Computing the Topological Entropy of Braids, Fast

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Surface Diffeomorphisms

- We consider a family of smooth diffeos $f_t : M \to M$, parametrised by $t \in [0, 1]$.
- f_1 leaves invariant n punctures in the surface M.
- The motion of the punctures traces a braid.
- This braid implies a minimum topological entropy for f_1.
- Given only the abstract braid, in terms of group generators, how do we compute its topological entropy?
- We want to do this for very large braids, so we need a method that is computationally efficient.

Topological Entropy

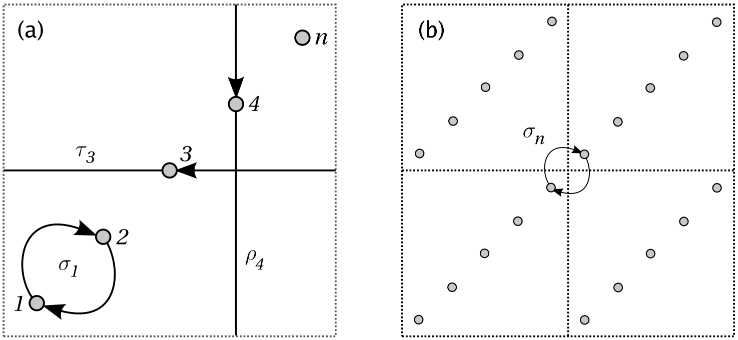
- Can compute the topological entropy in several ways:
  - Train-tracks algorithm such as Bestvina–Handel (1995).
  - (Poor) lower bound using Burau representation (Kolev 1989).
- Implementation of B–H algorithm due to Toby Hall takes Artin braid group generators as input.
- Prohibitive for even a small (~ 40) number of generators.
- No simple implementation for the torus.

The Algorithm of Moussafir

- Recently, Moussafir (2006) introduced a fast method that converges to the exact entropy of a braid.
- Uses Dynnikov coordinates (2002) to encode the number of crossings between a lamination and a reference line.
- Less information than train tracks, but at lower cost.
- Here: derive such an algorithm for braids on the torus.
- Allows a topological analysis of bi-periodic systems, such as the sine flow.

Braid Group Generators

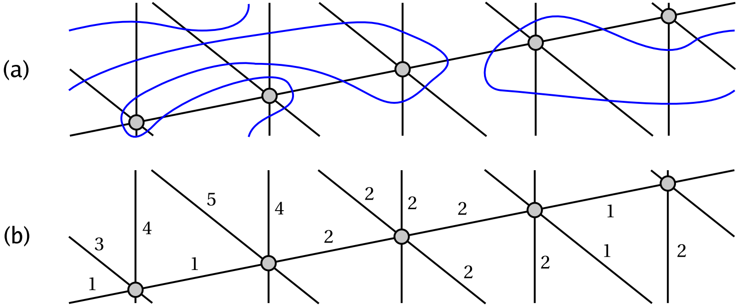
Birman (1969) defines three types of generators for the braid group on the torus.



The usual Artin braid group is a subgroup that only uses σ.

Lamination and Triangulation

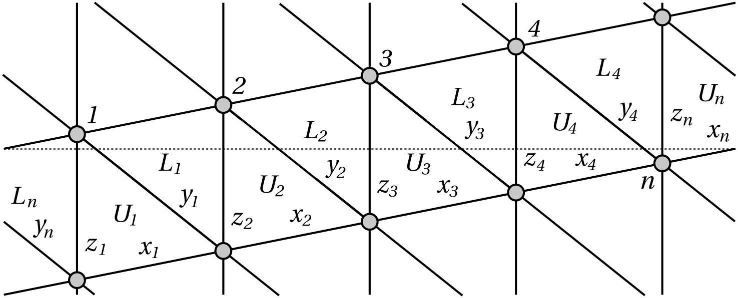
A lamination is an equivalence class of simple closed curves.



The triangulation links the punctures, but is not unique. We count the crossings between the lamination and triangulation.
The triangulation is static, but the lamination is evolved under the diffeomorphism.

Triangulation (2)

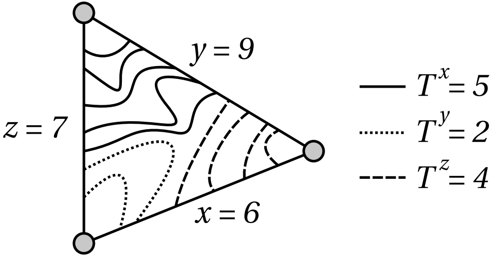
We often show multiple copies of the domain (universal cover). A dotted line separates copies.



U and L label triangles, x, y, z are crossing numbers.

Edge-linking Numbers

Any part of the lamination passing between edges x and y is counted by T^z, similarly for T^x and T^y.



$$T^x = \frac{1}{2}(y + z - x)$$
$$T^y = \frac{1}{2}(x + z - y)$$
$$T^z = \frac{1}{2}(x + y - z)$$

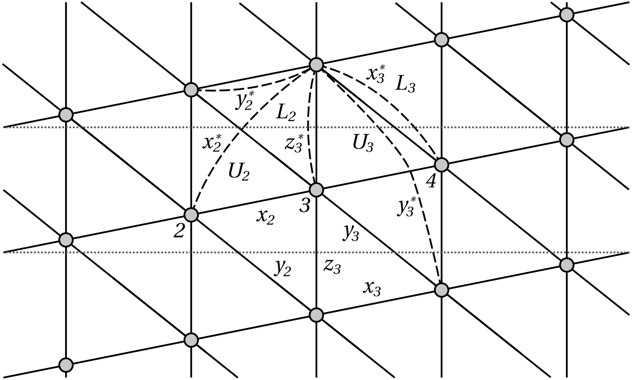
The lamination is always assumed to be 'pulled tight.'

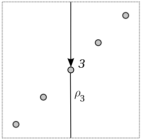
Crossing Update Rules

- The diffeomorphism changes the lamination as described using braid group generators for the punctures.
- The crossing numbers with the triangulation also change.
- The goal is to calculate the 'update rules' for the crossing numbers, given the generators.
- The preimage of edge e is e^*, which is a curve that becomes e after the braid operation.
- The number of crossings of e^* before the braid operation has to be equal to the number of crossings with e.
- But e^* is usually not part of the triangulation!
- Like playing Minesweeper: we know some numbers, we must deduce others.

Crossing Update Rules for ρ_3

Only edges x_2, y_2, x_3, y_3 and z_3 change.

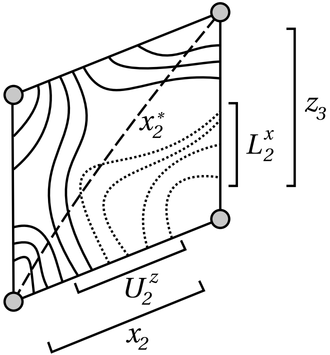




x_2 and y_3 are the preimages of y_2 and x_3, respectively, so $y_2^* = x_2$ and $x_3^* = y_3$. The edge z_3 is its own preimage, $z_3^* = z_3$.

The Quadrilateral Puzzle

The preimages of x_2 and y_3 are not edges in the triangulation. We must deduce their crossing numbers.



Curves entering x_2 and z_3 must cross x_2^*, unless they loop directly from x_2 to z_3. The number of these loops is exactly $\min(U_2^z, L_2^x)$. Hence the number of preimage crossings is $x_2 + z_3 - 2\min(U_2^z, L_2^x)$.

Crossing Update Rules for ρ_i

For general i, we have

$$\begin{aligned}
x^*_{i-1} &= x_{i-1} + z_i - 2\min(U^z_{i-1}, L^x_{i-1}) \\
y^*_{i-1} &= x_{i-1} \\
x^*_i &= y_i \\
y^*_i &= z_i + y_i - 2\min(U^y_i, L^z_i) \\
z^*_i &= z_i
\end{aligned}$$

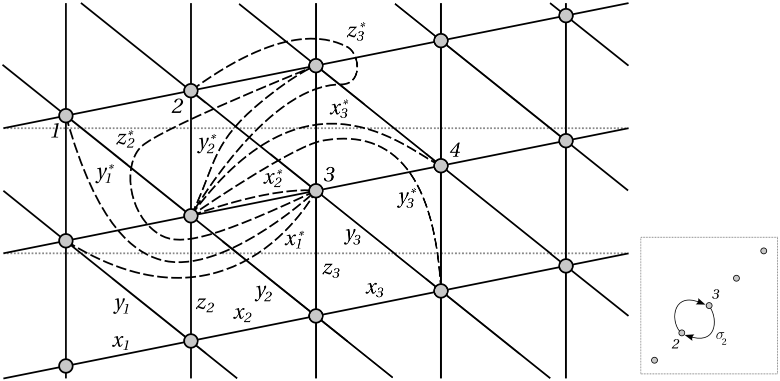
Crossing Update Rules for ρ_i^{-1}

Because of the π-rotational symmetry of the triangulation, we easily find the update rules for ρ_i^{-1} directly from that for ρ_i:

$$
\begin{aligned}
x_{i-1}^* &= y_{i-1} \\
y_{i-1}^* &= y_{i-1} + z_i - 2\min(U_{i-1}^z, L_{i-1}^y) \\
x_i^* &= z_i + x_i - 2\min(U_i^x, L_i^z) \\
y_i^* &= x_i \\
z_i^* &= z_i
\end{aligned}
$$

Crossing Update Rules for σ_2

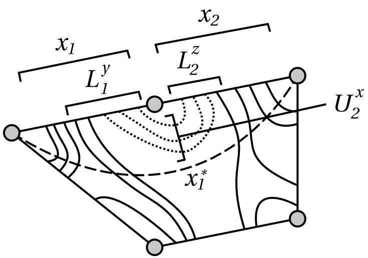
Two punctures are permuted clockwise! Many more edges involved.

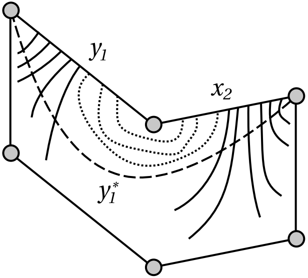


Edge x_2 is its own preimage, but all other preimages require some Minesweeping. Edge y_2^* is of the quadrilateral form.

Crossing Puzzle for x_1^* and y_1^*

The number of crossings with preimage x_1^* is given by the number of curves crossing x_1 and x_2, minus twice the number of loops directly between x_1 and x_2, given by $\min(L_1^y, U_2^x, L_2^z)$. Hence $x_1^* = x_1 + x_2 - 2\min(L_1^y, U_2^x, L_2^z)$.
The preimage problem for y_1^* is similar, but involves 4 triangles, so that $y_1^* = y_1 + x_2 - 2\min(U_1^z, L_1^y, U_2^x, L_2^z)$.

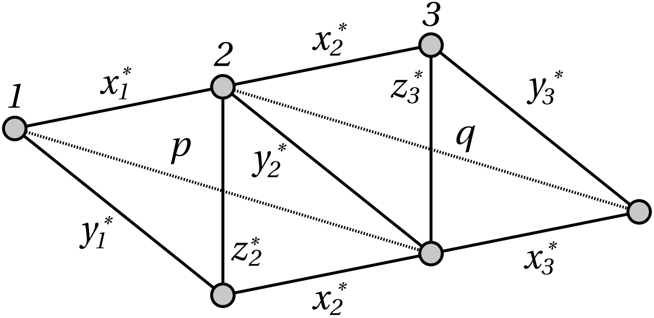




Preimages x_3^* and y_3^* are handled in the same way.

Crossing Puzzle for z_2^* and z_3^*

Still need to find z_2^* and z_3^*, which pass through 7 (!) triangles. We deduce z_2^* and z_3^* by invoking the quadrilateral solution with the *updated* crossing numbers. We introduce temporary edges p and q directly between punctures 1 and 3 and between 2 and 4.



The preimage of p is y_1, and the preimage of q is y_3. Since x_1^*, y_1^*, p, x_2^*, y_2^*, q, x_3^* and y_3^* are known, z_2^* and z_3^* follow.

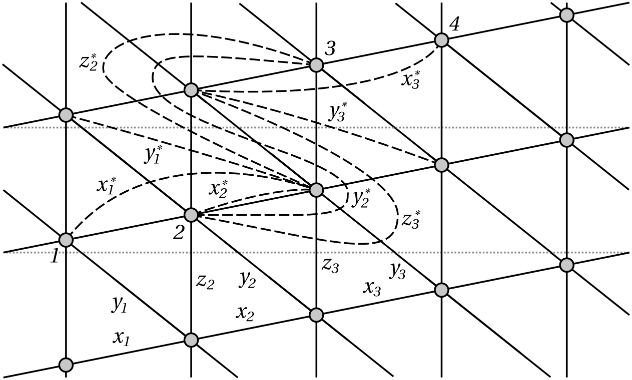
Crossing Update Rules for σ_i

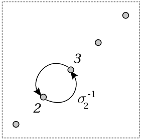
Putting it all together,

$$
\begin{aligned}
x^*_{i-1} &= x_{i-1} + x_i - 2\min(L^y_{i-1}, U^x_i, L^z_i) \\
y^*_{i-1} &= y_{i-1} + x_i - 2\min(U^z_{i-1}, L^y_{i-1}, U^x_i, L^z_i) \\
x^*_i &= x_i \\
y^*_i &= z_i + x_i - 2\min(U^x_i, L^z_i) \\
z^*_i &= x^*_{i-1} + y^*_{i-1} - \min(y^*_{i-1} + y_{i-1} - x^*_i, x^*_{i-1} + y_{i-1} - y^*_i) \\
x^*_{i+1} &= x_i + x_{i+1} - 2\min(U^z_i, L^x_i, U^y_{i+1}) \\
y^*_{i+1} &= x_i + y_{i+1} - 2\min(U^z_i, L^x_i, U^y_{i+1}, L^z_{i+1}) \\
z^*_{i+1} &= x^*_i + y^*_i - \min(x^*_i + y_{i+1} - y^*_{i+1}, y^*_i + y_{i+1} - x^*_{i+1}).
\end{aligned}
$$

Crossing Update Rules for σ_i^{-1}

Lack of reflection symmetry about a vertical line through the midpoint of two punctures means that it is not possible to deduce the update rules for σ_i^{-1} by a relabelling in the rules for σ_i.





The preimage curve y_i^* is the most complicated yet, as it passes through ten triangles.

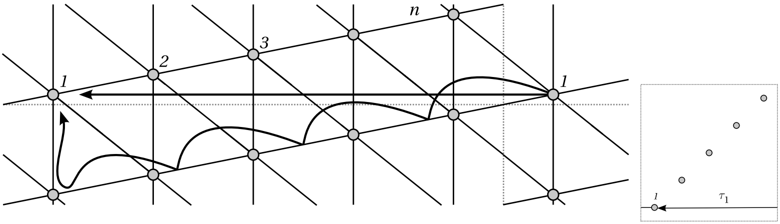
Crossing Update Rules for σ_i^{-1} (2)

However, things proceed pretty much as before, so we quote the result:

$$
\begin{aligned}
x_{i-1}^* &= x_{i-1} + x_i - 2\min(U_{i-1}^z, L_{i-1}^x, U_i^y) \\
y_{i-1}^* &= y_{i-1} + x_i - 2\min(L_{i-1}^x, U_i^y) \\
x_i^* &= x_i \\
y_i^* &= x_i^* + z_i^* - \min(z_i^* + y_i - x_i^*, x_i^* + y_i - z_{i+1}^*) \\
z_i^* &= x_i + y_i - 2\min(U_i^x, L_{i-1}^y, U_{i-1}^z, L_{i-1}^x, U_i^y) \\
x_{i+1}^* &= x_i + x_{i+1} - 2\min(L_i^y, U_{i+1}^x, L_{i+1}^z) \\
y_{i+1}^* &= y_i + x_{i+1} - 2\min(L_i^x, U_{i+1}^y) = x_i + y_{i+1} - 2\min(L_i^y, U_{i+1}^x) \\
z_{i+1}^* &= x_i + y_i - 2\min(L_i^y, U_{i+1}^x, L_{i+1}^z, U_{i+1}^y, L_i^x)
\end{aligned}
$$

Crossing Update Rules for τ_1

Our choice of triangulation makes ρ_i easy but τ_i quite complicated. Instead of attacking this problem directly, τ_1 can be achieved by a sequence of σ_i, including σ_n, followed by one ρ_1^{-1}.



The same technique works for τ_i^{-1}.

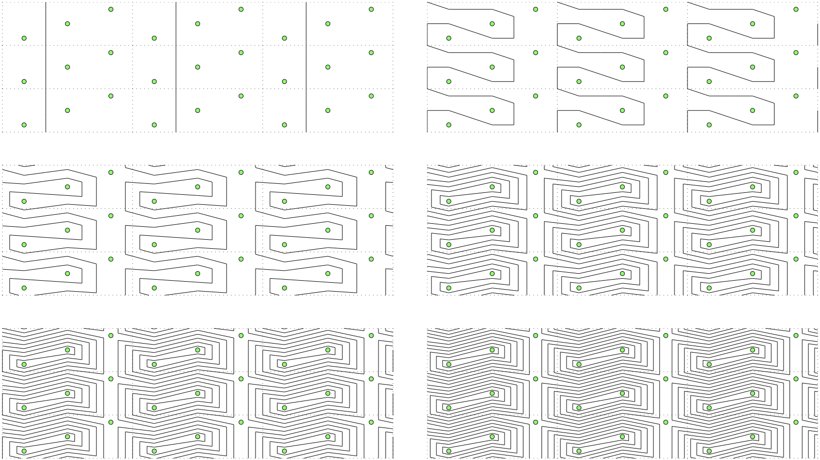
Crossing Update Rules for τ_i

To calculate the updated set of crossing numbers $\{x_i, y_i, z_i\}$ for τ_i do the following:

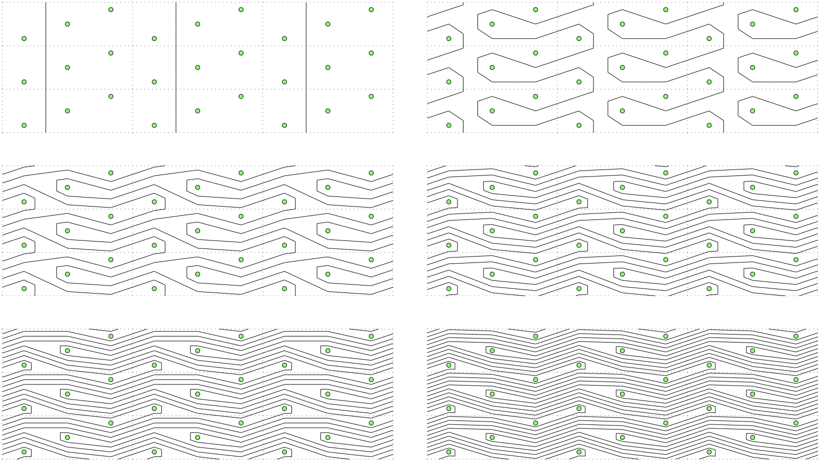
1. Perform, in turn, σ_{i-1}^{-1}, σ_{i-2}^{-1}, ..., σ_{i+2}^{-1} and σ_{i+1}^{-1}. Treat the indices 'modulo' n, so that σ_n^{-1} follows σ_1^{-1}.
2. Relabel $x_i \leftarrow x_{i+1}$, $y_i \leftarrow y_{i+1}$ and $z_i \leftarrow z_{i+1}$. This leaves all punctures except the ith one in the correct position.
3. Perform ρ_i^{-1}.

A Finite-Order Braid: σ_1

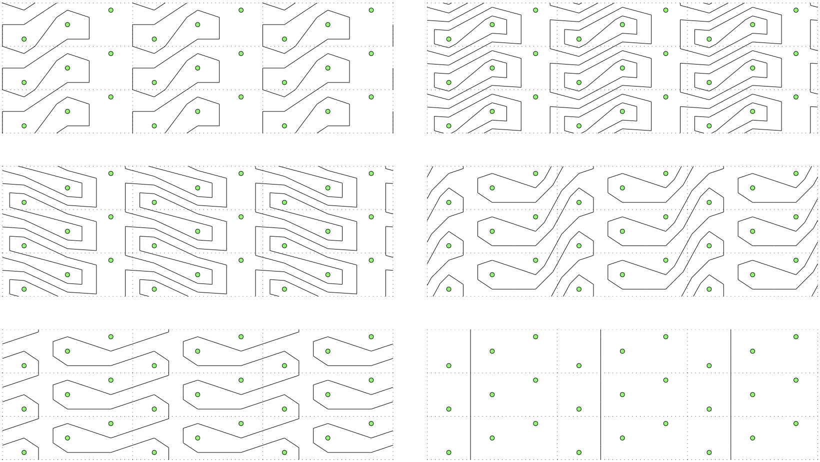
We can reconstruct a lamination from the crossing numbers, so our update rules allow us to draw it 'on the fly.'



Another Finite-Order Braid: τ_2



The Identity Braid $\sigma_1^{-2}\rho_1^{-1}\tau_2\rho_1\tau_2^{-1}$



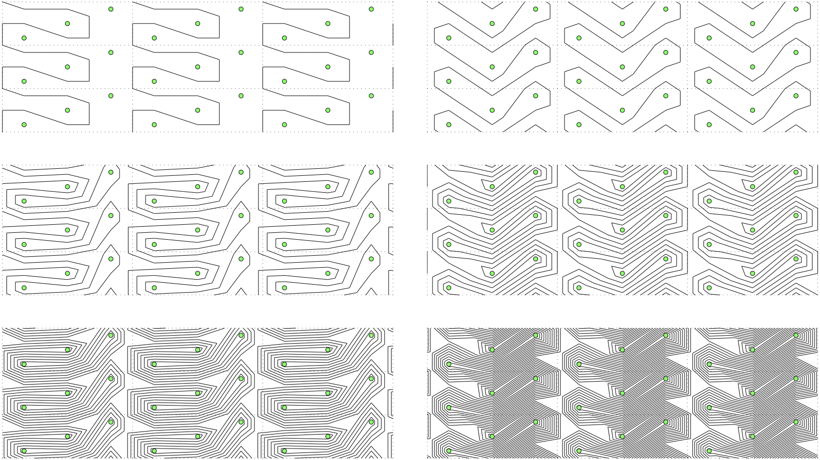
Calculating Topological Entropy

For braids on the sphere, Moussafir (2006) showed that the number of crossings grows at the same rate as the topological entropy implied by the braid. The same result applies here:

$$\lambda^{\dagger} = \log \sum_{i} (x_i^* + y_i^* + z_i^*) - \log \sum_{i} (x_i + y_i + z_i)$$

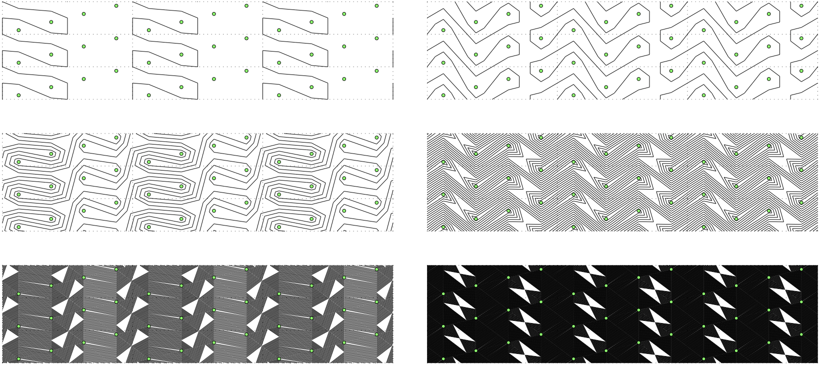
as the number of crossings goes to infinity.

The Golden Braid: $\sigma_1\sigma_2^{-1}$



Topological entropy is $2\log\left(\frac{1}{2}(1+\sqrt{5})\right)$.

The Silver Braid: $\sigma_1\sigma_3\sigma_2^{-1}\sigma_4^{-1}$



Topological entropy is $4\log\left(1+\sqrt{2}\right)$.

Convergence to the Topological Entropy

Convergence of $\lambda^\dagger$ to the exact braid entropy λ appears to be exponential (provided the braid has a pseudo-Anosov component).

iteration	total crossings	entropy $\lambda^\dagger$	error $\lvert\lambda - \lambda^\dagger\rvert$
1	24	2.48490664978800	0.72215947574891
2	154	1.85889877206568	0.09615159802660
3	912	1.77868738766070	0.01594021362162
4	5330	1.76546652708556	0.00271935304647
5	31080	1.76321328732169	0.00046611328261
6	181162	1.76282713309230	0.00007995905321
7	1055904	1.76276089245107	0.00001371841199
8	6154274	1.76274952773491	0.00000235369583
9	35869752	1.76274757786911	0.00000040383002
10	209064250	1.76274724332535	0.00000006928627
11	1218515760	1.76274718592673	0.00000001188765
12	7102030322	1.76274717607868	0.00000000203960
13	41393666184	1.76274717438903	0.00000000034994
14	241259966794	1.76274717409913	0.00000000006004
15	1406166134592	1.76274717404939	0.00000000001030
16	8195736840770	1.76274717404085	0.00000000000177
17	47768254910040	1.76274717403939	0.00000000000030
18	278413792619482	1.76274717403914	0.00000000000005
19	1622714500806864	1.76274717403910	0.00000000000001
20	9457873212221714	1.76274717403908	0.00000000000000

The Sine Flow

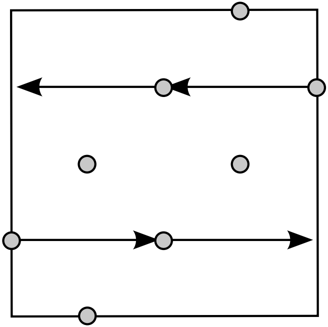
Area-preserving map obtained by integrating alternating sine flow, defined over $0 \le x, y < 1$, period T.

$$
\begin{aligned}
x_{n+\frac{1}{2}} &= x_n + \tfrac{1}{2} T \sin\left(2\pi y_n\right) \\
y_{n+\frac{1}{2}} &= y_n
\end{aligned}
$$

$$
\begin{aligned}
x_{n+1} &= x_{n+\frac{1}{2}} \\
x_{n+1} &= y_{n+\frac{1}{2}} + \tfrac{1}{2} T \sin\left(2\pi x_{n+\frac{1}{2}}\right)
\end{aligned}
$$

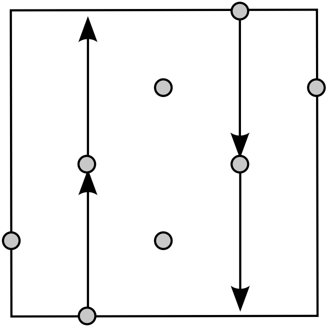
Exhibits wide range of behaviour as T is varied, from integrability to near-total chaos.

Periodic Orbits for $T = 1$



$\{(\frac{1}{2}, \frac{3}{4}), (1, \frac{3}{4})\}$

$\{(0, \frac{1}{4}), (\frac{1}{2}, \frac{1}{4})\}$

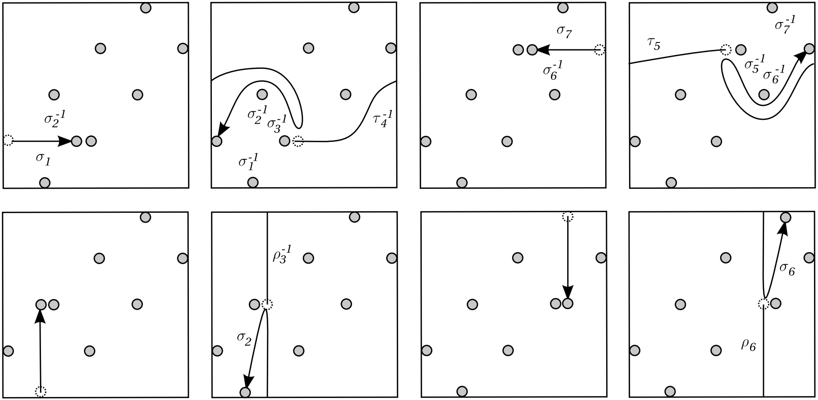


$\{(\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{2})\}$

$\{(\frac{3}{4}, \frac{1}{2}), (\frac{3}{4}, 1)\}$

Mapping to Braid Generators

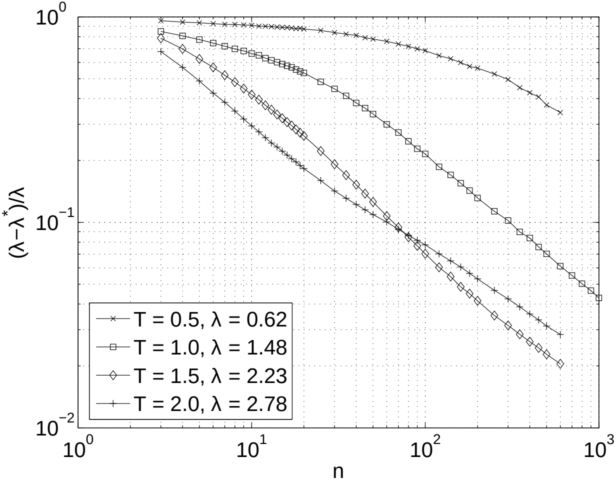
Encode the trajectories in terms of braid group generators.



The braid word is
$\sigma_1\sigma_2^{-1}\tau_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_7\sigma_6^{-1}\tau_5\sigma_5^{-1}\sigma_6^{-1}\sigma_7^{-1}\rho_3^{-1}\sigma_2\rho_6\sigma_6$, with entropy $1.21875572687\ldots$ (82% of the flow entropy).

Convergence to Flow Entropy

Now follow arbitrary trajectories [Gambaudo (1999), Thiffeault (2005)]: the resulting nonperiodic braid has entropy that converges to the 'true' entropy of the flow.



Conclusions

- Compute topological entropy of braids.
- Unlike train-tracks, not exact, but very accurate for braids with a pA component.
- Exact entropies by 'short-circuiting.'
- Fast! Use integer arithmetic or double precision.
- Allows the use of extremely long 'random braids' with millions of generators.
- An easy way to estimate the topological entropy of a flow. Even accessible experimentally!
- See `http://arXiv.org/nlin/0603003` for preprint.

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