Topological optimization

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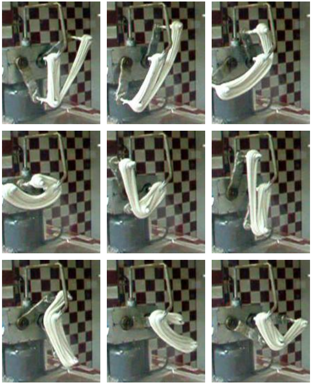
The taffy puller

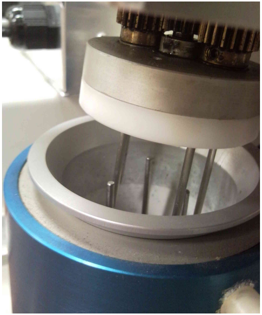


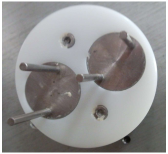
[Photo and movie by M. D. Finn.]

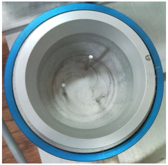
[movie 1]

The mixograph

Model experiment for kneading bread dough:







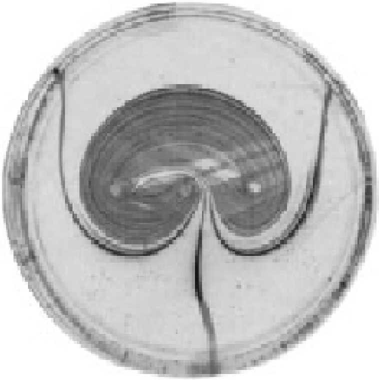
[Department of Food Science, University of Wisconsin. Photos by J-LT.]

Planetary mixers

In food processing, rods are often used for stirring.

[movie 2] ©BLT Inc.

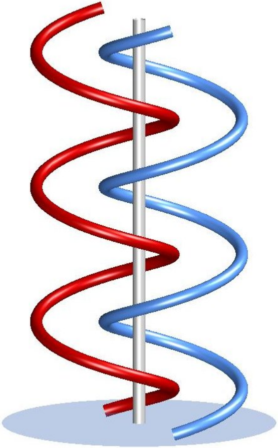
Experiment of Boyland, Aref & Stremler

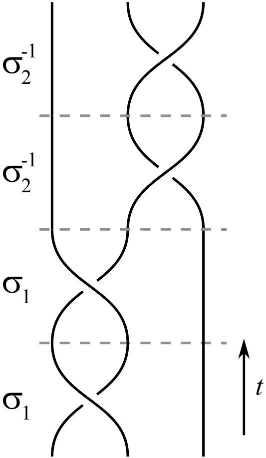


[movie 3] [movie 4]

[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

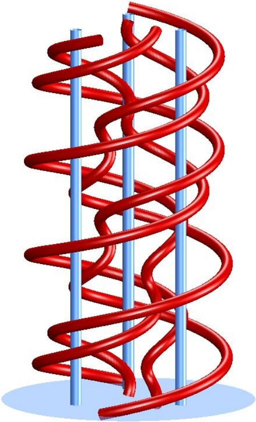
Braid description of taffy puller

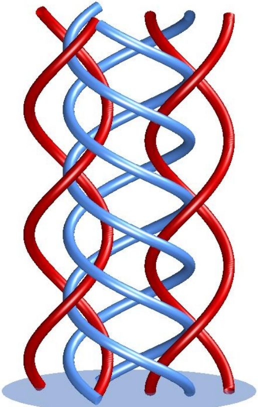




The three rods of the taffy puller in a space-time diagram. Defines a braid on $n = 3$ strands, $\sigma_1^2\sigma_2^{-2}$ (three periods shown).

Braid description of mixograph





$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

braid on B_7, the braid group on 7 strands.

Topological entropy of a braid

Burau representation for 3-braids:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$[\sigma_1^{-1}\,\sigma_2] = [\sigma_1^{-1}] \cdot [\sigma_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

This matrix has spectral radius $(3 + \sqrt{5})/2$ (Golden Ratio^2), and hence the topological entropy is $\log[(3 + \sqrt{5})/2]$.

This is the growth rate of a 'rubber band' caught on the rods.

This matrix trick only works for 3-braids, unfortunately.

Optimizing over generators

- Entropy can grow without bound as the length of a braid increases;
- A proper definition of optimal entropy requires a cost associated with the braid.
- Divide the entropy by the smallest number of generators required to write the braid word.
- For example, the braid $\sigma_1^{-1}\sigma_2$ has entropy $\log[(3+\sqrt{5})/2]$ and consists of two generators.
- Its Topological Entropy Per Generator (TEPG) is thus $\frac{1}{2}\log[(3+\sqrt{5})/2] = \log[\text{Golden Ratio}]$.
- Assume all the generators are used (stronger: irreducible).

Optimal braid

- In B_3 and B_4, the optimal TEPG is log[Golden Ratio].
- Realized by $\sigma_1^{-1}\sigma_2$ and $\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$, respectively.
- In B_n, $n > 4$, the optimal TEPG is $<$ log[Golden Ratio].

Why? Recall Burau representation:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

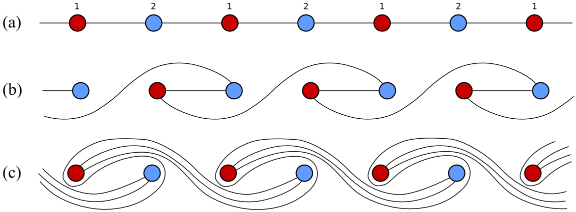
Its spectral radius provides a lower bound on entropy. However,

$$|[\sigma_1]| = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad |[\sigma_2]| = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

provides an upper bound! Need to find Joint Spectral Radius.

Periodic array of rods

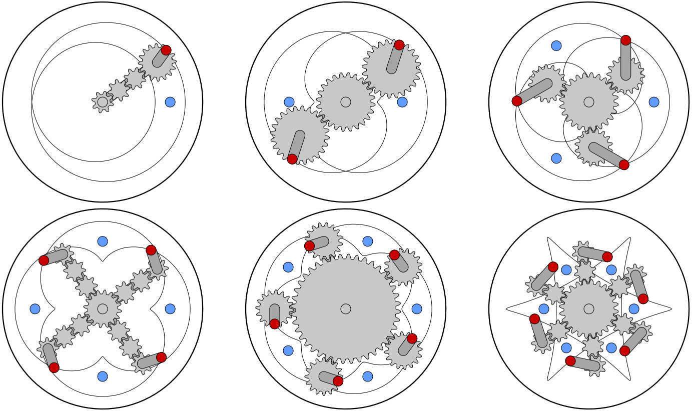
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).



- The entropy per 'switch' is $\log(1+\sqrt{2})$, the Silver Ratio!
- This is optimal for a periodic lattice of two rods (follows from D'Alessandro et al. (1999)).
- Also optimal if we assign cost by simultaneous operation.

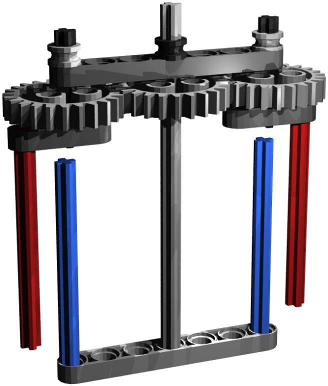
Silver mixers

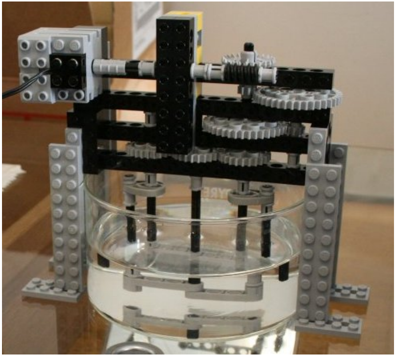
- The designs with entropy given by the Silver Ratio can be realized with simple gears.
- All the rods move at once: very efficient.



[movie 5]

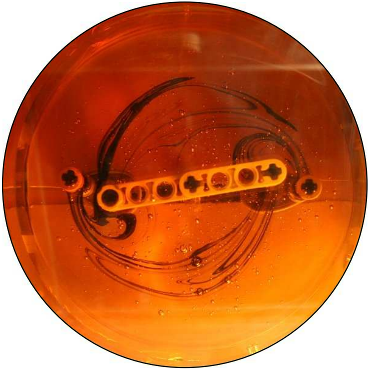
Build it!



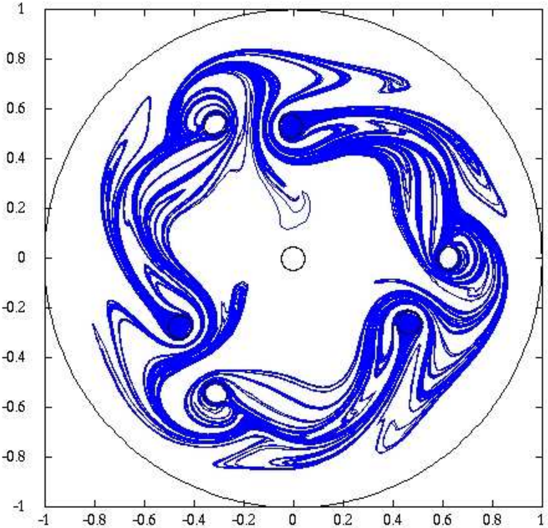


[movie 6] [movie 7]

Experiment: Silver mixer with four rods



Silver mixer with six rods



[movie 8]

The Minimizer problem

- On a given surface of genus g, which pA has the least λ?
- If the foliation is orientable (vector field), then things are much simpler;
- Action of the pA on first homology captures dilatation λ;
- Polynomials of degree $2g$;
- Procedure:
    - We have a guess for the minimizer;
    - Find all integer-coefficient, reciprocal polynomials that could have smaller largest root;
    - Show that they can't correspond to pAs;
    - For the smallest one that can, construct pA.

Newton's formulas

We need an efficient way to bound the number of polynomials with largest root smaller than λ. Given a reciprocal polynomial

$$P(x) = x^{2g} + a_1\, x^{2g-1} + \ldots + a_2\, x^2 + a_1\, x + 1$$

we have Newton's formulas for the traces,

$$\mathrm{Tr}(\phi_*^k) = -\sum_{m=1}^{k-1} a_m \mathrm{Tr}(\phi_*^{k-m}) - k a_k,$$

where

- ϕ is a (hypothetical) pA associated with $P(x)$;
- ϕ_* is the matrix giving the action of the pA ϕ on first homology;
- $\mathrm{Tr}(\phi_*)$ is its trace.

Bounding the traces

The trace satisfies

$$|\mathrm{Tr}(\phi_*^k)| = \left|\sum_{m=1}^{g}(\lambda_m^k + \lambda_m^{-k})\right| \leq g(r^k + r^{-k})$$

where λ_m are the roots of ϕ_*, and $r = \max_m(|\lambda_m|)$.

- Bound $\mathrm{Tr}(\phi_*^k)$ with $r < \lambda$, $k = 1, \dots, g$;
- Use these g traces and Newton's formulas to construct candidate $P(x)$;
- Overwhelming majority have fractional coeffs $\rightarrow$ discard!
- Carefully check the remaining polynomials:
  - Is their largest root real?
  - Is it strictly greater than all the other roots?
  - Is it really less than λ?
- Largest tractable case: $g = 8$ (10^{12} polynomials).

Lefschetz's fixed point theorem

This procedure still leaves a fair number of polynomials — though not enormous (10's to 100's, even for $g = 8$.)

The next step involves using Lefschetz's fixed point theorem to eliminate more polynomials:

$$L(\phi) = 2 - \mathrm{Tr}(\phi_*) = \sum_{p \in \mathrm{Fix}(\phi)} \mathrm{Ind}(\phi, p)$$

where

- $L(\phi)$ is the Lefschetz number;
- $\mathrm{Fix}(\phi)$ is set of fixed points of ϕ;
- $\mathrm{Ind}(\phi, p)$ is index of ϕ at p.

We can easily compute $L(\phi^k)$ for every iterate using Newton's formula.

Eliminating polynomials

Outline of procedure: for a surface of genus g,

- Use the Euler–Poincaré formula to list possible singularity data for the foliations;
- For each singularity data, compute possible contributions to the index (depending on how the singularities and their separatrices are permuted);
- Check if index is consistent with Lefschetz's theorem.

With this, we can reduce the number of polynomials to one or two!

Minimizers for orientable foliations

g	polynomial	minimizer
2	$X^4 - X^3 - X^2 - X + 1$	$\simeq 1.72208$ †
3	$X^6 - X^4 - X^3 - X^2 + 1$	$\simeq 1.40127$
4	$X^8 - X^5 - X^4 - X^3 + 1$	$\simeq 1.28064$
5	$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$	$\simeq 1.17628$ $*$
6	$X^{12} - X^7 - X^6 - X^5 + 1$	$\gtrsim 1.17628$
7	$X^{14} + X^{13} - X^9 - X^8 - X^7 - X^6 - X^5 + X + 1$	$\simeq 1.11548$
8	$X^{16} - X^9 - X^8 - X^7 + 1$	$\simeq 1.12876$

- † Zhirov (1995)'s result; also for nonorientable [Lanneau–T];
- $*$ Lehmer's number; realized by Leininger (2004)'s pA;
- For genus 6 we have not explicitly constructed the pA;
- Genus 6 is the first nondecreasing case.
- Genus 7 and 8: pA's found by Aaber & Dunfield (2010) and Kin & Takasawa (2010b) [$g = 7$]; Hironaka (2009) [$g = 8$].

Conclusions

- Having rods undergo 'braiding' motion guarantees a minimal amound of entropy (stretching of material lines).
- Can optimize to find the best rod motions, but depends on choice of 'cost function.'
- For two natural cost functions, the Golden Ratio and Silver Ratio pop up!
- Found orientable minimizer on surfaces of genus $g \leq 8$; only known nonorientable case is for genus 2.

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