On mix-norms and the rate of decay of correlations

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An evolving scalar field

Let $f^t(\boldsymbol{x}) = f(t, \boldsymbol{x})$ be a spatially-periodic mean-zero function bounded uniformly in $L^2(\mathbb{T}^d)$ for all $t > 0$.

For example, $f(t, \boldsymbol{x})$ might be a solution to the advection-diffusion equation

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f = D \Delta f,$$

with mean-zero $f^0 \in L^2(\mathbb{T}^d)$ and smooth incompressible (i.e. $\nabla \cdot \boldsymbol{u} = 0$) flow $\boldsymbol{u}(t, \boldsymbol{x})$.

Mix-norms

Mathew et al. [2003] used the $H^{-1/2}$ norm as a measure of mixing, and Lin et al. [2011] extended this to any negative Sobolev (e.g., H^{-q}) norm.

These are collectively known as mix-norms. Thus the magnitude of

$$\left\| f^t \right\|_{H^{-q}}$$

measures how well-mixed f^t is.

This viewpoint has proved very fruitful both for proving theorems and for applications: Doering and Thiffeault [2006], Shaw et al. [2007], Thiffeault [2012], Lunasin et al. [2012], Iyer et al. [2014], Kiselev and Xu [2016], Marcotte and Caulfield [2018], Miles and Doering [2018], Yao and Zlatoš [2017], Vermach and Caulfield [2018], Bedrossian and He [2020], Coti Zelati [2020] . . . .

Mix-norms and correlations

Correlations decay to zero iff any such mix-norm decays to zero. That is,

$$\lim_{t\to\infty} \langle f^t \,,\, g\rangle = 0 \quad \forall g \in L^2 \iff \lim_{t\to\infty} \left\| f^t \right\|_{H^{-q}} = 0, \text{ for any } q > 0.$$

In that sense decay of correlations and decay of mix-norms are ‘equivalent.’

But do correlations and mix-norms decay at the same rate?

Question

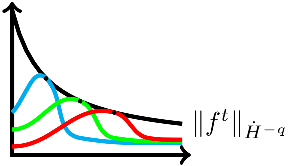
How is the rate of decay of a mix-norm related to the rate of decay of correlations?

Rate of mixing

The $\dot{H}^{-q}$ norm is defined via the duality equation

$$\|f\|_{\dot{H}^{-q}} = \sup_{g \in \dot{H}^q} \frac{|\langle f \,,\, g \rangle|}{\|g\|_{\dot{H}^q}} \,.$$

In our setting, this supremum can be realized. The mix-norm is the envelope of correlations with $\|g\|_{\dot{H}^q} = 1$.



The mix-norm is the point-wise smallest uniform rate of decay of correlations.

However, each correlation could potentially decay strictly faster than the mix-norm.

Different notions of the rate of decay of correlations

When studying a collection of functions converging to zero as $t \to \infty$, such as $|\langle f^t\,,\,g\rangle|$ for $g \in \dot{H}^q$, there are several common ways to define a rate of decay:

1. Correlations decay at the uniform rate $r(t)$ for $g \in \dot{H}^q$ if

$$\left|\left\langle f^t\,,\,g\right\rangle\right| \leq r(t)\,\|g\|_{\dot{H}^q}\ , \text{ for each } g \in \dot{H}^q\,.$$

2. Correlations decay at the asymptotic rate $\varrho(t)$ for $g \in \dot{H}^q$ if

$$\left|\left\langle f^t\,,\,g\right\rangle\right| = \mathcal{O}\left(\varrho\right), \text{ for each } g \in \dot{H}^q\,.$$

   That is,

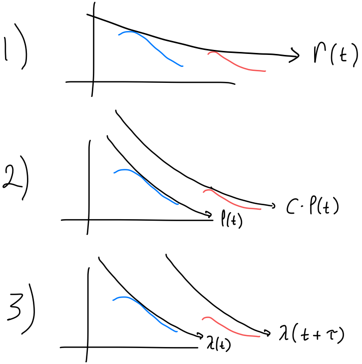
$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\,g\rangle|}{\varrho(t)} = C_g \in [0,\infty).$$

3. Correlations decay at the translational rate $\lambda(t)$ for $g \in \dot{H}^q$ if for each $g \in \dot{H}^q$ there exists $\tau_g \in \mathbb{R}$ such that for all $t > \tau_g$ we have

$$\left|\left\langle f^t\,,\,g\right\rangle\right| \leq \lambda(t-\tau_g)\,\|g\|_{\dot{H}^q}\ .$$

To compare the different notions of decay rate, we construct a test function g where the correlation $|\langle f^t\,,\,g\rangle|$ decays slowly.

Different notions: sketch



1. The uniform rate must work for all correlations;
2. The asymptoptic rate can be fit for each correlation, lifting the tail by multiplication by a constant;
3. The translational rate lifts the tail by translating.

q-recurrence

Denote $P_I f^t$ as the projection of f^t onto the Fourier modes $\boldsymbol{k} \in I$. Then

$$\left\| P_I f^t \right\|_{\dot{H}^{-q}}^2 = \sum_{\boldsymbol{k} \in I} k^{-2q} \, |\widehat{f^t}(\boldsymbol{k})|^2$$

measures the amount of mix-norm supported on I.

Definition

We say f^t is q-*recurrent* if there exists a finite set $I \subset \mathbb{Z}^d$ such that

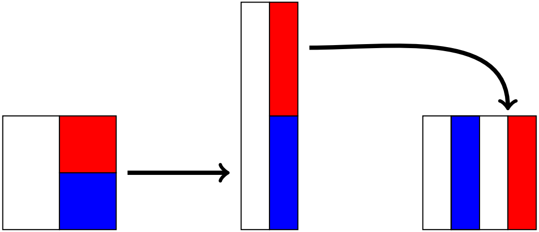
$$\limsup_{t \to \infty} \frac{\| P_I f^t \|_{\dot{H}^{-q}}}{\| f^t \|_{\dot{H}^{-q}}} > 0.$$

Functions that are not q-recurrent will be called q-*transient*.

As time progresses the Fourier energy could move off of I, but for q-recurrent functions a proportion of the Fourier energy always returns to populate the spatial scales in I.

Example: baker's map and q-transience

The baker's map $B : \mathbb{T}^2 \to \mathbb{T}^2$:



For the y-independent initial function $f^0(x, y) = 2\cos(2\pi x)$, applying the baker's map gives $f^n = f^0 \circ B^{-n} = 2\cos(2\pi\ 2^n x)$.

Given any finite set $I \in \mathbb{Z}^d$, it is clear that, as n increases, the Fourier energy will move off of I and never return. Therefore f^n is q-transient $\forall q > 0$.

Example: baker's map and q-transience (cont'd)

This is a one dimensional action on Fourier coefficients $f_k^n = f_{k_1,0}^n$ via an infinite dimensional matrix $A_{k\ell}$ as

$$f_k^{n+1} = \sum_{\ell} A_{k\ell}\, f_\ell^n$$

where

$$(A_{k\ell}) = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ 1 & & & & \\ & & & & \\ & 1 & & & \\ & & & & \\ & & 1 & & \\ & & & & \\ & & & 1 & \\ & & & & \ddots \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \end{array}$$

is populated by 1's along a subdiagonal of slope -2 and 0's everywhere else.

Example: baker-like action and q-recurrence

Consider the action on the Fourier coefficients of $f^n(x)$ via the infinite dimensional matrix

$$(\widetilde{A}_{k\ell}) = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \dots \end{array} \\ \left(\begin{array}{ccccc} a & & & & \\ b & & & & \\ & & & & \\ & 1 & & & \\ & & & & \\ & & 1 & & \\ & & & & \\ & & & 1 & \\ & & & & \ddots \end{array}\right) \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \vdots \end{array} \end{array}$$

where a, $b > 0$ are constants such that $a^2 + b^2 = 1$.

Example: baker-like action and q-recurrence (cont'd)

Nonzero coefficients of f^n:

	$k=1$	2	3	4	5	6	7	8	...
f_k^0	1								
f_k^1	a	b							
f_k^2	a^2	ab		b					
f_k^3	a^3	a^2b		ab				b	
$\vdots$									

The energy starts concentrated on the $k=1$ mode and subsequently splits between modes $k=1,2$ so that L^2 norm is preserved. After that, the $k=1$ mode continues to donate a proportion b of its energy to $k=2$ and the energy on $k=2$ is transported down the spectrum at the same rate as the baker's map ($k=2^n$).

From direct computation, we find f^n is q-transient for $q \leq \log_2(1/a)$ and q-recurrent for $q > \log_2(1/a)$.

Sine flow example

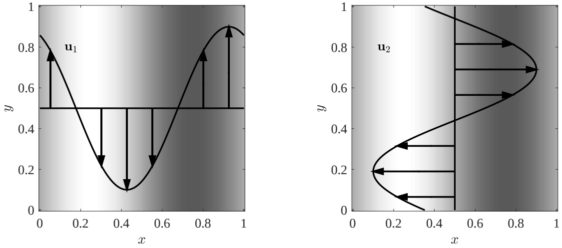
The sine flow is a two-dimensional time-periodic flow with a full period consisting of the shear flow

$$\boldsymbol{u}_1(t,x) = \sqrt{2}\,(0\,,\,\sin(2\pi x + \psi_1)), \qquad 0 \le t < 1/2,$$

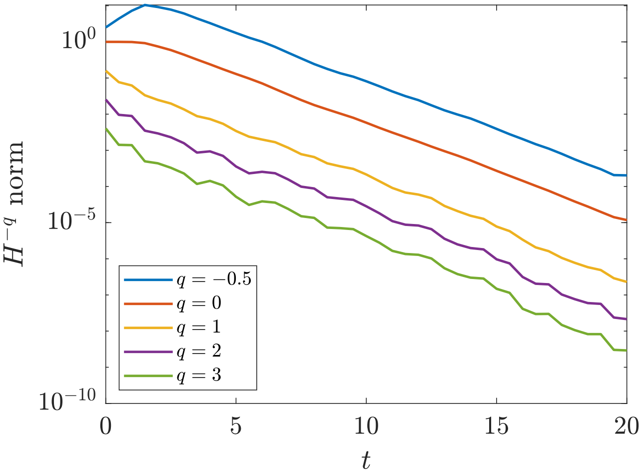
followed by

$$\boldsymbol{u}_2(t,y) = \sqrt{2}\,(\sin(2\pi y + \psi_2)\,,\,0), \qquad 1/2 \le t < 1,$$

with $(x,y) \in [0,1]^2$ and periodic spatial boundary conditions. Here ψ_1 and ψ_2 are random phases, uniformly distributed in $[0,2\pi]$, chosen independently at every period.



Sine flow example (cont'd)



Advection-diffusion equation with $\boldsymbol{u}$ given by the random sine flow, the rate of decay of the mix-norms is independent of q. The the initial condition is $f^0(x) = \sqrt{2}\cos(2\pi x)$, and the diffusivity is $D = 10^{-5}$.

Results: independence on q

In general, if f^t is q-recurrent then the decay rate of the mix-norm is independent of q in the following sense:

Theorem

If f^t is q-recurrent, then it is also q'-recurrent for any $q' > q$. Moreover, we have

$$\limsup_{t\to\infty} \frac{\|f^t\|_{\dot{H}^{-q'}}}{\|f^t\|_{\dot{H}^{-q}}} > 0\,.$$

Then together with the trivial estimate

$$\left\|f^t\right\|_{\dot{H}^{-q'}} \leq \left\|f^t\right\|_{\dot{H}^{-q}}$$

we conclude that $\|f^t\|_{\dot{H}^{-q'}}$ is Big-O but not Little-O of $\|f^t\|_{\dot{H}^{-q}}$.

Results: constructing a test function 1

q-recurrence is the property that allows us to construct a test function achieving the decay rate of the mix-norm:

Theorem

Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. Then f^t is q-recurrent if and only if there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{\|f^t\|_{\dot{H}^{-q}}} > 0.$$

The proof is by construction (see paper).

Results: constructing a test function 2

In general, we may find a correlation function with decay rate arbitrarily close to the mix-norm:

Theorem

Let f^t be a mean-zero function in $L^2(\mathbb{T}^d)$ with $\|f^t\|_{\dot{H}^{-q}} > 0$ for all $t > 0$. For any positive function $h(t)$ such that $h(t) = o\left(\|f^t\|_{\dot{H}^{-q}}\right)$, there is a function $g \in \dot{H}^q$ such that

$$\limsup_{t\to\infty} \frac{|\langle f^t\,,\, g\rangle|}{h(t)} > 0.$$

Results: Answering the original question

Having constructed these slowly decaying correlations, we can prove the following corollary:

Corollary

1. *For any asymptotic rate* ϱ*, we have*

$$\limsup_{t\to\infty} \frac{\varrho(t)}{\|f^t\|_{H^{-q}}} > 0\,.$$

2. *For any translational rate* λ *satisfying* $\limsup_{t\to\infty} \lambda(t-\tau)/\lambda(t)$ *finite for any* $\tau \in \mathbb{R}$*, we have*

$$\limsup_{t\to\infty} \frac{\lambda(t)}{\|f^t\|_{H^{-q}}} > 0\,.$$

We conclude the mix-norm is asymptotically the smallest uniform, asymptotic, and translational rate of decay of correlations.

Conclusion

These results answer the question we posed at the outset:

Question

How is the rate of decay of a mix-norm related to the rate of decay of correlations?

Answer

- For q-recurrent f^t, there is a test function g for which we achieve the decay rate of the mix-norm.
- For q-transient f^t, we can get arbitrarily close.

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