Transport and mixing by viscous vortex rings

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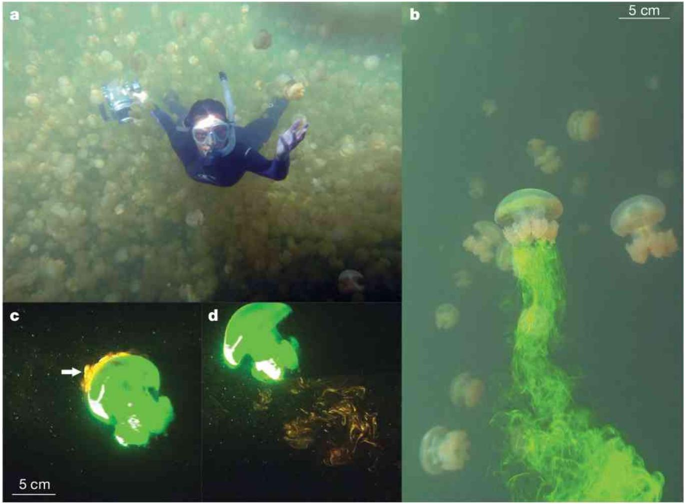
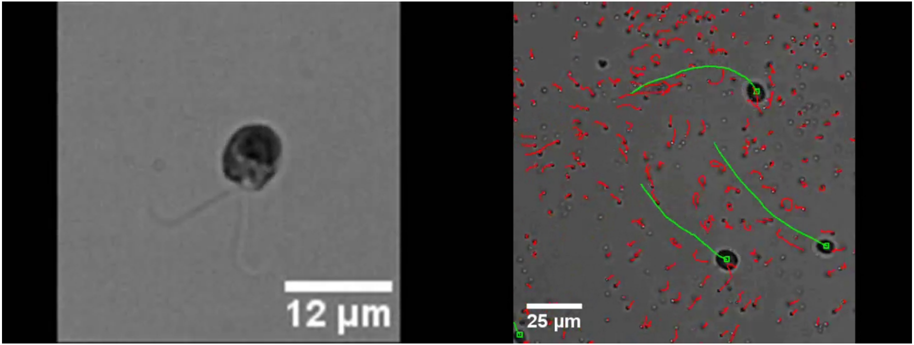
Small vortices generated by jellyfish

Katija & Dabiri (2009)



play movie (Palau's Jellyfish Lake.)

Chlamydomonas reinhardtii

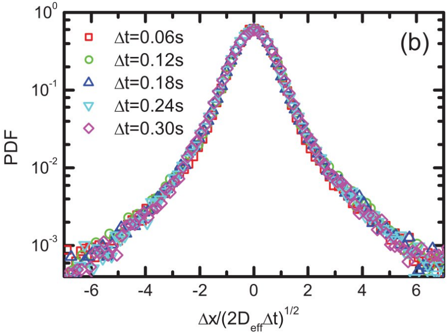


play movie

[Guasto, J. S., Johnson, K. A., & Gollub, J. P. (2010). *Phys. Rev. Lett.* **105**, 168102]

Probability density of displacements

Experiments with microswimmers:



- Measure pdf of displacements of small particles.
- Non-Gaussian pdf with 'exponential' tails.
- 'Diffusive scaling' possibly a short time effect [Thiffeault (2015)].

[Leptos, K. C., Guasto, J. S., Gollub, J. P., Pesci, A. I., & Goldstein, R. E. (2009). *Phys. Rev. Lett.* **103**, 198103]

Probability density of displacements for a swimmer

Finite-path drift function $\boldsymbol{\Delta}_\lambda(\boldsymbol{\eta})$ for a fluid particle, initially at $\mathbf{x} = \boldsymbol{\eta}$, affected by a single swimmer moving at velocity $\mathbf{U}$:

$$\boldsymbol{\Delta}_\lambda(\boldsymbol{\eta}) = \int_0^{\lambda/U} \mathbf{u}(\mathbf{x}(s) - \mathbf{U}s)\,\mathrm{d}s, \qquad \dot{\mathbf{x}} = \mathbf{u}(\mathbf{x} - \mathbf{U}t), \quad \mathbf{x}(0) = \boldsymbol{\eta}\,.$$

Assuming homogeneity and isotropy, we obtain the probability density of displacements,

$$p_{\mathbf{R}^1_\lambda}(\mathbf{r}) = \frac{1}{\Omega\, r^{d-1}} \int_V \delta(r - \Delta_\lambda(\boldsymbol{\eta}))\,\frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}$$

where $\Omega = \Omega(d)$ is the area of the unit sphere in d dimensions.

Here $\mathbf{R}^1_\lambda$ is a random variable that gives the displacement of the particle from its initial position after being affected by a single swimmer with path length λ.

Variance

The second moment (variance) of $\mathbf{R}_\lambda^1$ is

$$\langle (R_\lambda^1)^2 \rangle = \int_V r^2\, p_{\mathbf{R}_\lambda^1}(\mathbf{r})\, \mathrm{d}V_{\mathbf{r}} = \int_V \Delta_\lambda^2(\boldsymbol{\eta})\, \frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}.$$

Let $\mathbf{R}_\lambda^N$ be the random particle displacement due to N swimmers;

$$\langle (R_\lambda^N)^2 \rangle = N \langle (R_\lambda^1)^2 \rangle = n \int_V \Delta_\lambda^2(\boldsymbol{\eta})\, \mathrm{d}V_{\boldsymbol{\eta}}$$

with $n = N/V$ the number density of swimmers.

If the integral above exists then the particle motion is diffusive after we allow for random re-orientation of the swimmers.

Probability density for x displacement

We integrate over y and z to get the pdf for one coordinate x only:

$$p_{X_\lambda^1}(x) = \tfrac{1}{2} \int_V \frac{1}{\Delta_\lambda(\boldsymbol{\eta})} \left[\Delta_\lambda(\boldsymbol{\eta}) > |x|\right] \frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}$$

where $[A]$ is an indicator function: it is 1 if A is satisfied, 0 otherwise.

Now we want $p_{X_\lambda^N}(x)$, the pdf for N swimmers. The road to this is through the characteristic function:

$$\langle \mathrm{e}^{\mathrm{i}kX_\lambda^1} \rangle = \int_{-\infty}^{\infty} p_{X_\lambda^1}(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x = \int_V \operatorname{sinc}\left(k\Delta_\lambda(\boldsymbol{\eta})\right) \frac{\mathrm{d}V_{\boldsymbol{\eta}}}{V}$$

where $\operatorname{sinc} x := x^{-1} \sin x$.

(In 2D, replace sinc by Bessel function $J_0(x)$.)

Characteristic function

To help integrals converge nicely later, it is better to work with

$$\gamma(x) := 1 - \operatorname{sinc} x.$$

Then,

$$\langle \mathrm{e}^{\mathrm{i}kX_\lambda^1} \rangle = 1 - \Gamma_\lambda(k)/Vol$$

where

$$\boxed{\Gamma_\lambda(k) := \int_V \gamma(k\Delta_\lambda(\boldsymbol{\eta}))\,\mathrm{d}V_{\boldsymbol{\eta}}}$$

Many swimmers

The sum of many displacements has distribution given by a convolution of individual distributions.

The characteristic function for N swimmers is thus $\langle \mathrm{e}^{\mathrm{i}kX_\lambda^N} \rangle = \langle \mathrm{e}^{\mathrm{i}kX_\lambda^1} \rangle^N$:

$$\begin{aligned} \langle \mathrm{e}^{\mathrm{i}kX_\lambda^1} \rangle^N &= (1 - \Gamma_\lambda(k)/V)^{nV} \\ &\sim \exp\left(-n\,\Gamma_\lambda(k)\right), \quad V \to \infty \end{aligned}$$

where we used $N = nV$.

We take the inverse Fourier transform of $\langle \mathrm{e}^{\mathrm{i}kX_\lambda^1} \rangle^N$ to finally obtain

$$\boxed{p_{X_\lambda}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-n\,\Gamma_\lambda(k)\right) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k}$$

A model swimmer

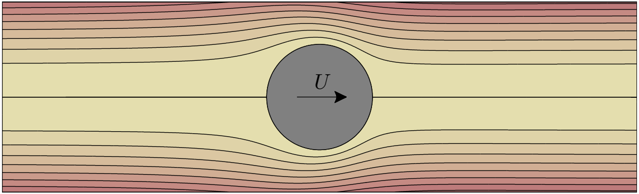
This is as far as we can go without introducing a model swimmer.

We take a squirmer, with axisymmetric streamfunction:

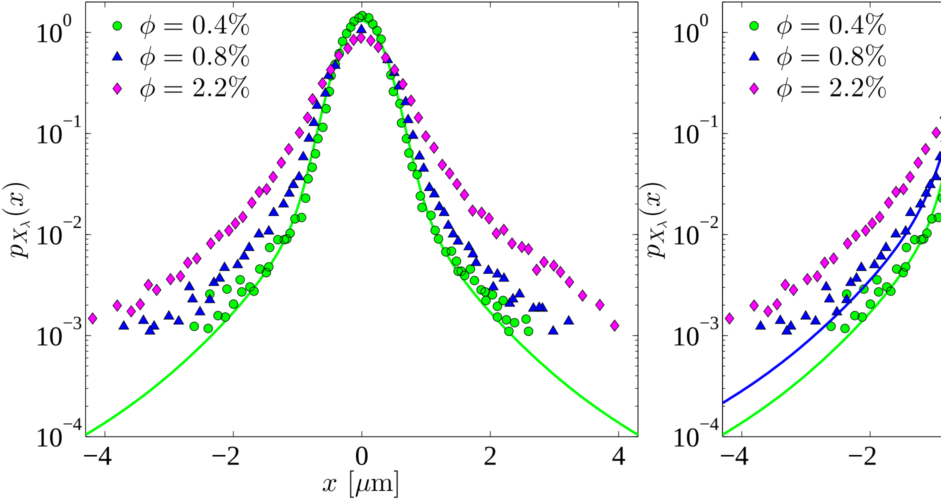
$$\Psi_{\mathsf{sf}}(\rho, z) = \tfrac{1}{2}\rho^2\, U \left\{ -1 + \frac{\ell^3}{(\rho^2 + z^2)^{3/2}} + \tfrac{3}{2}\frac{\beta \ell^2 z}{(\rho^2 + z^2)^{3/2}} \left(\frac{\ell^2}{\rho^2 + z^2} - 1 \right) \right\}$$

[See for example Lighthill (1952); Blake (1971); Ishikawa *et al.* (2006); Ishikawa & Pedley (2007b); Drescher *et al.* (2009)]

We use the stresslet strength $\beta = 0.5$, which is close to a treadmiller:



Comparing to Leptos *et al.*



Fit the stresslet strength $\beta = 0.5$ to one curve. Only fitted parameter is the stresslet strength $\beta = 0.5$.

[Thiffeault 2015]

Transport by vortices

- For the jellyfish, vortex rings are a convenient building block.
- Inviscid vortex filament: trapped region (atmosphere) leads to infinite transport.
- Unlike 2D, velocity depends logarithmically on core size.
- We must add some viscosity to regularize:
  - How far does a vortex go?
  - How does the trapped region change in time?
- Use a model vortex rather than numerical solution, since we need to go to large distances to evaluate $\int \Delta^2 \, \mathrm{d}V$.
- Some references: [Phillips (1956); Tung (1967); Maxworthy (1972); Stanaway *et al.* (1988); Saffman (1992); Shariff & Leonard (1992); Dabiri & Gharib (2004); Dabiri (2006); Shadden *et al.* (2006); Fukumoto & Kaplanski (2008); Fukumoto (2010)]

Viscous model

Equation for azimuthal component of the vorticity $\zeta(\rho, z, t)$:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (u_\rho \zeta)}{\partial \rho} + \frac{\partial (u_z \zeta)}{\partial z} = \nu \left(\frac{\partial^2 \zeta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \zeta}{\partial \rho} - \frac{\zeta}{\rho^2} + \frac{\partial^2 \zeta}{\partial z^2} \right)$$

The velocity components are given in terms of the streamfunction $\Psi(\rho, z, t)$,

$$u_\rho = -\frac{1}{\rho}\frac{\partial \Psi}{\partial z}, \qquad u_z = \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho}$$

The boundary conditions are

$$\Psi(0, z, t) = \zeta(0, z, t) = 0$$

$$\zeta\,, \Psi \longrightarrow 0, \qquad \text{as } r \to \infty$$

[Fukumoto & Kaplanski (2008); Fukumoto (2010)]

Viscous model: vortex filament

We follow Fukumoto & Kaplanski (2008) and neglect the inertial terms.
Initial condition corresponding to a vortex filament of radius ρ_0 at $z = 0$:

$$\zeta_0(\rho, z) = \Gamma_0 \, \delta(\rho - \rho_0) \, \delta(z)$$

Solution:

$$\zeta(\rho, z, t) = \frac{\Gamma_0 \, \rho_0}{4\sqrt{\pi}(\nu t)^{3/2}} \, \exp\left(-\frac{\rho^2 + \rho_0^2 + z^2}{4\nu t}\right) I_1\left(\frac{\rho\rho_0}{2\nu t}\right)$$

Notice that ζ is even in z, so the vorticity centroid

$$Z(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \zeta \, \rho^2 z \, \mathrm{d}\rho \, \mathrm{d}z \Bigg/ \int_{-\infty}^{\infty} \int_0^{\infty} \zeta \, \rho^2 \, \mathrm{d}\rho \, \mathrm{d}z$$

is zero for all times.

Viscous model: vortex motion

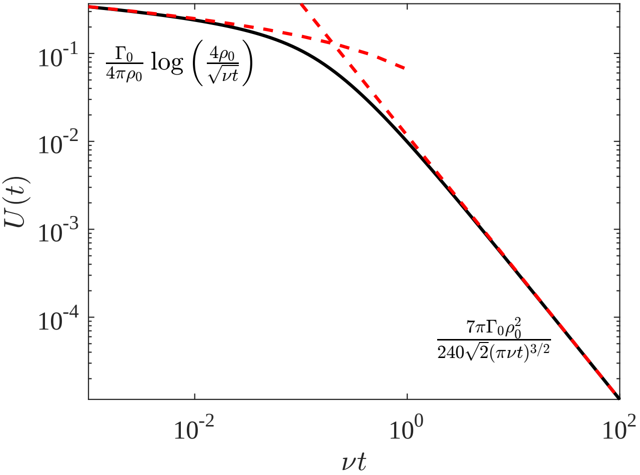
However, the time derivative $U(t) = \dot{Z}$ evaluated directly using the full equations of motion is nonzero.

Fukumoto & Kaplanski (2008) use this to estimate $U(t)$:

$$U(t) = \frac{\Gamma_0 \rho_0^2}{96\sqrt{2\pi}(\nu t)^{3/2}} \Bigg\{ {}_2F_2\left(\tfrac{3}{2}, \tfrac{3}{2}; \tfrac{5}{2}, 3; -\tfrac{\rho_0^2}{2\nu t}\right) - \tfrac{36}{5}\, {}_2F_2\left(\tfrac{3}{2}, \tfrac{5}{2}; 2, \tfrac{7}{2}; -\tfrac{\rho_0^2}{2\nu t}\right) + \frac{72\nu t}{\rho_0^2} \exp\!\left(-\tfrac{\rho_0^2}{4\nu t}\right) I_1\!\left(\tfrac{\rho_0^2}{4\nu t}\right) \Bigg\}$$

This is a bit messy, but has the advantage that it's uniformly valid for all times.

Viscous model: vortex motion (cont'd)

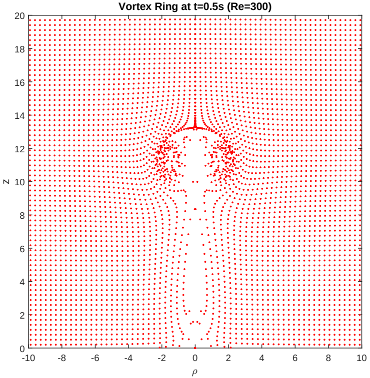


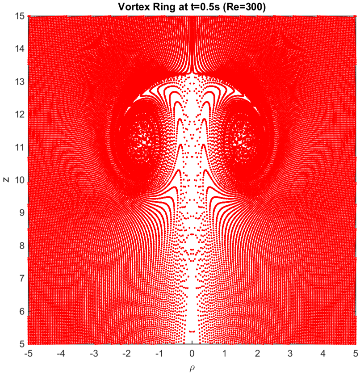
A crucial fact is that $U(t)$ decays as $t^{-3/2}$, so the total vortex displacement is finite (unlike in 2D): $\boxed{Z_\infty \sim 5\Gamma_0\rho_0/24\pi\nu}$

Particle drift

The Fukumoto & Kaplanski (2008) solution is (nearly) analytic, so it allows us to

- Easily do particle advection;
- Derive far-field asymptotics.





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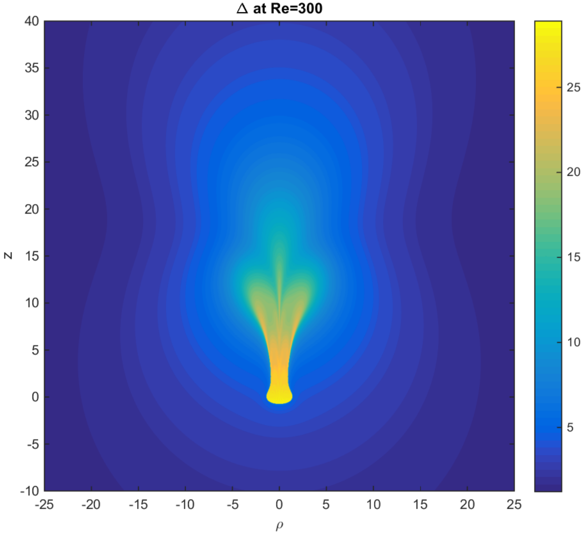
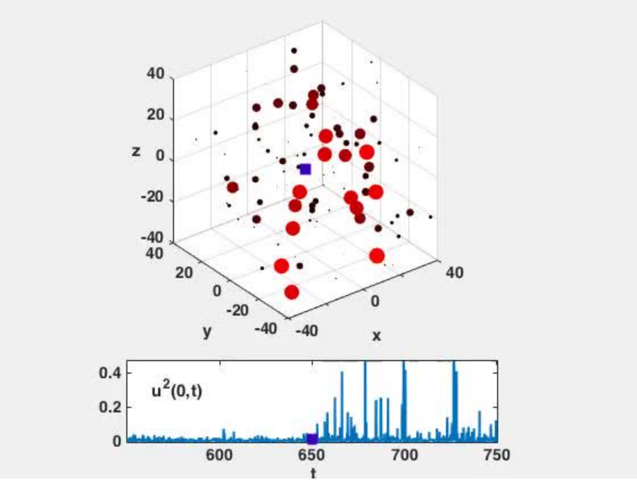
The drift function $\Delta(\rho, z)$



- $\Delta(\rho, z)$ is the net particle displacement as a function of initial position.
- Largest displacements come from particles initially in the vortex.
- Broadens as particle are entrained.
- Eventually peters out.

Simulation of vortex rings: velocity fluctutations

Add vortices at a constant rate μ (vortices / unit time · unit vol).



play movie

Velocity fluctuations: analytic

Let $\mathbf{V}_0$ be the induced fluid velocity at the origin; then

$$\mathbb{E}\|\mathbf{V}_0\|^2 = \mu \int_0^\infty \int_V v_T^2(\mathbf{r})\,\mathrm{d}V\,\mathrm{d}T$$

where v_T is the velocity field of one vortex ring and T is the age of the vortex.

Surprisingly, for Fukumoto & Kaplanski's model this can be integrated exactly to obtain the variance:

$$\mathbb{E}\|\mathbf{V}_0\|^2 = \mu \int_0^\infty \int_V v_T^2\,\mathrm{d}V\,\mathrm{d}T = \boxed{\frac{\mu I^2}{3\pi^2 \nu \rho_0}}$$

where $I = \pi \Gamma_0 \rho_0^2$ is the hydrodynamic impulse.

Conclusions

- We have a theory that works very well in explaining mixing by microorganisms.
- Predicts effective diffusivity as well as detailed pdf of particle displacements.
- Based on Lagrangian drift due to single 'swimmer.'
- This should have broader applicability: try vortex rings.
- Use a simple model, which allows analytic progress, and compare to simulations.
- Compute energy fluctuations: often used as a proxy for mixing.
- A full accounting of the effective diffusion is still lacking. . . ongoing work.

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