A Bound on Mixing Efficiency

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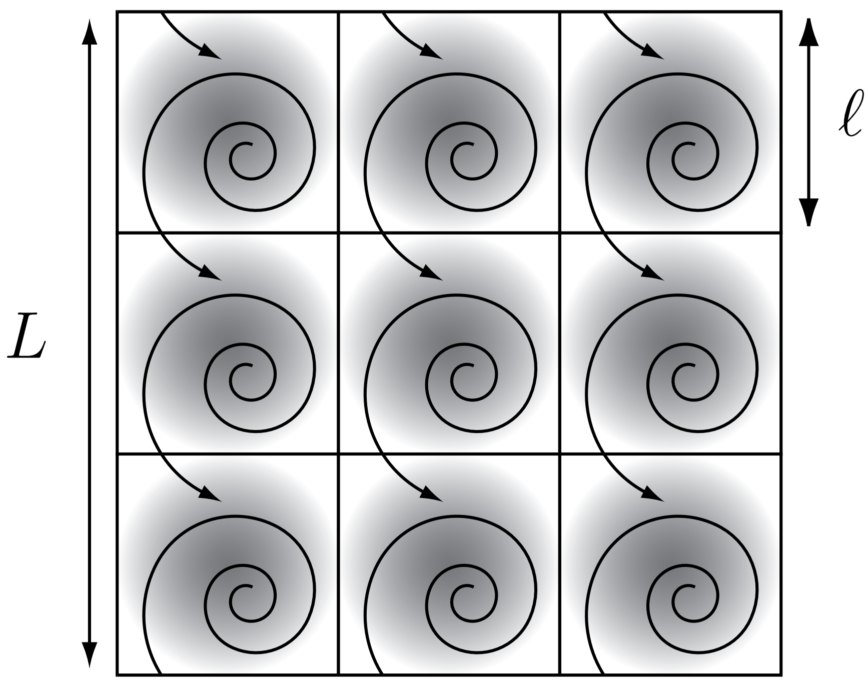
Summary

- Derive upper bound on the mixing efficiency for a scalar under the influence of advection and diffusion with a body source (e.g. , differential heating between equator and poles).
- Inspired by work on Navier–Stokes by Doering & Foias (2002).
- Mixing efficiency measured in terms of an equivalent diffusivity.
- The precise value of the bound on the equivalent diffusivity depends only on the functional shape of both the source and the advecting field.
- Direct numerical simulations performed for a simple advecting flow to test the bounds.

The Setup



- Periodic system (2 or 3 dimensions)
- Stirring and source of scalar variance at scale ℓ
- System of size L
- Velocity field regarded as given: could be time-dependent and even turbulent
- Source distribution and strength could also be time-dependent

Advection–Diffusion Equation

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \, \Delta \theta + s \, , \qquad \text{(A-D)}$$

where κ is the molecular diffusivity and $s(\boldsymbol{x}, t)$ is a source function with zero spatial mean.

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To characterise the fluctuations in θ, we use the variance,

$$\Theta^2 \equiv \left\langle L^{-d}\,\|\theta\|^2_{L^2(\mathbb{T}^d)}\right\rangle$$

The angle brackets $\langle\cdot\rangle$ denote a long-time average, and $\|\cdot\|_{L^2(\mathbb{T}^d)}$ is the L^2 norm on $\mathbb{T}^d$.

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$$s(\boldsymbol{x},t) = S\,\Phi(\boldsymbol{x}/\ell, t/\tau), \quad \left\langle L^{-d}\,\|\Phi\|^2_{L^2(\mathbb{T}^d)}\right\rangle = 1,$$

$$\boldsymbol{u}(\boldsymbol{x},t) = U\,\boldsymbol{\Upsilon}(\boldsymbol{x}/\ell, t/\tau), \quad \left\langle L^{-d}\,\|\boldsymbol{\Upsilon}\|^2_{L^2(\mathbb{T}^d)}\right\rangle = 1.$$

The Bounds

We will restrict to a time-independent source, so $\partial_t \Phi = 0$.

(See paper for more general case.)

Introduce an arbitrary function Ψ that satisfies

$$\left\langle L^{-d} \int_{\mathbb{T}^d} \Psi(\boldsymbol{x}/\ell)\, \Phi(\boldsymbol{x}/\ell)\, \mathrm{d}^d x \right\rangle = 1,$$

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Multiply A-D by Ψ and space-time average. After some integration by parts and use of normalisations,

$$S = -\left\langle L^{-d} \int_{\mathbb{T}^d} (\boldsymbol{u} \cdot \nabla \Psi + \kappa\, \Delta \Psi)\, \theta\, \mathrm{d}^d x \right\rangle.$$

The Bounds (continued)

The Cauchy–Schwartz inequality,

$$\|fg\|_{L^1(\mathbb{T}^d)} \leq \|f\|_{L^2(\mathbb{T}^d)}\,\|g\|_{L^2(\mathbb{T}^d)}$$

implies the bound

$$S \leq \left\langle L^{-d}\,\|\boldsymbol{u}\cdot\nabla\Psi + \kappa\,\Delta\Psi\|^2_{L^2(\mathbb{T}^d)}\right\rangle^{1/2}\Theta.$$

The unknown θ has now been extracted from the average and appears only as a norm that characterises its fluctuations.

We may think of this as a lower bound on the fluctuations in θ in terms of the source strength S, the vigor of stirring U, and their shape Φ and Υ.

The Bounds (continued)

Substituting the scaled variables $T = t/\tau$ and $\boldsymbol{y} = \boldsymbol{x}/\ell$, we have

$$S \leq \frac{U\Theta}{\ell} \left\langle \|\Omega\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2}$$

where $\mathbb{I} = [0, 1]$ is the unit interval and

$$\Omega(\boldsymbol{y}, T) \equiv -\boldsymbol{\Upsilon}(\boldsymbol{y}, T) \cdot \nabla_{\boldsymbol{y}} \Psi(\boldsymbol{y}) - \frac{1}{\mathrm{Pe}} \Delta_{\boldsymbol{y}} \Psi(\boldsymbol{y}).$$

Here the Péclet number is $\mathrm{Pe} = U\ell/\kappa$.

In principle the bound could be sharpened by varying the arbitrary function Ψ. Requires solution of the associated Euler–Lagrange equation for the specific problem at hand (*i.e.*, Φ and Υ).

The Bounds (continued)

We give up a bit of sharpness (inconsequential at large Pe) by using the triangle inequality, and find

$$S \leq \frac{U\Theta}{\ell}\left(c_1 + \mathrm{Pe}^{-1}\, c_2\right),$$

where $\quad c_1 \equiv \left\langle \|\boldsymbol{\Upsilon}\cdot\nabla_{\boldsymbol{y}}\Psi\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2}, \quad c_2 \equiv \left\langle \|\Delta_{\boldsymbol{y}}\Psi\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2}.$

are dimensionless constants, independent of Pe and Θ.

c_1 depends explicitly on the stirring shape-function $\boldsymbol{\Upsilon}$ and implicitly on the source shape function Φ through the normalisation condition on Ψ.

Bound on Equivalent Diffusivity

Express bound in terms of an equivalent diffusivity:

$$\kappa_{\text{eq}} \equiv \frac{S\ell^2}{\Theta} \leq c_1 \; U\ell \; + c_2 \, \kappa,$$

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- $\kappa_{\mathrm{eq}} = \kappa$ for $U = 0$, which is the purely diffusive case (after a trivial rescaling, not included above).
- The scaling $U\ell$ is often used as a rough estimate for turbulent diffusivity, but here we have an explicit prefactor that depends on the stirring and source distribution.

Global Upper Bound

For a given source distribution Φ, we can use the Hölder inequality

$$\|fg\|_{L^1(\mathbb{T}^d)} \leq \|f\|_{L^{p}(\mathbb{T}^d)} \, \|g\|_{L^{q}(\mathbb{T}^d)} \,, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

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and write

$$c_1 = \left\langle \|\boldsymbol{\Upsilon} \cdot \nabla_{\boldsymbol{y}} \Psi\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2} \leq \left\langle \|\boldsymbol{\Upsilon}\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2} \sup_{\boldsymbol{y},t} |\nabla_{\boldsymbol{y}} \Psi|$$

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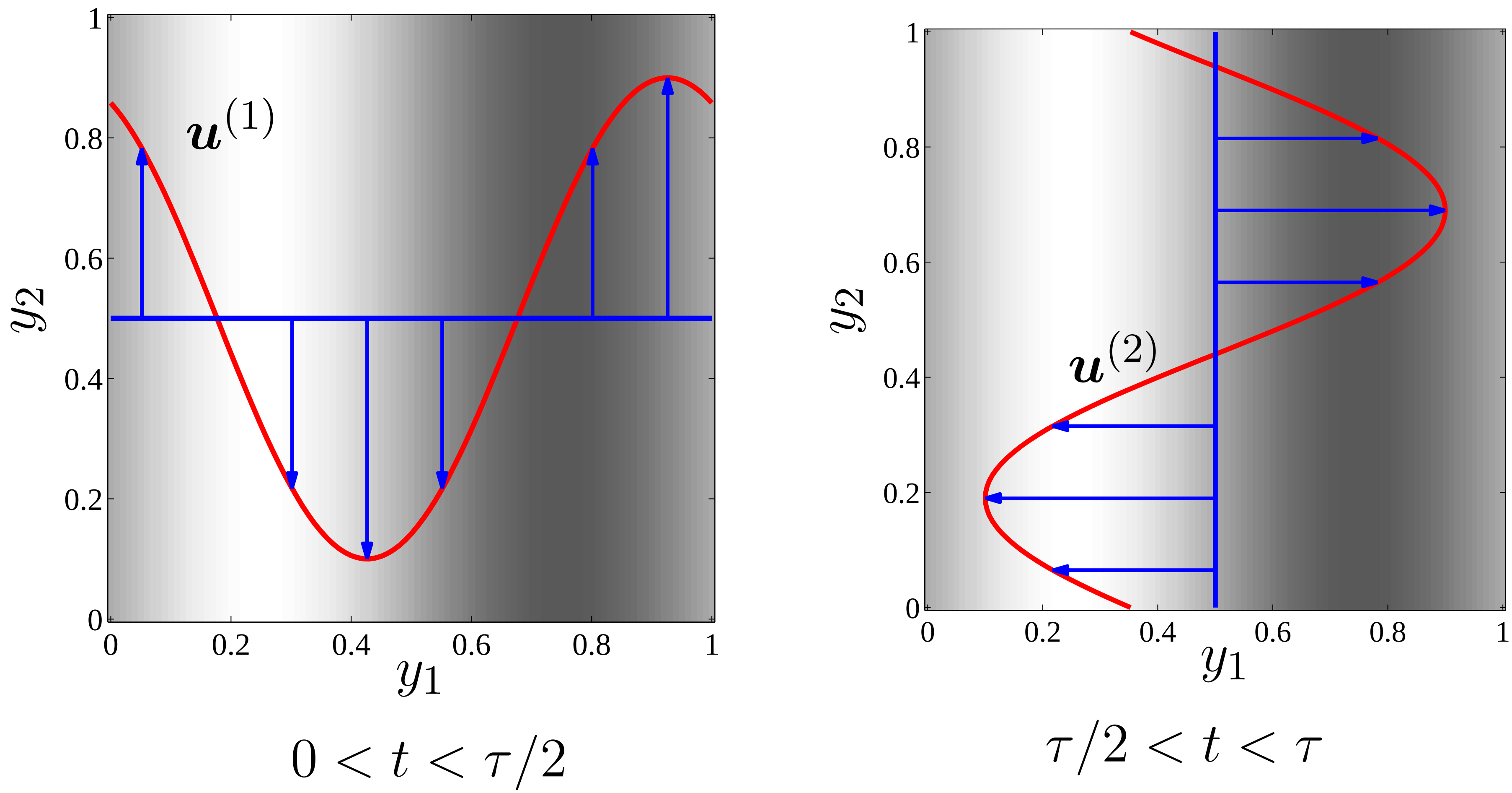
$$\boxed{c_1 \leq \sup_{\boldsymbol{y},t} |\nabla_{\boldsymbol{y}} \Psi|}$$

This bound is valid for any stirring velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$.
No flow that can be more efficient than this!

Example: Random Sine Flow

Alternating horizontal and vertical sine shear flows, with randomised phase. Source distribution (shaded) is fixed.



Calculating the Bound

Shape function for the source:

$$\Phi = \sqrt{2}\,\sin 2\pi y_1\,,$$

and for the velocity field:

$$\Upsilon = \begin{cases} \sqrt{2}\;(0\;,\;\sin(2\pi y_1 + \chi_1))\,, & n < T < (n+\frac{1}{2})\,; \\ \sqrt{2}\;(\sin(2\pi y_2 + \chi_2)\;,\;0)\,, & (n+\frac{1}{2}) < T < (n+1)\,. \end{cases}$$

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We choose $\Psi = \Phi$, though in principle this could be optimised.

$$\boldsymbol{\Upsilon}\cdot\nabla_{\boldsymbol{y}}\Psi = \begin{cases} 0, & n < T < (n+\frac{1}{2})\,; \\ 4\pi\ \sin(2\pi y_2 + \chi_2)\cos 2\pi y_1\,, & (n+\frac{1}{2}) < T < (n+1)\,. \end{cases}$$

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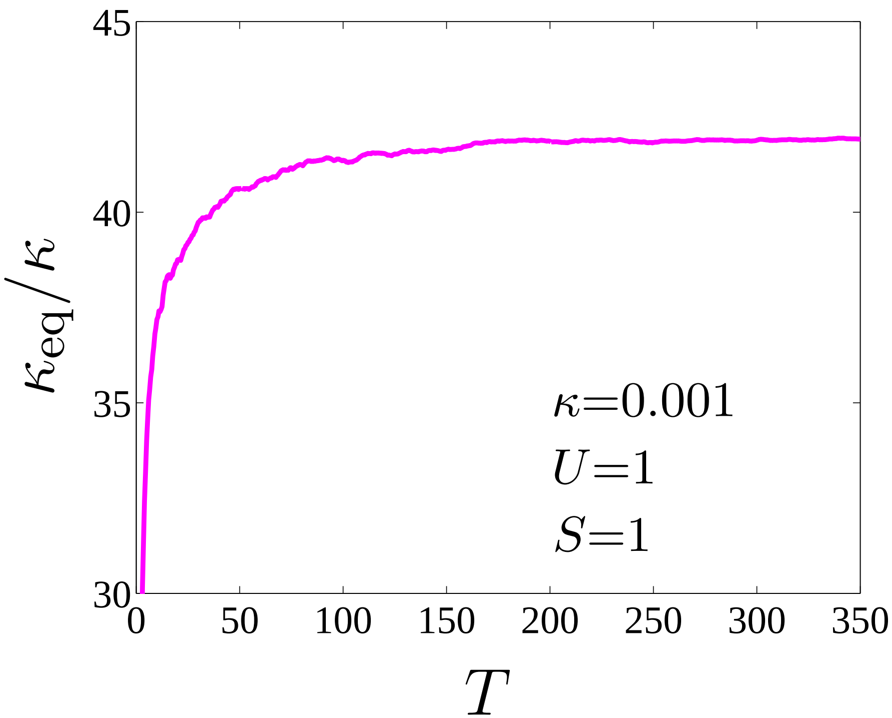
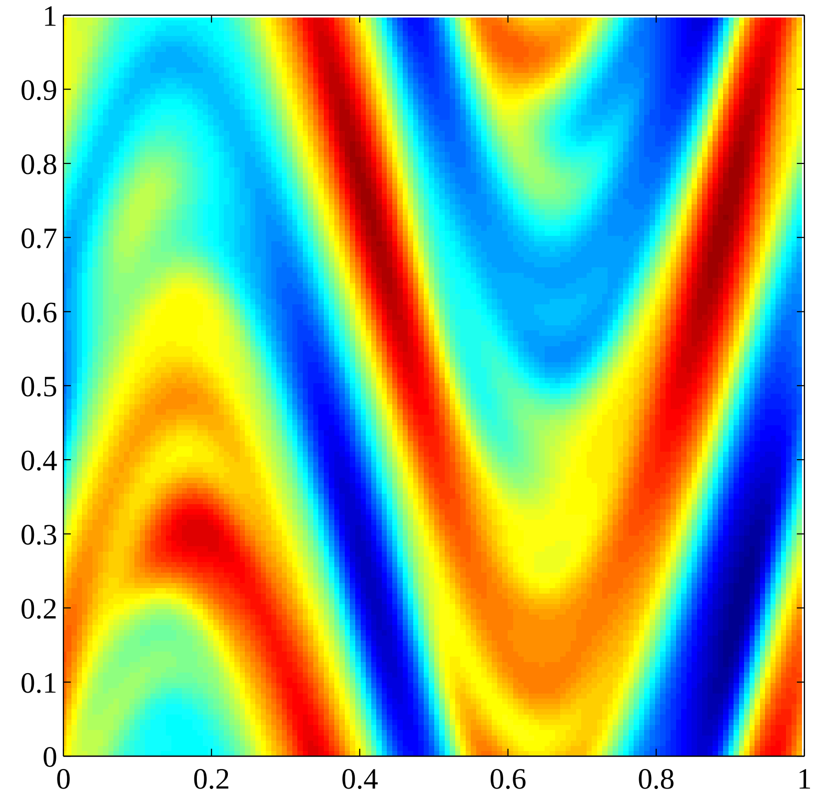
After including a trivial rescaling of $(2\pi)^{-2}$ (so that κ_{eq} reduces to κ in the absence of stirring), we find

$$\boxed{\kappa_{\mathrm{eq}} \le \frac{UL}{2\sqrt{2}\,\pi} + \kappa}$$

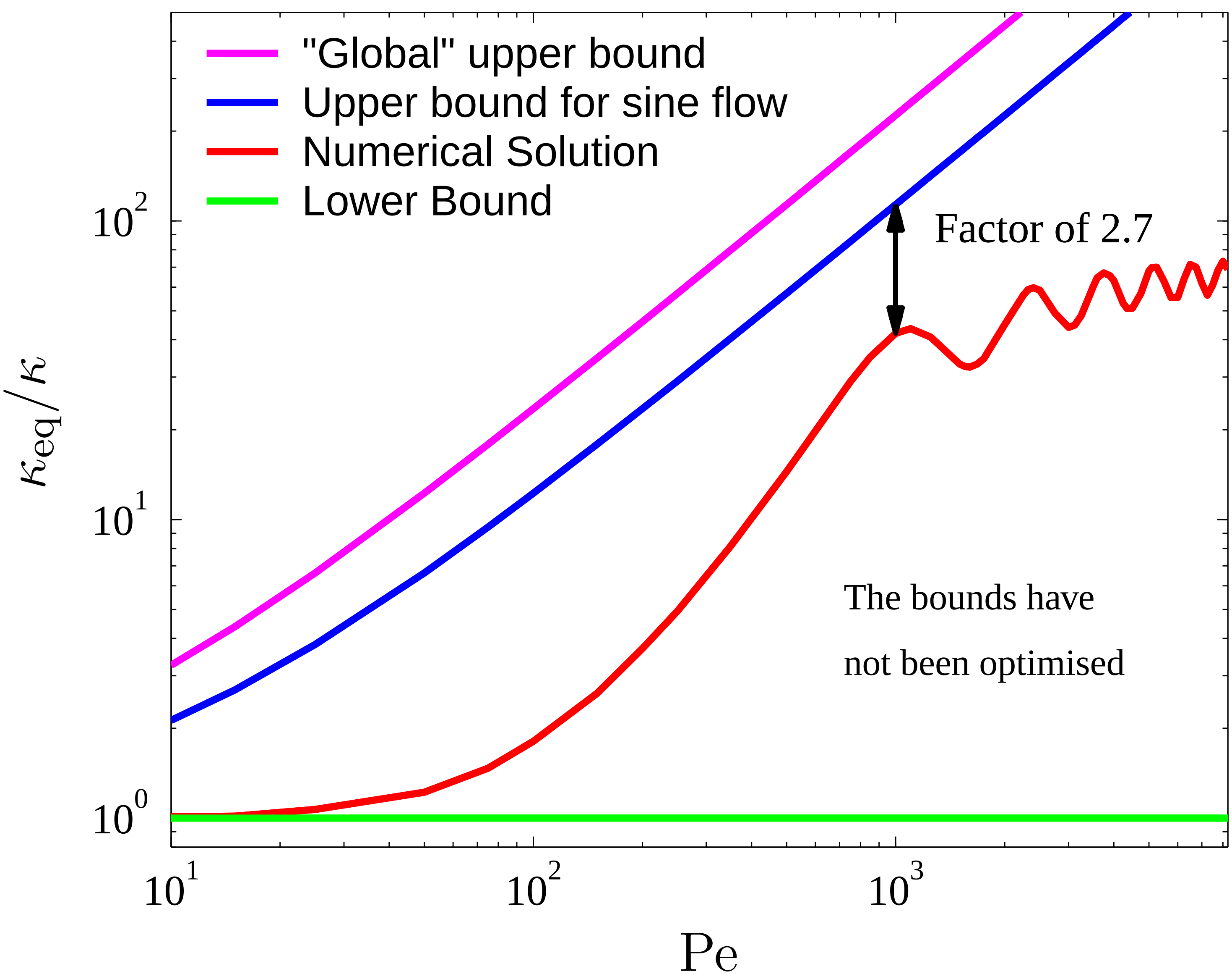
Computing the Equivalent Diffusivity





$T = 350.5$

Comparison with Numerical Results



Conclusions

- Simple bound on fluctuations in the concentration of a scalar, active or passive.
- The constants involved in the bound only depend on the shape of the source distribution and stirring field.
- Suffers from the same problems as all bounding approaches: Hard to tease out physics!
- Nevertheless, important to know where one expects the answer to lie, and little work is required for bounding.
- Exploit this to try and optimise mixing configurations: focus on the bound rather than a full solution of the A-D equation.