

Topology, Braids, and Mixing

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with

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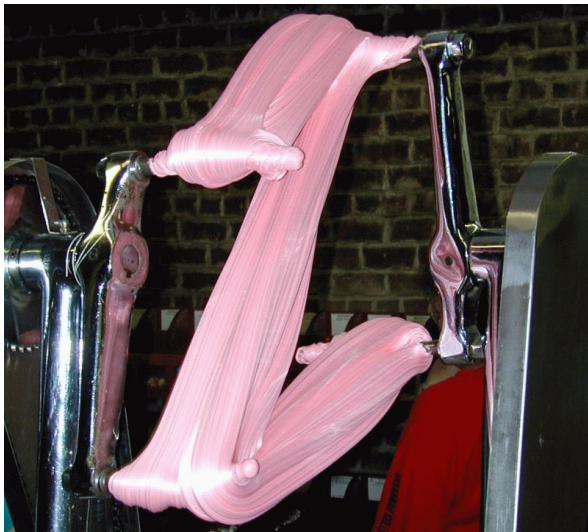
AMMP Colloquium, 1 November 2005

The Taffy Puller

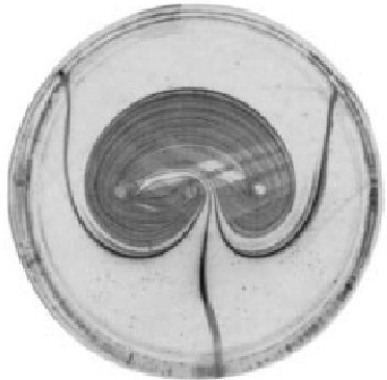


[movie 1]

The Four-pronged Taffy Puller



Experiment of Boyland, Aref, & Stremler

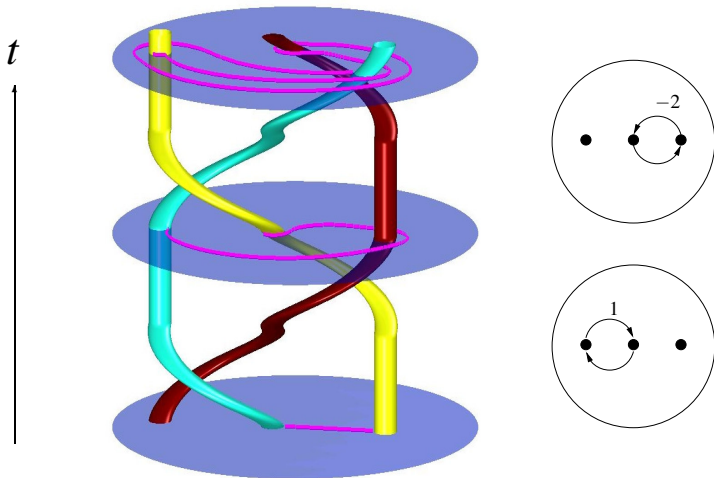


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

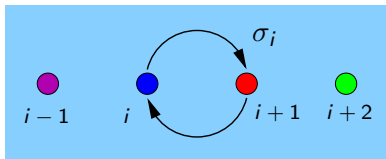
[P. L. Boyland, M. A. Stremler, and H. Aref, *Physica D* **175**, 69 (2003)]

[movie 2] [movie 3]

The Connection with Braids



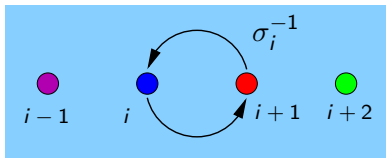
Generators of the n -Braid Group



A generator of Artin's braid group B_n on n strands, denoted

$$\sigma_i, \quad i = 1, \dots, n - 1$$

is the clockwise interchange of the i th and $(i + 1)$ th rod.



B_n is a **finitely-generated group**, with an infinite number of elements, called **words**.

These generators are used to characterise the topological motion of the rods.

Presentation of Artin's Braid Group

The generators obey the **presentation**

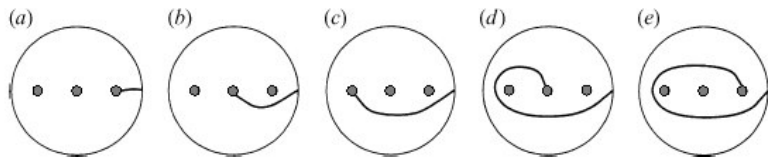
$$\left| \dots \right. \left. \dots \right| = \left| \dots \right. \left. \dots \right| \quad \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i$$

$$\left| \dots \right. \left. \dots \right| = \left| \dots \right. \left. \dots \right| \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1$$

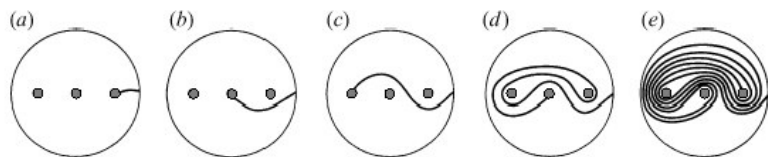
A presentation means that these are the **only rules** obeyed by the generators that are not the consequence of elementary group properties.

The Two BAS Stirring Protocols

$\sigma_1\sigma_2$ protocol



$\sigma_1^{-1}\sigma_2$ protocol

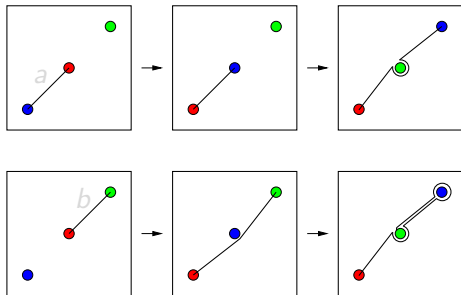
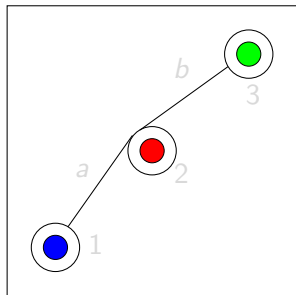


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Train-Tracks

What is the growth rate of an “elastic band” tied to the rods?

Train-tracks give the answer.



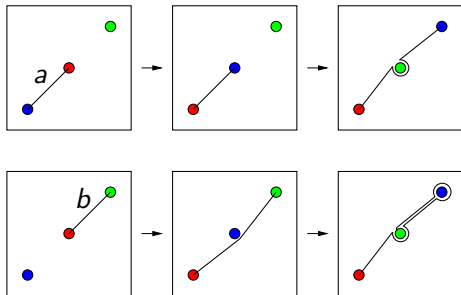
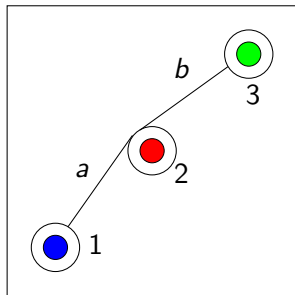
Elastic band has edges (letters) and infinitesimal loops (numbers).
As the rods are moved, the edges and loops are mapped as

$$a \mapsto a2b, \quad b \mapsto a2b3b, \quad 1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2.$$

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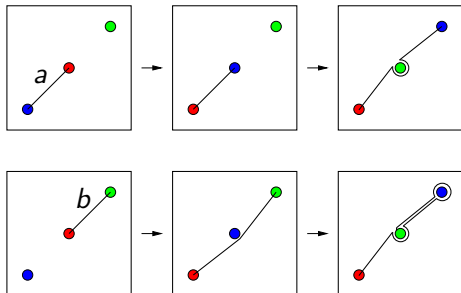
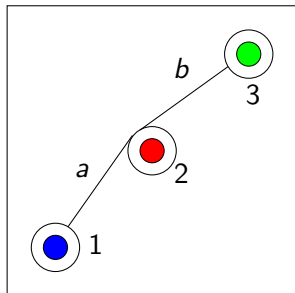
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The Evolution of Edges: Topological Entropy

The edges and loops are mapped according to

$$a \mapsto a2b, \quad b \mapsto a2b3b, \quad 1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2.$$

A crucial point is that edges are separated by loops: no cancellations can occur. A [transition matrix](#) can be formed:

$$M = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array} \right] \begin{array}{l} \leftarrow a \\ \leftarrow b \\ \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \end{array}$$

The largest eigenvalue gives the asymptotic growth factor of the elastic, 2.6180. The logarithm of this is the [topological entropy](#) of the braid.

[M. Bestvina and M. Handel, *Topology* **34**, 109 (1995)]

The Difference between BAS's Two Protocols

- Practically speaking, the topological entropy of a braid is a lower bound on the **line-stretching exponent** of the flow!
- The first (**bad**) stirring protocol has zero topological entropy.
- The second (**good**) protocol has topological entropy $\log[(3 + \sqrt{5})/2] = 0.96 > 0$.
- **So for the second protocol the length of a line joining the rods grows exponentially!**
- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).

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One Rod Mixer: The Kenwood Chef



10270584

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More than a Mixer
at CHRISTMAS

and all through the year...

The **Kenwood** ^{USA}
Chef

Your Servant Madam!

The Kenwood "Chef" not only
★ MIXES, KNEADS and WHIPS
but, with its wide range of optional
attachments, it
★ LIQUIDISES, BLENDS, PUREES,
EXTRACTS JUICES
★ SLICES, MINCES, GRINDS, SHEVES
★ PEELS POTATOES and
★ Even OPENS CANS

Yes, you need the Kenwood
"Chef" in your home every day
of the year. It makes every meal
a festive occasion, with tempting
dishes and drinks having the true
professional touch.

What a wonderful Christmas gift...
especially if it arrives in good time to
prepare the Christmas spread!

Standard Pack **£18:7:6**
complete with Mixing Bowl,
K-Extractor, Whisk, Dough
Hook, Rubber Spatula and
Plastic Dust-Cover.

Optional attachments extra

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The **ROTO-BLEND**
Gives you health — Why
neglect it when you bake?
This specially made-to-order
mixer attachment is made
of steel and is perfect
for mixing fruit and veg-
etables, dairy and other
products. It can be used
for mincing, blending, pureeing
and liquidising. It is also
ideal for preparing soups, dips,
sauces and dips, and
other drinks and condiments.

4940-7

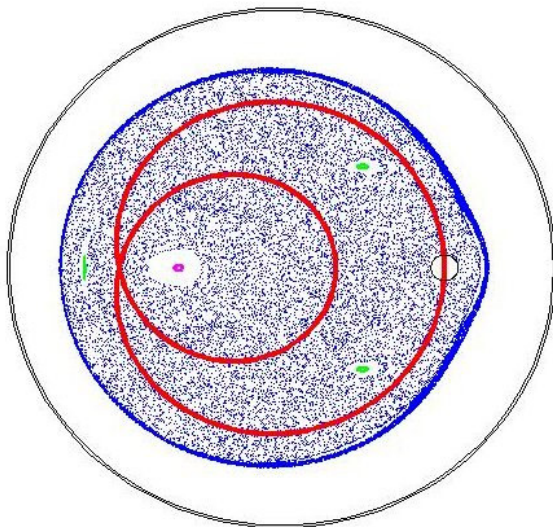
The **KENMEK '55'**
The Continental Mixer!
Shaves the work off hours in
a few seconds, assisting
kneading and full kneading.
So simple to use, it blends
breads and vegetables, drinks,
condiments and soups, and
saves time and the trouble
of mixing dough and condiments.

4940-7

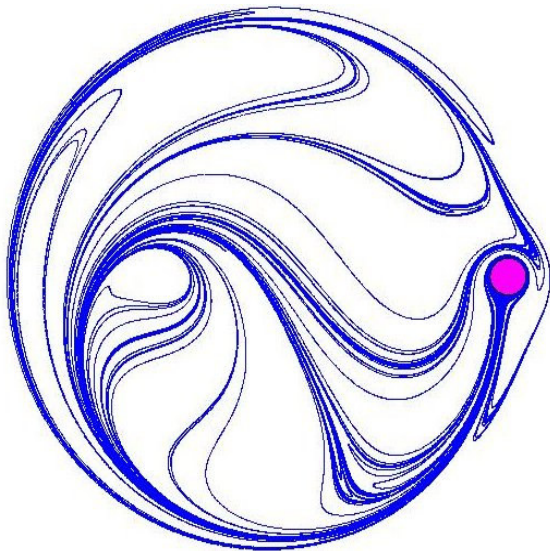
The **MINOR**
A highly efficient Portable
Mixer — for use
in Hall, in room, in tent,
in field — wherever it is
needed. It is small, light,
easy to carry, and has
enough power to mix
breads, cakes, soups, dips,
and other drinks and condiments.

4940-7

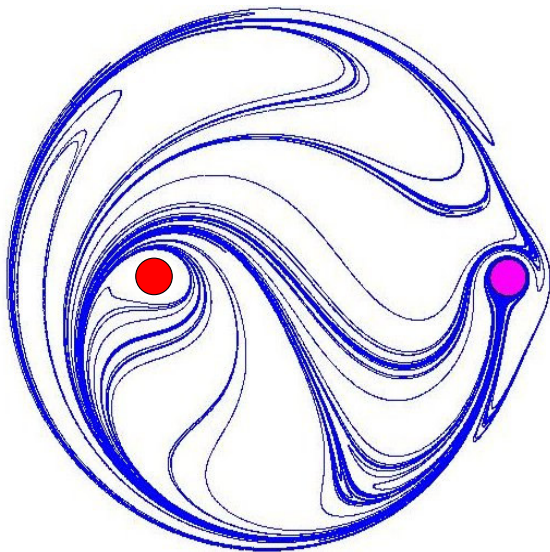
Poincaré Section



Stretching of Lines: A Ghostly Rod?

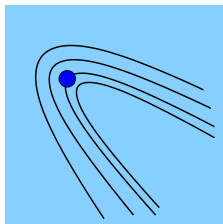
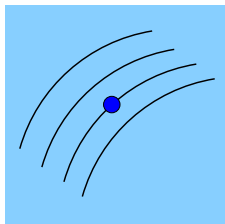


Stretching of Lines: A Ghostly Rod?



Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (blue dot).



Material lines must bend around the orbit: **it acts just like a “rod”!**

[J-LT, *Phys. Rev. Lett.* **94**, 084502 (2005)]

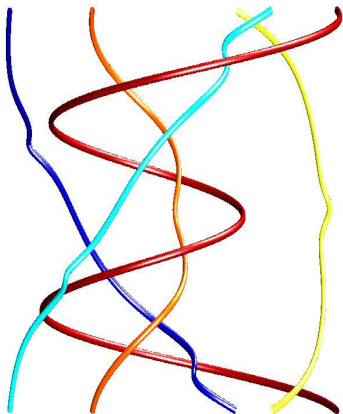
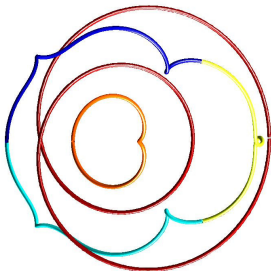
Today: focus on **periodic orbits**.

How do they braid around each other?

Motion of Islands

Make a braid from the motion of the rod and the **periodic islands**.

Most (74%) of the line-stretching is accounted for by this braid.

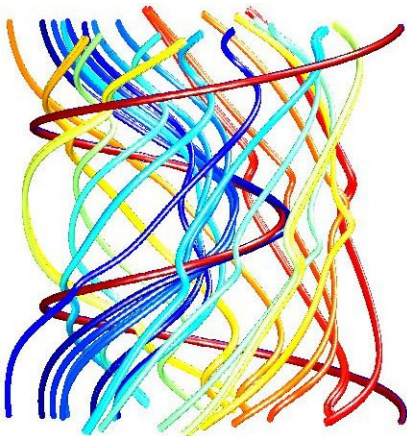


[G-T-F, "Topological Mixing with Ghost Rods," preprint, 2005.]

Motion of Islands and Unstable Periodic Orbits

Now we also include **unstable** periodic orbits as well as the stable ones (islands).

Almost all (99%) of the line-stretching is accounted for by this braid.



Conclusions

- Topological chaos involves moving **obstacles** in a 2D flow, which create nontrivial braids.
- A braid with positive topological entropy **guarantees** chaos in some region.
- Periodic orbits make great obstacles (in periodic flows), especially islands.
- This is a good way to “explain” the chaos in a flow — accounts for stretching of material lines.
- Other studies:
 - braids on the torus and sphere;
 - random braids;
 - optimisation via braids;
 - applications to open flows. . .

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References

- T–G–F, “The Size of Ghost Rods,” in *Proceedings of the Workshop on Analysis and Control of Mixing* (Springer-Verlag, 2006, in press).
- G–T–F, “Topological Mixing with Ghost Rods,” preprint, 2005.
- F–T–G, “Topological Chaos in Spatially Periodic Mixers,” in submission, 2005.
- T, “Measuring Topological Chaos,” *Physical Review Letters* **94** (8), 084502, March 2005.

Preprints and slides available at www.ma.imperial.ac.uk/~jeanluc