Transport and Mixing in Complex and Turbulent Flows

Progress since the 2010 IMA workshop

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

Institute for Mathematics and its Applications
Minneapolis, MN, 9 May 2014

Supported by NSF

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Three personal examples

To be able to prepare in time, I decided to narrow the focus to a few papers written at the IMA, with several participants:

Lin, Z., Doering, C. R., & Thiffeault, J.-L. (2011a). *J. Fluid Mech.* **675**, 465–476

Thomases, B., Shelley, M., & Thiffeault, J.-L. (2011). *Physica D,* **240**, 1602–1614

Lin, Z., Thiffeault, J.-L., & Childress, S. (2011b). *J. Fluid Mech.* **669**, 167–177

Zhi George Lin was a postdoc here and is now at Zhejiang University.

I'll discuss the papers and some of their impact.

I apologize for the shameless focus on my papers. . .

Optimization of mixing

Inspired by 'mix-norm' of Mathew *et al.* (2005): $H^{-1/2} \to H^{-1}$.

J. Fluid Mech. (2011), *vol.* 675, *pp.* 465–476. © Cambridge University Press 2011 465
doi:10.1017/S0022112011000292

Optimal stirring strategies for passive scalar mixing

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(Received 4 September 2010; revised 30 October 2010; accepted 12 January 2011; first published online 10 March 2011)

We address the challenge of optimal incompressible stirring to mix an initially inhomogeneous distribution of passive tracers. As a quantitative measure of mixing we adopt the H^{-1} norm of the scalar fluctuation field, equivalent to the (square root of the) variance of a low-pass filtered image of the tracer concentration field. First we establish that this is a useful gauge even in the absence of molecular diffusion: its vanishing as $t \to \infty$ is evidence of the stirring flow's mixing properties in the sense of ergodic theory. Then we derive absolute limits on the total amount of mixing

Direct descendant of IMA

JOURNAL OF MATHEMATICAL PHYSICS **53**, 115611 (2012)

Optimal mixing and optimal stirring for fixed energy, fixed power, or fixed palenstrophy flows

Evelyn Lunasin,[1] Zhi Lin,[2,a)] Alexei Novikov,[3] Anna Mazzucato,[3]
and Charles R. Doering[4]
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[2]*Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang 310013, People's Republic of China*
[3]*Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802, USA*
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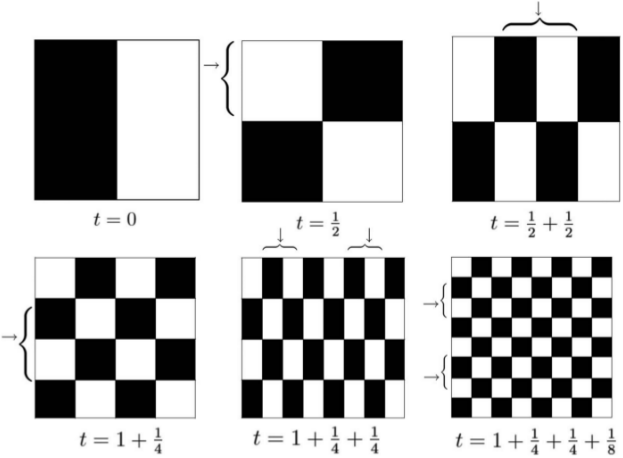
We consider passive scalar mixing by a prescribed divergence-free velocity vector field in a periodic box and address the following question: Starting from a given initial inhomogeneous distribution of passive tracers, and given a certain energy budget, power budget, or finite palenstrophy budget, what incompressible flow field best mixes the scalar quantity? We focus on the optimal stirring strategy recently proposed by Lin *et al.* ["Optimal stirring strategies for passive scalar mixing," J. Fluid Mech. **675**, 465 (2011)] that determines the flow field that instantaneously maximizes the depletion of the H^{-1} mix-norm. In this work, we bridge some of the gap between the best available *a priori* analysis and simulation results. After recalling some previous analysis, we present an explicit example demonstrating finite-time perfect mixing with a finite energy constraint on the stirring flow. On the other hand, using a recent result by Wirosoetisno *et al.* ["Long time stability of a classical efficient scheme for two dimensional Navier-Stokes equations," SIAM J. Numer. Anal. **50**(1), 126–150 (2012)] we establish that the H^{-1} mix-norm decays at most exponentially

Checkerboard maps provide nice bounds



$t = 0$

$t = \frac{1}{2}$

$t = \frac{1}{2} + \frac{1}{2}$

$t = 1 + \frac{1}{4}$

$t = 1 + \frac{1}{4} + \frac{1}{4}$

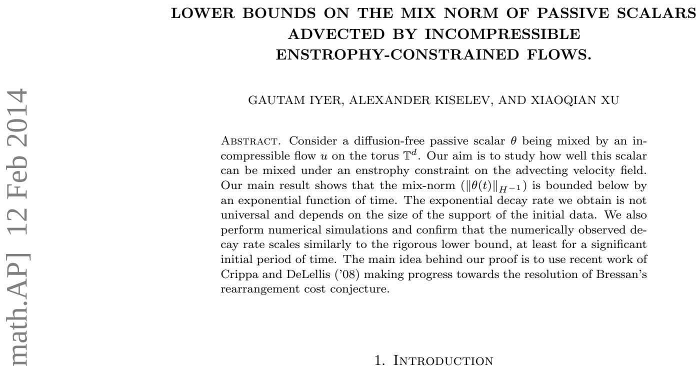
$t = 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{8}$

FIG. 1. Perfect mixing in finite time $t = 2 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 2$, with $\|\mathbf{u}\|_{L^2}$ constant in time.

Lunasin *et al.* (2012)

H^{-1} led to renewed interest in bounds on mixing

(Two workshop participants with a student.)



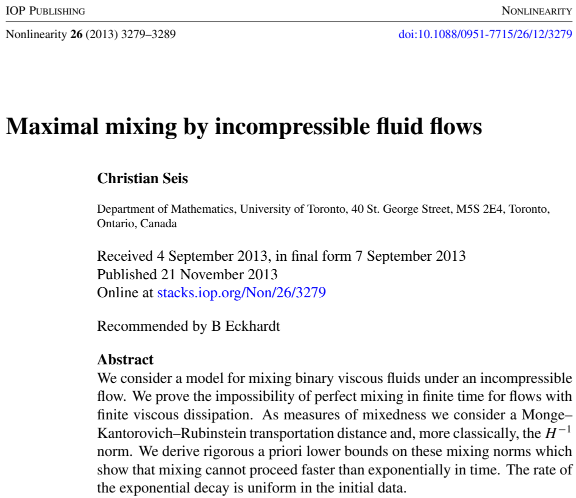
LOWER BOUNDS ON THE MIX NORM OF PASSIVE SCALARS ADVECTED BY INCOMPRESSIBLE ENSTROPHY-CONSTRAINED FLOWS.

GAUTAM IYER, ALEXANDER KISELEV, AND XIAOQIAN XU

ABSTRACT. Consider a diffusion-free passive scalar θ being mixed by an incompressible flow u on the torus $\mathbb{T}^d$. Our aim is to study how well this scalar can be mixed under an enstrophy constraint on the advecting velocity field. Our main result shows that the mix-norm ($\|\theta(t)\|_{H^{-1}}$) is bounded below by an exponential function of time. The exponential decay rate we obtain is not universal and depends on the size of the support of the initial data. We also perform numerical simulations and confirm that the numerically observed decay rate scales similarly to the rigorous lower bound, at least for a significant initial period of time. The main idea behind our proof is to use recent work of Crippa and DeLellis ('08) making progress towards the resolution of Bressan's rearrangement cost conjecture.

1. INTRODUCTION

More H^{-1}...



IOP PUBLISHING NONLINEARITY
Nonlinearity **26** (2013) 3279–3289 doi:10.1088/0951-7715/26/12/3279

Maximal mixing by incompressible fluid flows

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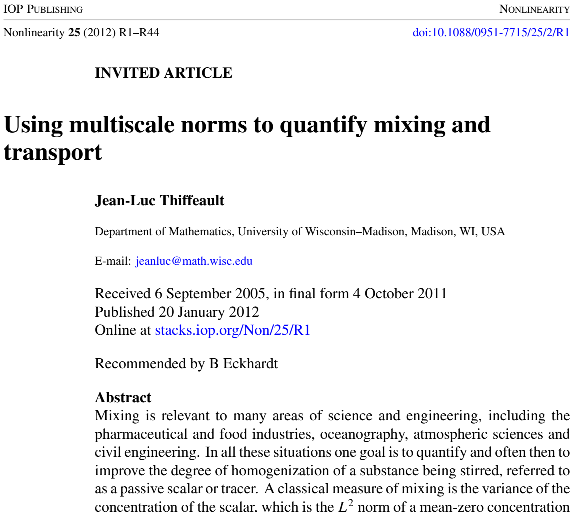
Published 21 November 2013
Online at stacks.iop.org/Non/26/3279

Recommended by B Eckhardt

Abstract

We consider a model for mixing binary viscous fluids under an incompressible flow. We prove the impossibility of perfect mixing in finite time for flows with finite viscous dissipation. As measures of mixedness we consider a Monge–Kantorovich–Rubinstein transportation distance and, more classically, the H^{-1} norm. We derive rigorous a priori lower bounds on these mixing norms which show that mixing cannot proceed faster than exponentially in time. The rate of the exponential decay is uniform in the initial data.

Eventually, a review paper



IOP Publishing NONLINEARITY

Nonlinearity **25** (2012) R1–R44 doi:10.1088/0951-7715/25/2/R1

INVITED ARTICLE

Using multiscale norms to quantify mixing and transport

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Received 6 September 2005, in final form 4 October 2011
Published 20 January 2012
Online at stacks.iop.org/Non/25/R1

Recommended by B Eckhardt

Abstract

Mixing is relevant to many areas of science and engineering, including the pharmaceutical and food industries, oceanography, atmospheric sciences and civil engineering. In all these situations one goal is to quantify and often then to improve the degree of homogenization of a substance being stirred, referred to as a passive scalar or tracer. A classical measure of mixing is the variance of the concentration of the scalar, which is the L^2 norm of a mean-zero concentration

Mixing in viscoelastic flows

Follow-up papers have focused on the flow, not mixing per se.

Physica D 240 (2011) 1602–1614

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

A Stokesian viscoelastic flow: Transition to oscillations and mixing

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ARTICLE INFO

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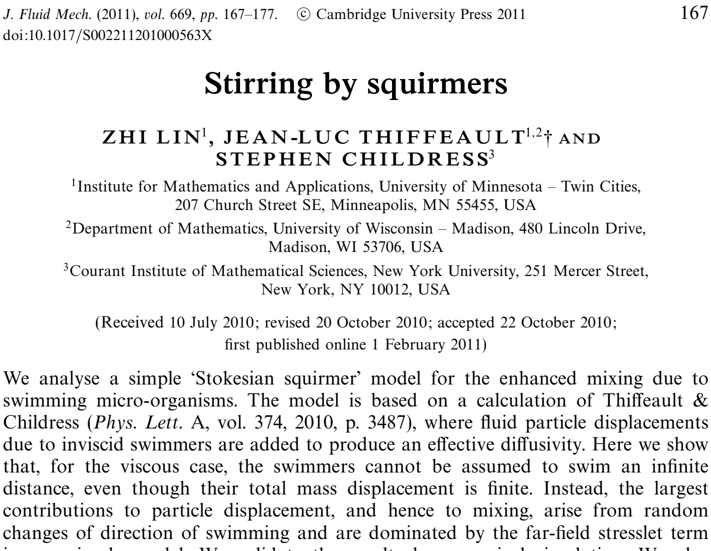
Instability
Mixing
Microfluidics

ABSTRACT

To understand observations of low Reynolds number mixing and flow transitions in viscoelastic fluids, we study numerically the dynamics of the Oldroyd-B viscoelastic fluid model. The fluid is driven by a simple time-independent forcing that, in the absence of viscoelastic stresses, creates a cellular flow with extensional stagnation points. We find that at $\mathcal{O}(1)$ Weissenberg number, these flows lose their slaving to the forcing geometry of the background force, become oscillatory with multiple frequencies, and show continual formation and destruction of small-scale vortices. This drives flow mixing, the details of which we closely examine. These new flow states are dominated by a single-quadrant vortex, which may be

Biomixing



J. Fluid Mech. (2011), *vol.* 669, *pp.* 167–177. © Cambridge University Press 2011 167
doi:10.1017/S002211201000563X

Stirring by squirmers

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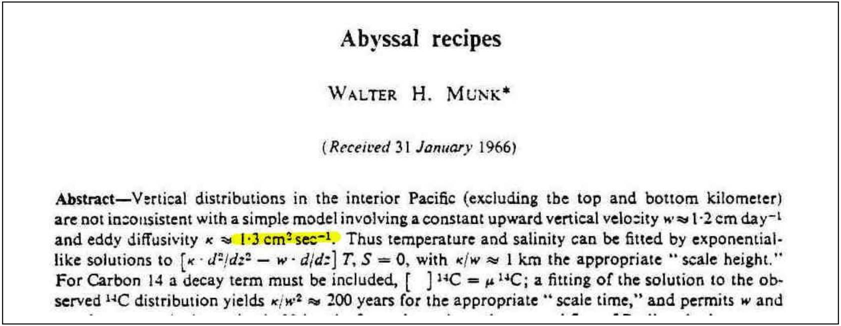
first published online 1 February 2011)

We analyse a simple 'Stokesian squirmer' model for the enhanced mixing due to swimming micro-organisms. The model is based on a calculation of Thiffeault & Childress (*Phys. Lett.* A, vol. 374, 2010, p. 3487), where fluid particle displacements due to inviscid swimmers are added to produce an effective diffusivity. Here we show that, for the viscous case, the swimmers cannot be assumed to swim an infinite distance, even though their total mass displacement is finite. Instead, the largest contributions to particle displacement, and hence to mixing, arise from random changes of direction of swimming and are dominated by the far-field stresslet term

Munk's Idea

Though it had been mentioned earlier, the first to seriously consider the role of ocean biomixing was Walter Munk (1966):



Abyssal recipes

WALTER H. MUNK*

(*Received* 31 *January* 1966)

Abstract—Vertical distributions in the interior Pacific (excluding the top and bottom kilometer) are not inconsistent with a simple model involving a constant upward vertical velocity $w \approx 1 \cdot 2$ cm day^{-1} and eddy diffusivity $\kappa \approx 1 \cdot 3$ cm^2 sec^{-1}. Thus temperature and salinity can be fitted by exponential-like solutions to $[\kappa \cdot d^2/dz^2 - w \cdot d/dz]\, T, S = 0$, with $\kappa/w \approx 1$ km the appropriate "scale height." For Carbon 14 a decay term must be included, $[\quad]\,{}^{14}C = \mu\,{}^{14}C$; a fitting of the solution to the observed ^{14}C distribution yields $\kappa/w^2 \approx 200$ years for the appropriate "scale time," and permits w and

"…I have attempted, without much success, to interpret [the eddy diffusivity] from a variety of viewpoints: from mixing along the ocean boundaries, from thermodynamic and biological processes, and from internal tides."

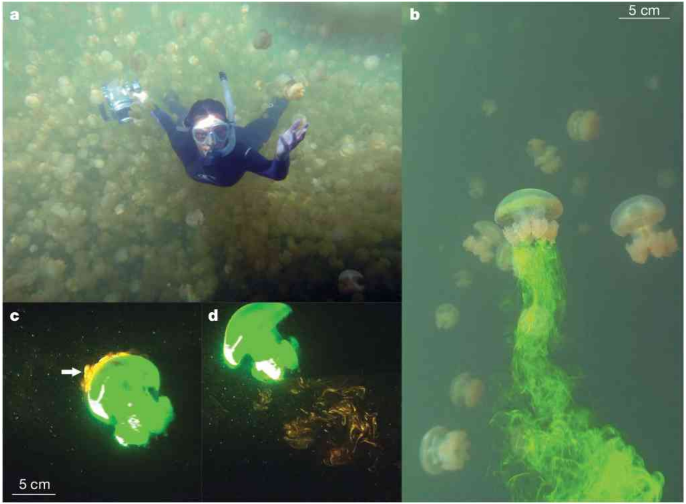
Ocean biomixing: Basic observations

The idea lay dormant for almost 40 years; then

- Huntley & Zhou (2004) analyzed swimming of 100 (!) species, ranging from bacteria to blue whales. Typical turbulent energy production is $\sim 10^{-5}\ \mathrm{W\ kg^{-1}}$. Total is comparable to energy dissipation by major storms.
- Another estimate comes from the solar energy captured: $63\ \mathrm{TeraW}$, something like 1% of which ends up as mechanical energy (Dewar *et al.*, 2006).
- Kunze *et al.* (2006) find that turbulence levels during the day in an inlet were 2 to 3 orders of magnitude greater than at night, due to swimming krill.
- However, Kunze has failed to find this effect again on subsequent cruises. Visser (2007) has questioned whether small-scale turbulence can lead to overturning.

In situ experiments

Katija & Dabiri (2009) looked at jellyfish:



play movie (Palau's Jellyfish Lake.) Correct length scale is path length?

Displacement by a moving body

86 Mr. J. Clerk-Maxwell *on* [Mar. 10,

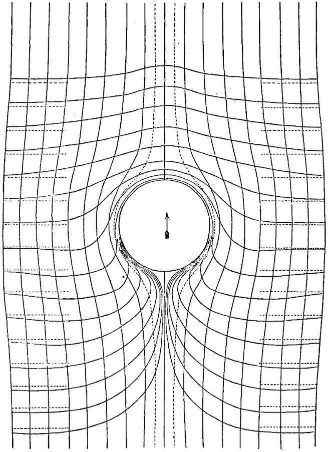


FIG. 1.
Fluid flowing past a fixed cylinder.

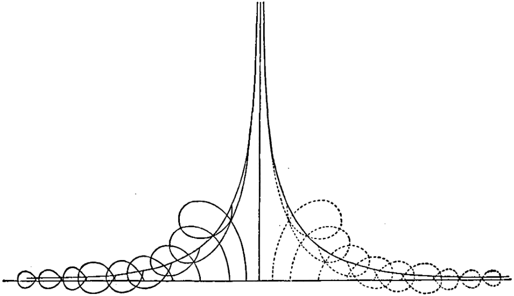


FIG. 2.
Paths of particles of the fluid when a cylinder moves through it.

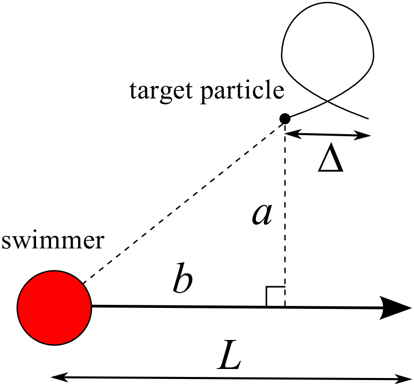
Maxwell (1869); Darwin (1953); Eames *et al.* (1994)

A sequence of kicks

Inspired by Einstein's theory of diffusion (Einstein, 1956): a test particle initially at $\mathbf{x}(0) = 0$ undergoes N encounters with an axially-symmetric swimming body:

$$\mathbf{x}(t) = \sum_{k=1}^{N} \Delta_L(a_k, b_k)\,\hat{\mathbf{r}}_k$$

$\Delta_L(a, b)$ is the displacement, a_k, b_k are impact parameters, and $\hat{\mathbf{r}}_k$ is a direction vector.



($a > 0$, but b can have either sign.)

After squaring and averaging, assuming isotropy:

$$\left\langle |\mathbf{x}|^2 \right\rangle = N \left\langle \Delta_L^2(a,b) \right\rangle$$

where a and b are treated as random variables with densities

$$\mathrm{d}\mathbf{A}/V = 2\,\mathrm{d}a\,\mathrm{d}b/V \quad (\text{2D}) \quad \text{or} \qquad 2\pi a\,\mathrm{d}a\,\mathrm{d}b/V \quad (\text{3D})$$

Replace average by integral:

$$\left\langle |\mathbf{x}|^2 \right\rangle = \frac{N}{V} \int \Delta_L^2(a,b)\,\mathrm{d}\mathbf{A}$$

Writing $n = 1/V$ for the number density (there is only one swimmer) and $N = Ut/L$ (L/U is the time between steps):

$$\left\langle |\mathbf{x}|^2 \right\rangle = \frac{Unt}{L} \int \Delta_L^2(a,b)\,\mathrm{d}\mathbf{A}$$

Effective diffusivity

Putting this together,

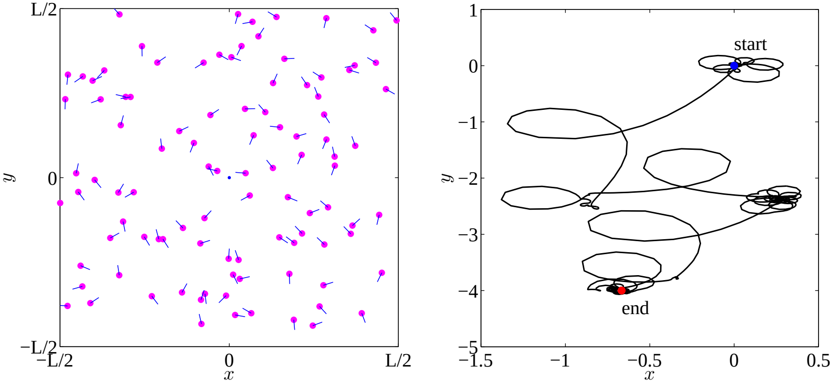
$$\left\langle|\mathbf{x}|^2\right\rangle = \frac{2Unt}{L}\int \Delta_L^2(a,b)\,\mathrm{d}a\,\mathrm{d}b = 4\kappa t, \qquad \text{2D}$$

$$\left\langle|\mathbf{x}|^2\right\rangle = \frac{2\pi Unt}{L}\int \Delta_L^2(a,b)a\,\mathrm{d}a\,\mathrm{d}b = 6\kappa t, \qquad \text{3D}$$

which defines the effective diffusivity κ.

If the number density is low $(nL^d \ll 1)$, then encounters are rare and we can use this formula for a collection of particles.

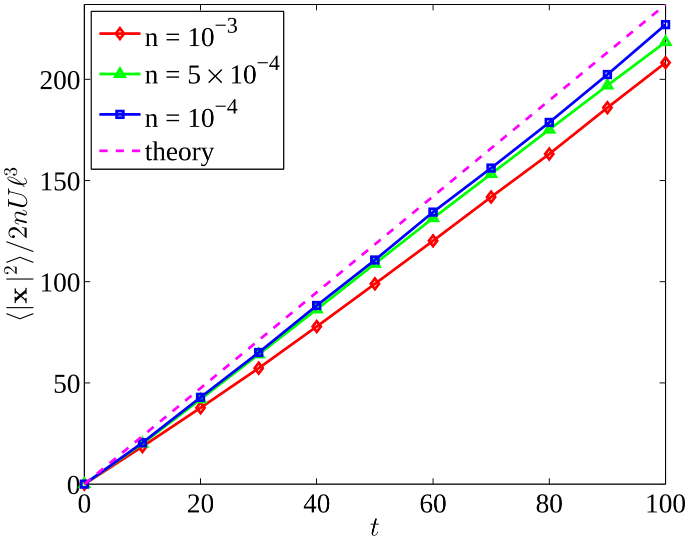
A ‘gas’ of swimmers



play movie

100 cylinders, box size = 1000

How well does the dilute theory work?

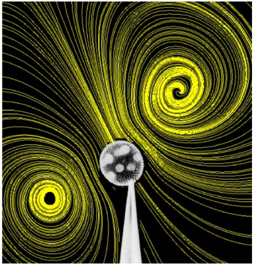


Squirmers: The viscous world

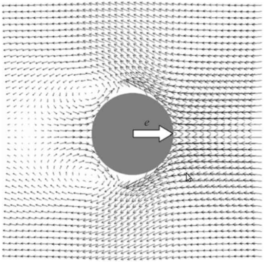
Considerable literature on transport due to microorganisms: Wu & Libchaber (2000); Hernandez-Ortiz *et al.* (2005); Saintillian & Shelley (2007); Ishikawa & Pedley (2007); Underhill *et al.* (2008); Ishikawa (2009); Leptos *et al.* (2009)

Lighthill (1952), Blake (1971), and more recently Ishikawa *et al.* (2006) have considered squirmers:

- Sphere in Stokes flow;
- Steady velocity specified at surface, to mimic cilia;
- Steady swimming condition imposed (no net force on fluid).



(Drescher *et al.*, 2009)



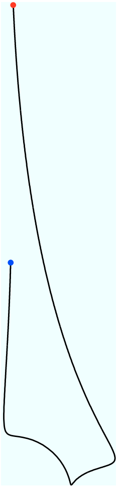
(Ishikawa *et al.*, 2006)

Particle motion for squirmer

A particle near the squirmer's swimming axis initially (blue) moves towards the squirmer.

After the squirmer has passed the particle follows in the squirmer's wake.

(The squirmer moves from bottom to top.)

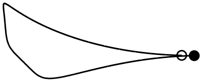


play movie

Squirmers: Trajectories

The two peaks in the displacement plot come from ‘incomplete’ trajectories:

$b/L = 0$

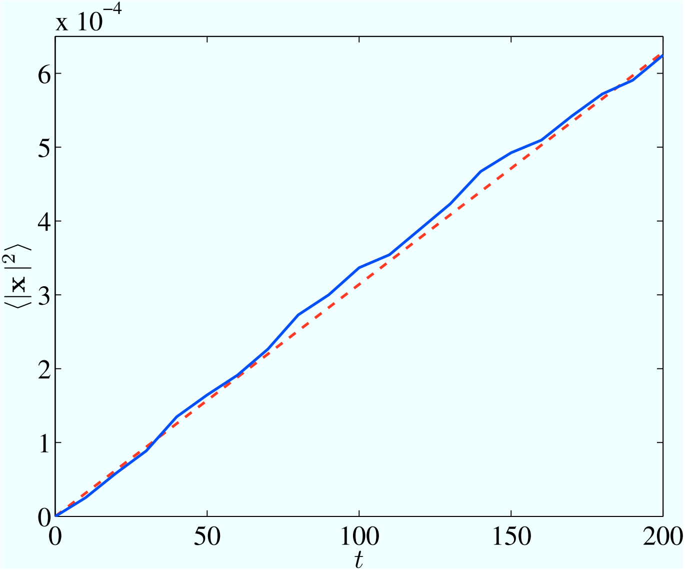


$b/L = 0.5$

$b/L = 1$

For long path length, the effective diffusivity is independent of the swimming path length, and yet the dominant contribution arises from the finiteness of the path (uncorrelated turning directions).

Squirmers: Transport



Some recent papers on biomixing I

Doostmohammadia, A., Stocker, R., & Ardekani, A. M. (2011). *Proc. Natl. Acad. Sci. USA,* **109** (10), 3856–3861

Eckhardt, B. & Zammert, S. (2012). *Eur. Phys. J. E,* **35**, 96

Kunze, E. (2011). *J. Mar. Res.* **69** (4-6), 591–601

Jepson, A., Martinez, V. A., Schwarz-Linek, J., Morozov, A., & Poon, W. C. K. (2013). *Phys. Rev. E,* **88**, 041002

Katija, K. (2012). *J. Exp. Biol.* **215**, 1040–1049

Khurana, N., Blawzdziewicz, J., & Ouellette, N. T. (2011). *Phys. Rev. Lett.* **106**, 198104

Kurtuldu, H., Guasto, J. S., Johnson, K. A., & Gollub, J. P. (2011). *Proc. Natl. Acad. Sci. USA,* **108** (26), 10391–10395

Lambert, R. A., Picano, F., Breugem, W.-P., & Brandt, L. (2013). *J. Fluid Mech.* **733**, 528–557

Leshansky, A. M. & Pismen, L. M. (2010). *Phys. Rev. E,* **82**, 025301

Miño, G. L., Dunstan, J., Rousselet, A., Clément, E., & Soto, R. (2013). *J. Fluid Mech.* **729**, 423–444

Morozov, A. & Marenduzzo, D. (2014). *Soft Matter,* **10**, 2748–2758

Noss, C. & Lorke, A. (2014). *Limnol. Oceanogr.* **59** (3), 724–732

Parra-Rojas, C. & Soto, R. (2013). *Phys. Rev. E,* **87**, 053022

Parra-Rojas, C. & Soto, R. (2014). arXiv:1404.4857

Some recent papers on biomixing II

Pushkin, D. O., Shum, H., & Yeomans, J. M. (2013). *J. Fluid Mech.* **726**, 5–25

Pushkin, D. O. & Yeomans, J. M. (2013). *Phys. Rev. Lett.* **111**, 188101

Pushkin, D. O. & Yeomans, J. M. (2014). arxiv:1403.2619

Rousseau, S., Kunze, E., Dewey, R., Bartlett, K., & Dower, J. (2010). *J. Phys. Ocean.* **40** (9), 2107–2121

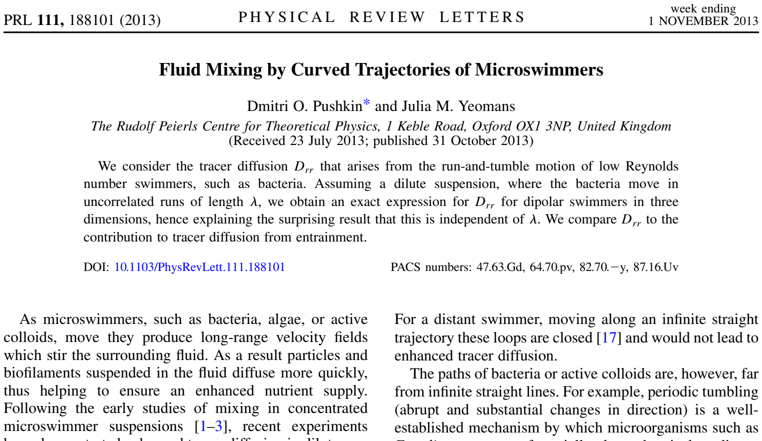
Saintillan, D. (2010). *Physics,* **3**, 84

Saintillan, D. & Shelley, M. J. (2012). *J. Roy. Soc. Interface,* **9**, 571–585

New effects are being included

In particular, Pushkin and Yeomans confirm that in many cases transport is dominated by entrainment rather than drift:

$$D_{\mathrm{entr}} \sim \frac{1}{2d}\, nU\ell\, V_{\mathrm{entr}}$$



PRL **111,** 188101 (2013) PHYSICAL REVIEW LETTERS week ending
1 NOVEMBER 2013

Fluid Mixing by Curved Trajectories of Microswimmers

Dmitri O. Pushkin[*] and Julia M. Yeomans
The Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, United Kingdom

(Received 23 July 2013; published 31 October 2013)

We consider the tracer diffusion D_{rr} that arises from the run-and-tumble motion of low Reynolds number swimmers, such as bacteria. Assuming a dilute suspension, where the bacteria move in uncorrelated runs of length λ, we obtain an exact expression for D_{rr} for dipolar swimmers in three dimensions, hence explaining the surprising result that this is independent of λ. We compare D_{rr} to the contribution to tracer diffusion from entrainment.

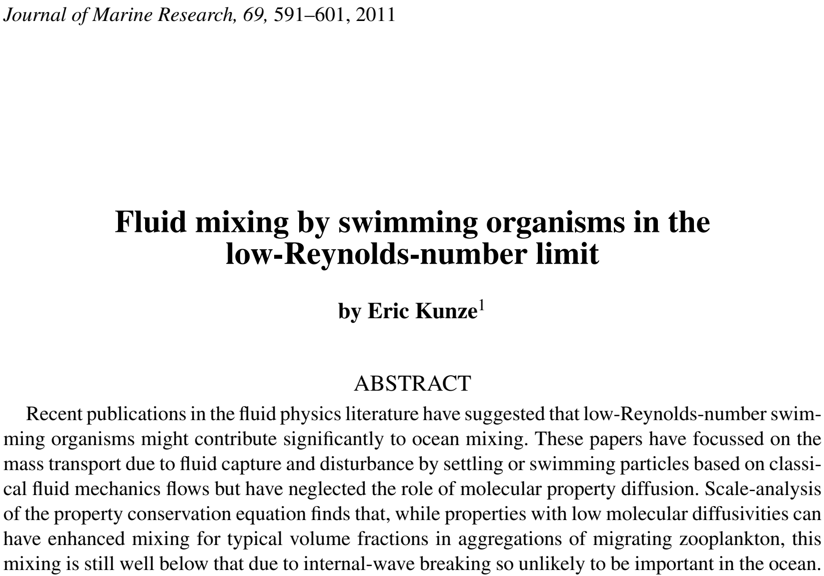
DOI: 10.1103/PhysRevLett.111.188101 PACS numbers: 47.63.Gd, 64.70.pv, 82.70.−y, 87.16.Uv

As microswimmers, such as bacteria, algae, or active colloids, move they produce long-range velocity fields which stir the surrounding fluid. As a result particles and biofilaments suspended in the fluid diffuse more quickly, thus helping to ensure an enhanced nutrient supply. Following the early studies of mixing in concentrated microswimmer suspensions [1–3], recent experiments

For a distant swimmer, moving along an infinite straight trajectory these loops are closed [17] and would not lead to enhanced tracer diffusion.

The paths of bacteria or active colloids are, however, far from infinite straight lines. For example, periodic tumbling (abrupt and substantial changes in direction) is a well-established mechanism by which microorganisms such as

Some dampened enthusiasm for ocean biomixing



Journal of Marine Research, 69, 591–601, 2011

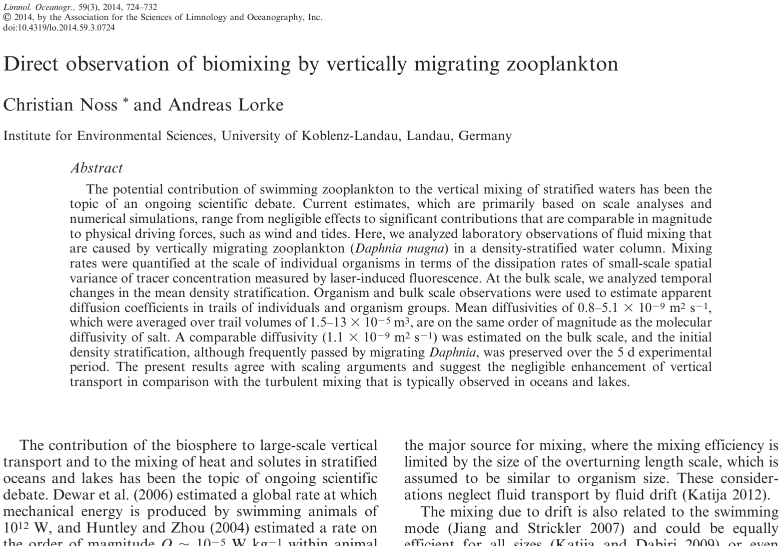
Fluid mixing by swimming organisms in the low-Reynolds-number limit

by Eric Kunze[1]

ABSTRACT

Recent publications in the fluid physics literature have suggested that low-Reynolds-number swimming organisms might contribute significantly to ocean mixing. These papers have focussed on the mass transport due to fluid capture and disturbance by settling or swimming particles based on classical fluid mechanics flows but have neglected the role of molecular property diffusion. Scale-analysis of the property conservation equation finds that, while properties with low molecular diffusivities can have enhanced mixing for typical volume fractions in aggregations of migrating zooplankton, this mixing is still well below that due to internal-wave breaking so unlikely to be important in the ocean.

Recent experiments back Kunze



Limnol. Oceanogr., 59(3), 2014, 724–732

doi:10.4319/lo.2014.59.3.0724

Direct observation of biomixing by vertically migrating zooplankton

Christian Noss * and Andreas Lorke

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Abstract

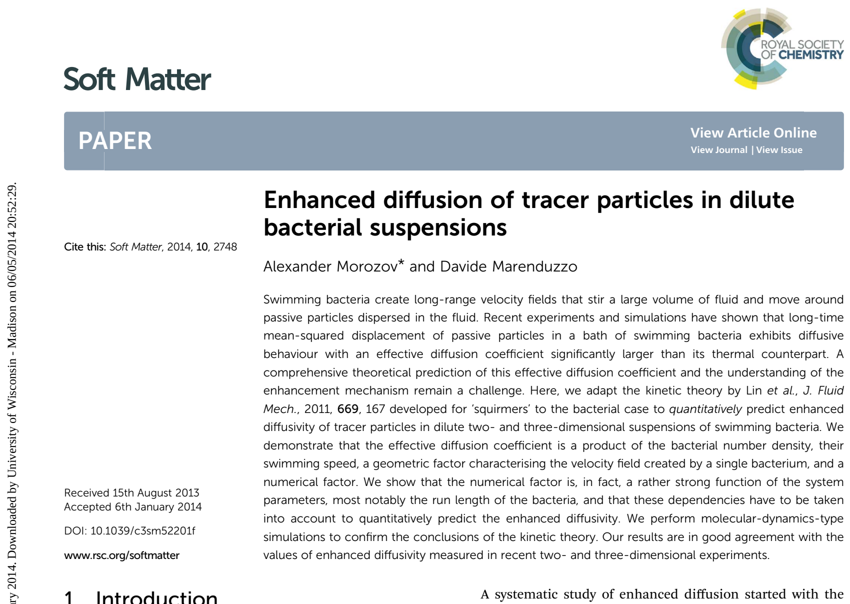
The potential contribution of swimming zooplankton to the vertical mixing of stratified waters has been the topic of an ongoing scientific debate. Current estimates, which are primarily based on scale analyses and numerical simulations, range from negligible effects to significant contributions that are comparable in magnitude to physical driving forces, such as wind and tides. Here, we analyzed laboratory observations of fluid mixing that are caused by vertically migrating zooplankton (*Daphnia magna*) in a density-stratified water column. Mixing rates were quantified at the scale of individual organisms in terms of the dissipation rates of small-scale spatial variance of tracer concentration measured by laser-induced fluorescence. At the bulk scale, we analyzed temporal changes in the mean density stratification. Organism and bulk scale observations were used to estimate apparent diffusion coefficients in trails of individuals and organism groups. Mean diffusivities of 0.8–5.1 × 10^{-9} m^2 s^{-1}, which were averaged over trail volumes of 1.5–13 × 10^{-5} m^3, are on the same order of magnitude as the molecular diffusivity of salt. A comparable diffusivity (1.1 × 10^{-9} m^2 s^{-1}) was estimated on the bulk scale, and the initial density stratification, although frequently passed by migrating *Daphnia*, was preserved over the 5 d experimental period. The present results agree with scaling arguments and suggest the negligible enhancement of vertical transport in comparison with the turbulent mixing that is typically observed in oceans and lakes.

The contribution of the biosphere to large-scale vertical transport and to the mixing of heat and solutes in stratified oceans and lakes has been the topic of ongoing scientific debate. Dewar et al. (2006) estimated a global rate at which mechanical energy is produced by swimming animals of 10^{12} W, and Huntley and Zhou (2004) estimated a rate on the order of magnitude $Q \sim 10^{-5}$ W kg^{-1} within animal

the major source for mixing, where the mixing efficiency is limited by the size of the overturning length scale, which is assumed to be similar to organism size. These considerations neglect fluid transport by fluid drift (Katija 2012).

The mixing due to drift is also related to the swimming mode (Jiang and Strickler 2007) and could be equally efficient for all sizes (Katija and Dabiri 2009) or even

More refinements of the model



Soft Matter

ROYAL SOCIETY OF CHEMISTRY

PAPER

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Enhanced diffusion of tracer particles in dilute bacterial suspensions

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Swimming bacteria create long-range velocity fields that stir a large volume of fluid and move around passive particles dispersed in the fluid. Recent experiments and simulations have shown that long-time mean-squared displacement of passive particles in a bath of swimming bacteria exhibits diffusive behaviour with an effective diffusion coefficient significantly larger than its thermal counterpart. A comprehensive theoretical prediction of this effective diffusion coefficient and the understanding of the enhancement mechanism remain a challenge. Here, we adapt the kinetic theory by Lin *et al.*, *J. Fluid Mech.*, 2011, **669**, 167 developed for 'squirmers' to the bacterial case to *quantitatively* predict enhanced diffusivity of tracer particles in dilute two- and three-dimensional suspensions of swimming bacteria. We demonstrate that the effective diffusion coefficient is a product of the bacterial number density, their swimming speed, a geometric factor characterising the velocity field created by a single bacterium, and a numerical factor. We show that the numerical factor is, in fact, a rather strong function of the system parameters, most notably the run length of the bacteria, and that these dependencies have to be taken into account to quantitatively predict the enhanced diffusivity. We perform molecular-dynamics-type simulations to confirm the conclusions of the kinetic theory. Our results are in good agreement with the values of enhanced diffusivity measured in recent two- and three-dimensional experiments.

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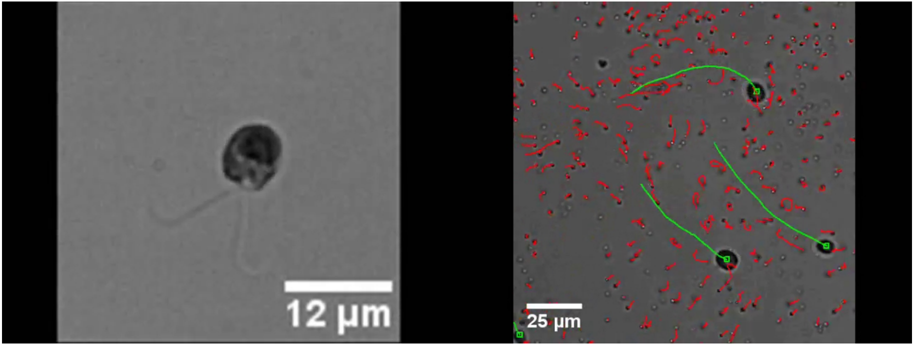
www.rsc.org/softmatter

1 Introduction

A systematic study of enhanced diffusion started with the

Chlamydomonas reinhardtii

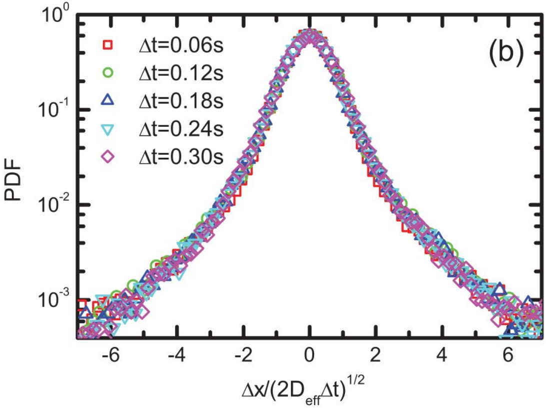


play movie

[Guasto, J. S., Johnson, K. A., & Gollub, J. P. (2010). *Phys. Rev. Lett.* **105**, 168102]

Probability density of displacements

Non-Gaussian PDF with ‘exponential’ tails:



[Leptos, K. C., Guasto, J. S., Gollub, J. P., Pesci, A. I., & Goldstein, R. E. (2009). *Phys. Rev. Lett.* **103**, 198103]

Probability density of displacements

Leptos *et al.* (2009) claim a reasonable fit of their PDF with the form

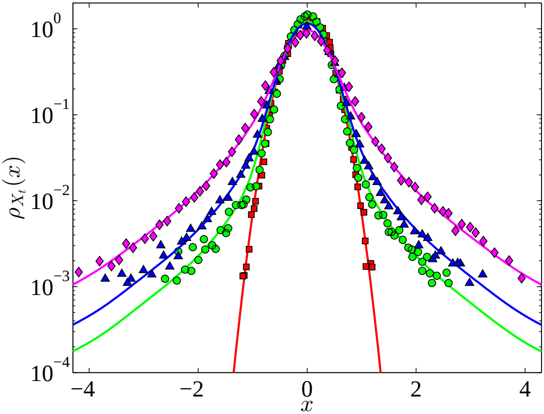
$$P_{\Delta t}(\Delta x) = \frac{1-f}{\sqrt{2\pi}\,\delta_g}\,\mathrm{e}^{-(\Delta x)^2/2\delta_g^2} + \frac{f}{2\delta_e}\,\mathrm{e}^{-|\Delta x|/\delta_e}$$

They observe the scalings $\delta_g \sim A_g(\Delta t)^{1/2}$ and $\delta_e \sim A_e(\Delta t)^{1/2}$, where A_g and A_e depend on ϕ.

They call this a diffusive scaling, since $\Delta x \sim \Delta t^{1/2}$. Their point is that this is strange, since the distribution is not Gaussian.

Commonly observed in diffusive processes that are a combination of trapped and hopping dynamics (Wang *et al.*, 2012).

PDF: Theory vs experiment



The normalized PDF for experimental data (dashed) agrees well with simple swimmer models. Eckhardt & Zammert (2012) have a phenomenological model.

Summary

H^{-1} as a measure of mixing:

- Allows analytical progress (bounds).
- Related to 'Bressan conjecture.'

Viscoelastic mixing:

- A fairly open area. . .
- Unfortunately has not caught on yet.

Biomixing:

- Settled in the ocean case?
- Interest has shifted to microswimmers.
- Many more lab experiments.
- We understand the details better (probabilistic tools).
- The approach developed at the IMA has been used and refined by others. Compares well to data and numerical experiments.

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