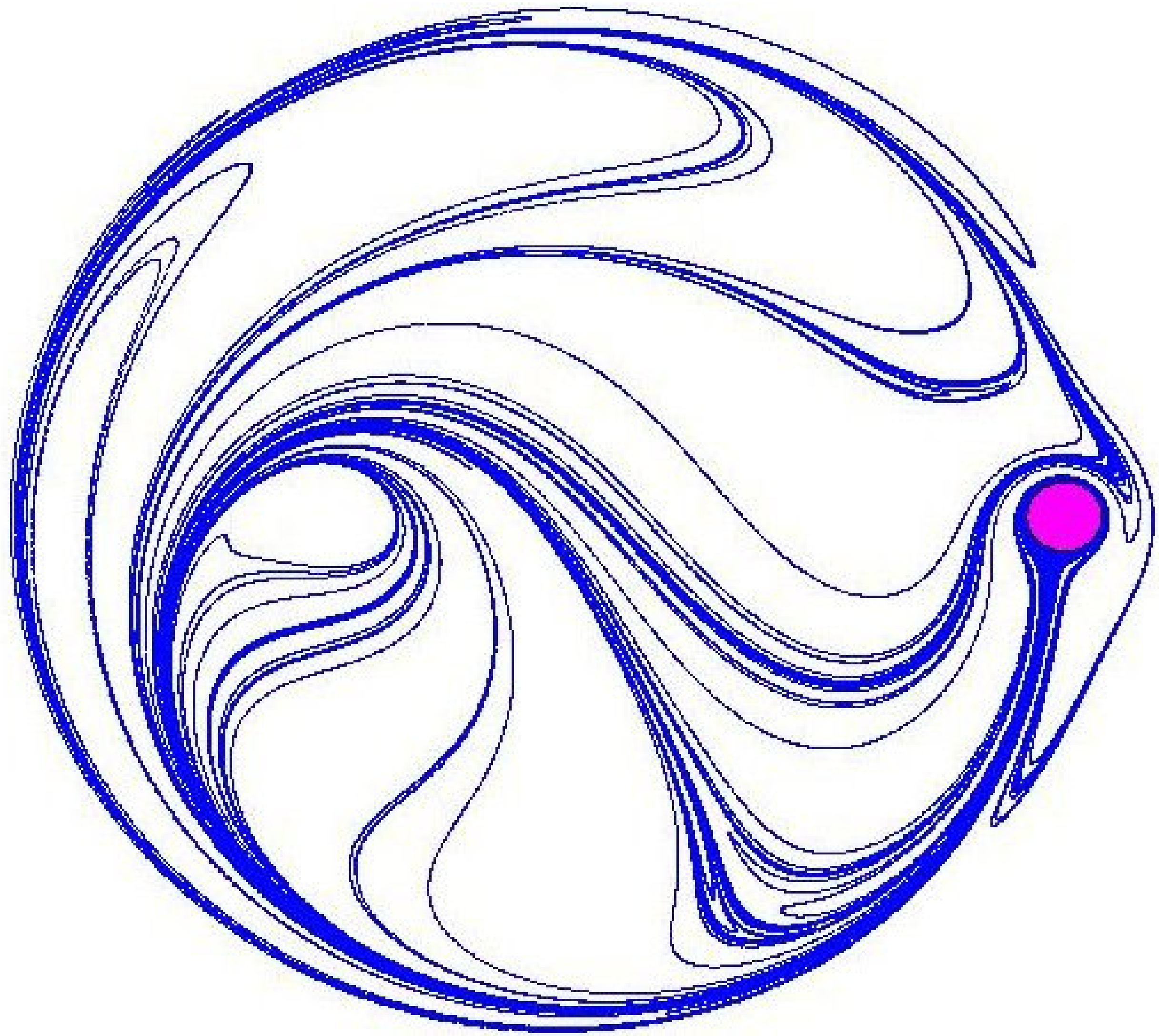
Well, it really should…

One Rod Mixer: The Kenwood Chef

Poincaré Section



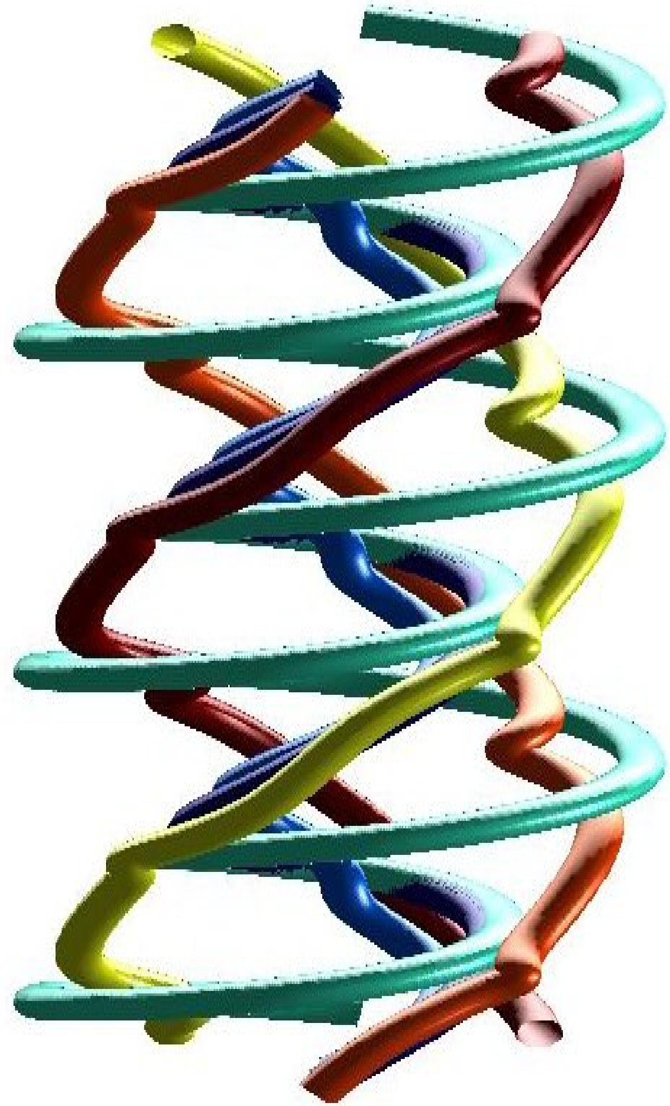
Stretching of Lines



Motion of Islands

Make a braid from the motion of the rod and the periodic islands.

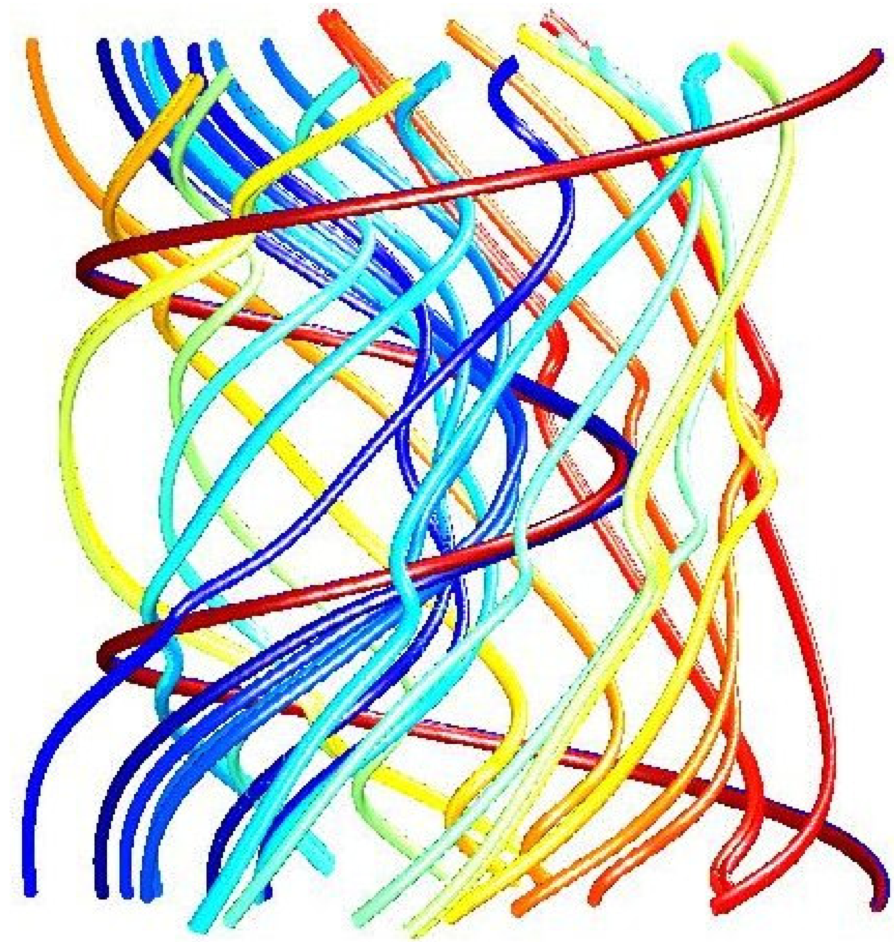
Most (74%) of the topological entropy is accounted for by this braid.



Motion of Islands and Unstable Periodic Orbits

Now we also include unstable periods orbits as well as the stable ones (islands).

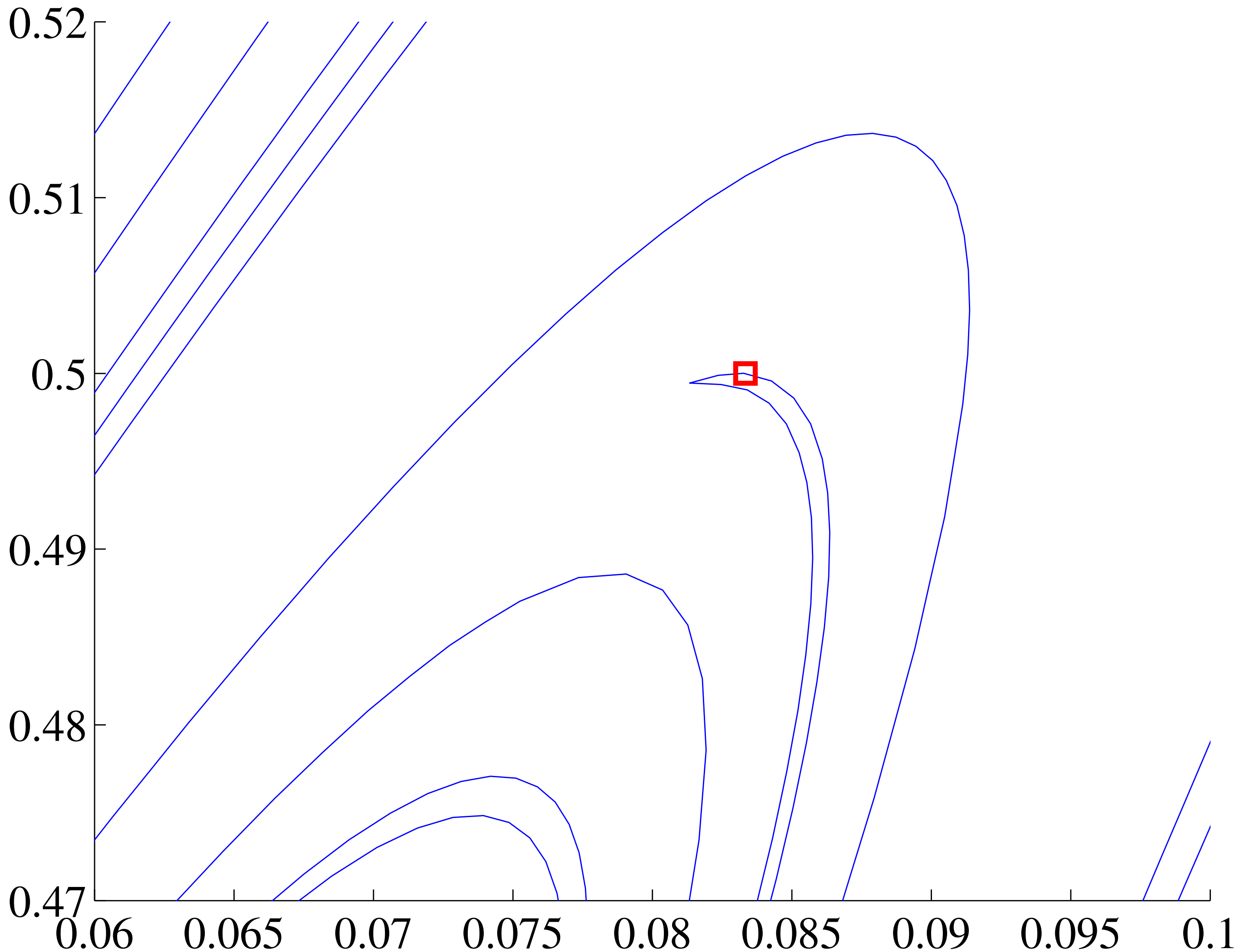
Almost all (99%) of the topological entropy is accounted for by this braid.



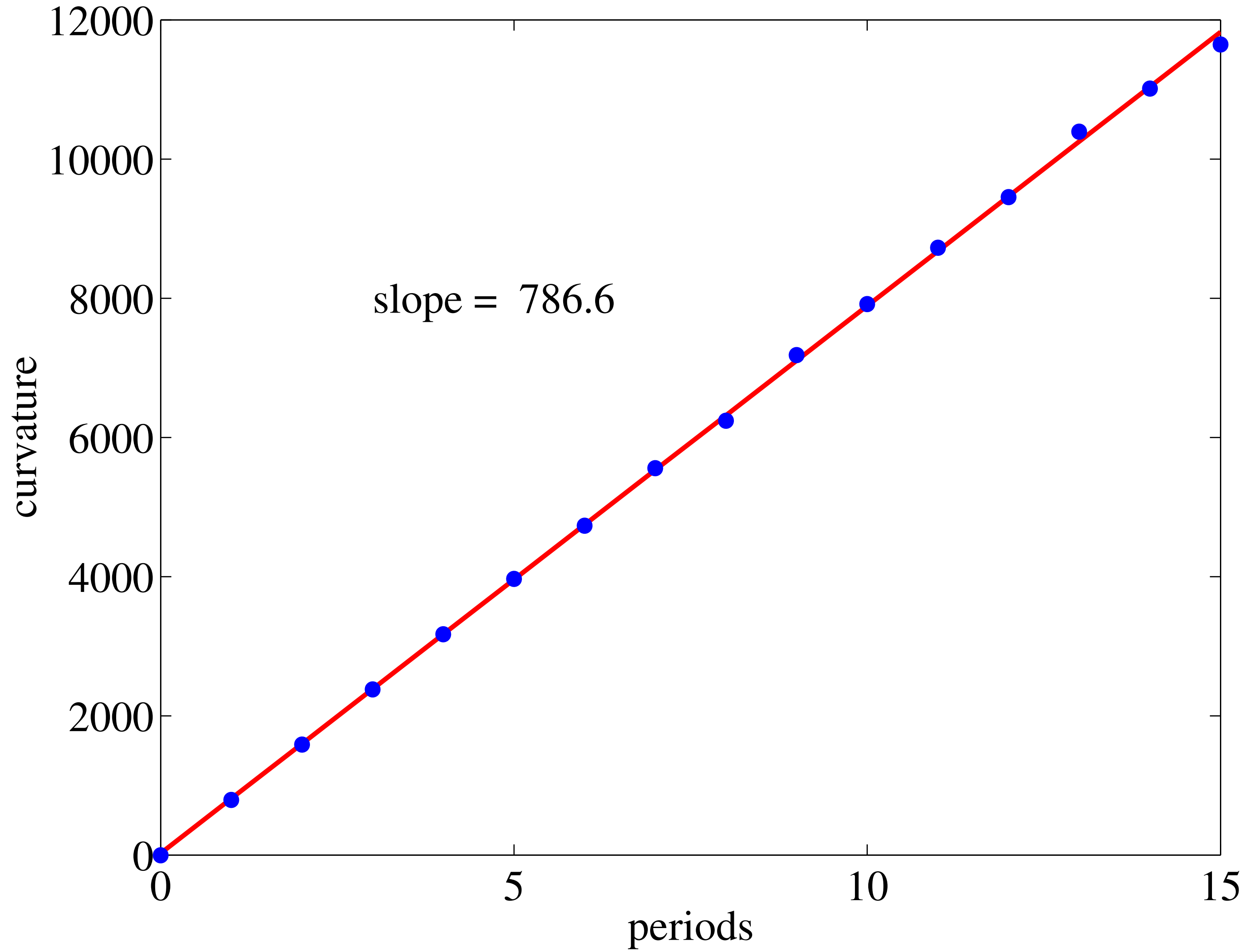
Periodic Orbits as Rods

[movie: sf_periodic.avi]

Blowup of the Tip



Curvature of the Tip



[Preliminary result: not sure this is generic.]

Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- The complexity of a braid can be represented by the largest eigenvalue of a product of matrices—the braiding factor.
- Any collection of n particles can potentially braid.
- The complexity of the braid is a good measure of chaos.
- No need for infinitesimal separation of trajectories or derivatives of the velocity field.
- Many issues to investigate: faithfulness of representation, improved lower bound for topological entropy, reducibility of braids, size of "ghost rod" for periodic orbits…
- See J-LT, *Phys. Rev. Lett.* **94**, 084502 (2005).