Topology, Braids, and Mixing

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with
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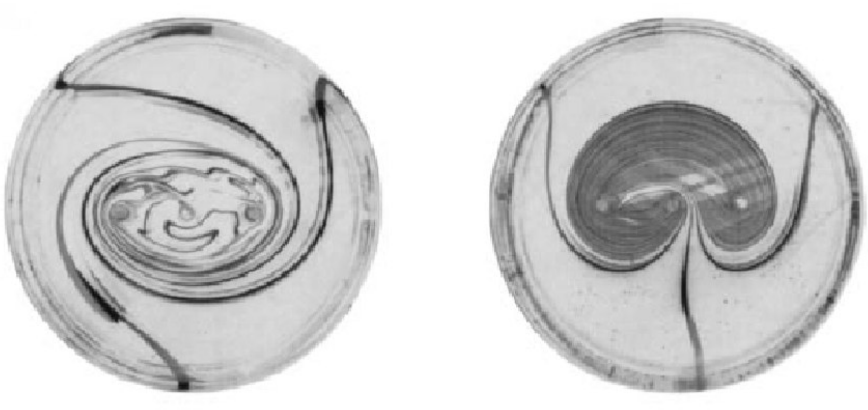
The Taffy Puller

[movie 1]

The Four-pronged Taffy Puller

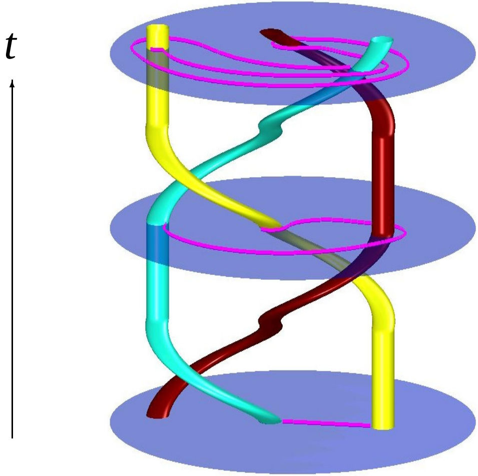
Experiment of Boyland, Aref, & Stremler

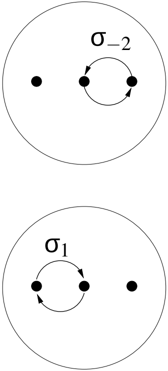


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]
[P. L. Boyland, M. A. Stremler, and H. Aref, *Physica D* **175**, 69 (2003)]

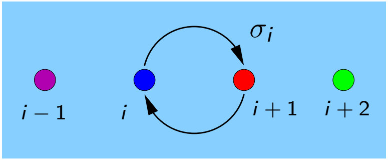
[movie 2] [movie 3]

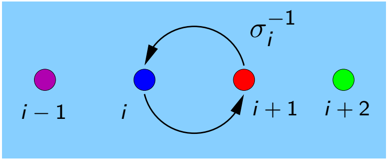
The Connection with Braids





Generators of the n-Braid Group





A generator of Artin's braid group B_n on n strands, denoted

$$\sigma_i\,, \quad i = 1, \ldots, n-1$$

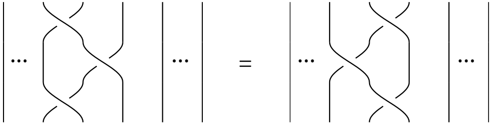
is the clockwise interchange of the i th and $(i+1)$th rod.

B_n is a finitely-generated group, with an infinite number of elements, called words.

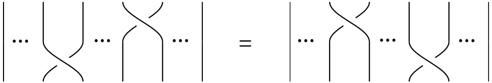
These generators are used to characterise the topological motion of the rods.

Presentation of Artin's Braid Group

The generators obey the presentation



$$\sigma_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\sigma_i$$

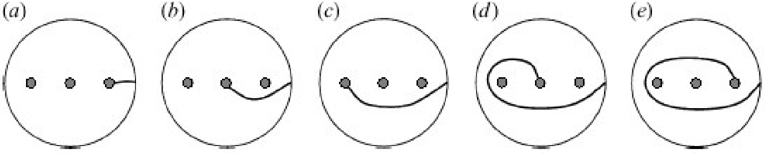


$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| > 1$$

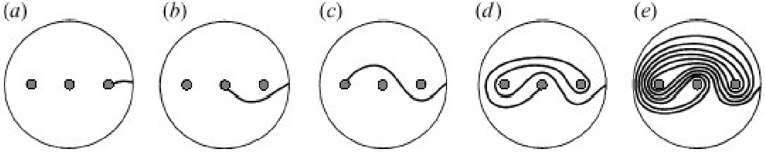
A presentation means that these are the only rules obeyed by the generators that are not the consequence of elementary group properties.

The Two BAS Stirring Protocols

$\sigma_1\sigma_2$ protocol



$\sigma_1^{-1}\sigma_2$ protocol

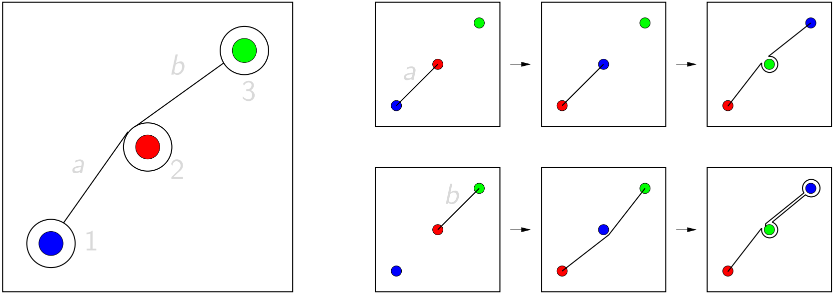


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Train-Tracks

What is the growth rate of an "elastic band" tied to the rods?

Train-tracks give the answer.



Elastic band has edges (letters) and infinitesimal loops (numbers). As the rods are moved, the edges and loops are mapped as

$$a \mapsto a2b, \quad b \mapsto a2b3b, \quad 1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2.$$

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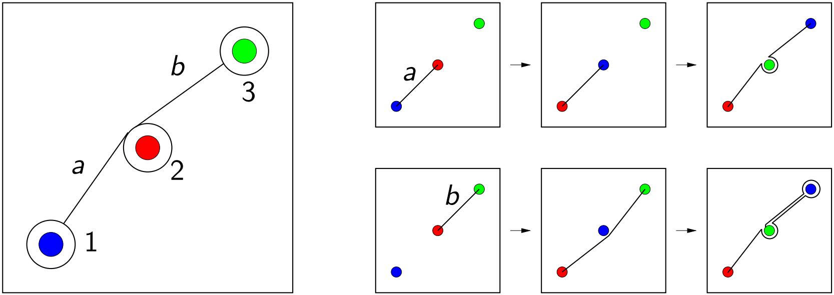

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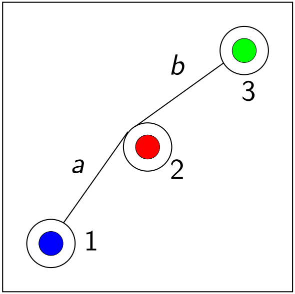

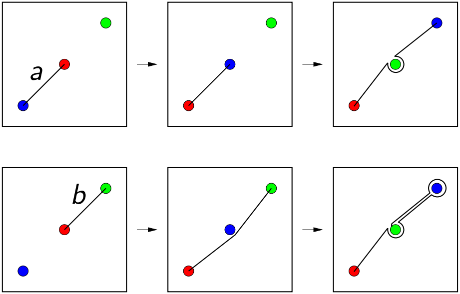

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The Evolution of Edges: Topological Entropy

The edges and loops are mapped according to

$$a \mapsto a2b, \quad b \mapsto a2b3b, \quad 1 \mapsto 3, \quad 2 \mapsto 1, \quad 3 \mapsto 2.$$

A crucial point is that edges are separated by loops: no cancellations can occur. A transition matrix can be formed:

$$M = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a \\ \leftarrow b \\ \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \end{array}$$

The largest eigenvalue gives the asymptotic growth factor of the elastic, 2.6180. The logarithm of this is the topological entropy of the braid.

[M. Bestvina and M. Handel, *Topology* **34**, 109 (1995)]

The Difference between BAS's Two Protocols

- Practically speaking, the topological entropy of a braid is a lower bound on the line-stretching exponent of the flow!
- The first (bad) stirring protocol has zero topological entropy.
- The second (good) protocol has topological entropy $\log[(3+\sqrt{5})/2] = 0.96 > 0$.
- So for the second protocol the length of a line joining the rods grows exponentially!
- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).

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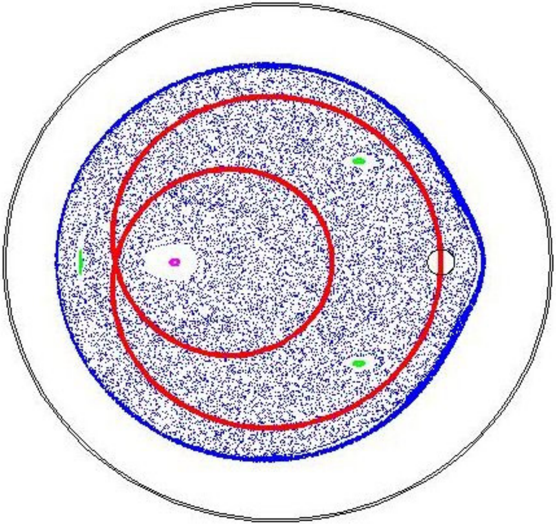
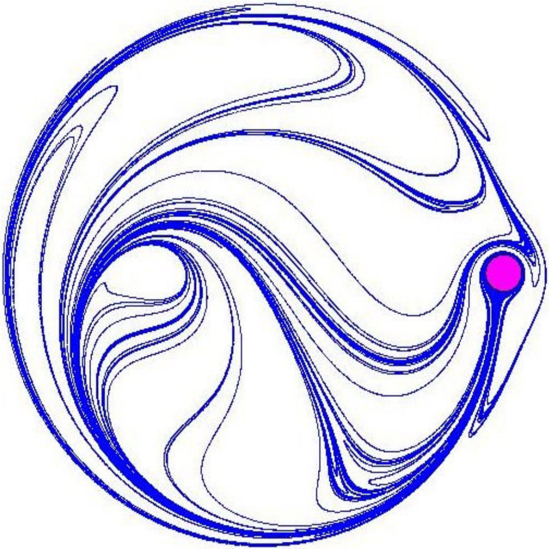
One Rod Mixer: The Kenwood Chef

Poincaré Section



Stretching of Lines: A Ghostly Rod?



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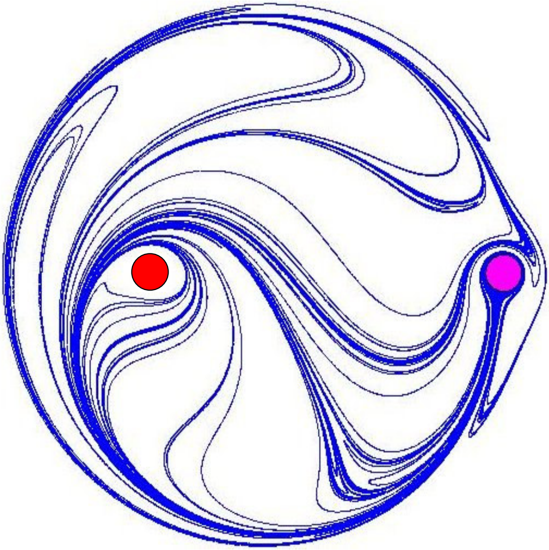
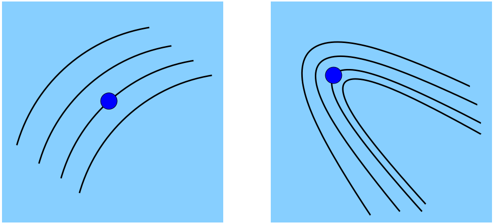


Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (blue dot).



Material lines must bend around the orbit: it acts just like a "rod"!

[J-LT, *Phys. Rev. Lett.* **94**, 084502 (2005)]

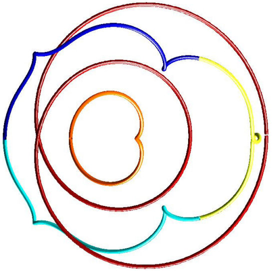
Today: focus on periodic orbits.

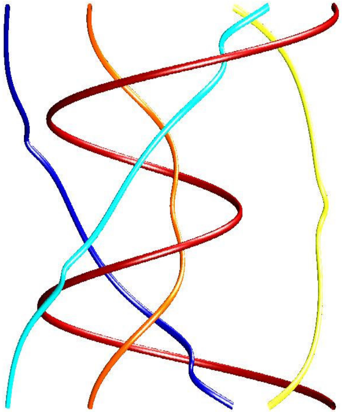
How do they braid around each other?

Motion of Islands

Make a braid from the motion of the rod and the periodic islands.

Most (74%) of the line-stretching is accounted for by this braid.



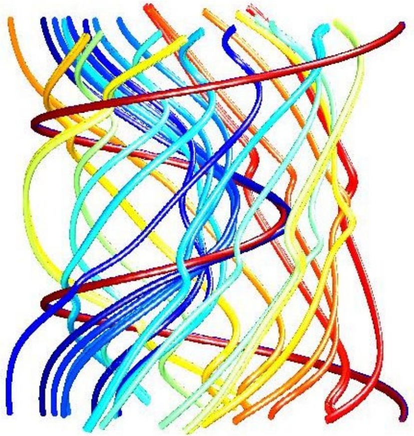


[G–T–F, "Topological Mixing with Ghost Rods," preprint, 2005.]

Motion of Islands and Unstable Periodic Orbits

Now we also include unstable periods orbits as well as the stable ones (islands).

Almost all (99%) of the line-stretching is accounted for by this braid.



Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- A braid with positive topological entropy guarantees chaos in some region.
- Periodic orbits make great obstacles (in periodic flows), especially islands.
- This is a good way to “explain” the chaos in a flow — accounts for stretching of material lines.
- Other studies:
    - braids on the torus and sphere;
    - random braids;
    - optimisation via braids;
    - applications to open flows...

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References

- T–G–F, "The Size of Ghost Rods," in *Proceedings of the Workshop on Analysis and Control of Mixing* (Springer-Verlag, 2006, in press).
- G–T–F, "Topological Mixing with Ghost Rods," preprint, 2005.
- F–T–G, "Topological Chaos in Spatially Periodic Mixers," in submission, 2005.
- T, "Measuring Topological Chaos," *Physical Review Letters* **94** (8), 084502, March 2005.

Preprints and slides available at www.ma.imperial.ac.uk/˜jeanluc