

Topological tools for the real world

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the taffy puller



Taffy is a type of candy.

Needs to be **pulled**: this aerates it and makes it lighter and chewier.

We can assign a **growth**: length multiplier per period.

(Here $(1 + \sqrt{2})^2$... more later.)

[movie by M. D. Finn]

play movie



making candy cane



[*Wired*: This Is How You Craft 16,000 Candy Canes in a Day]

four-pronged taffy puller

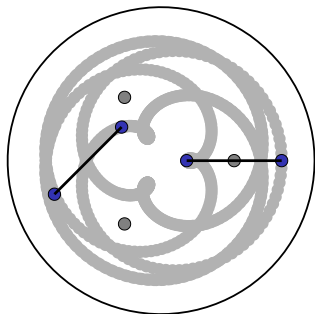


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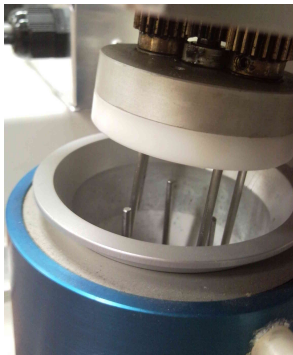
<http://www.youtube.com/watch?v=Y7t1LHdsquVM>

[studied in detail by Halbert & Yorke (2013)]

Experimental device for kneading bread dough:



play movie



[Department of Food Science, University of Wisconsin. Photos by J-LT.]

the mixograph as a braid



Encode the topological information as a sequence of **generators of the Artin braid group B_n** .

Equivalent to the 7-braid

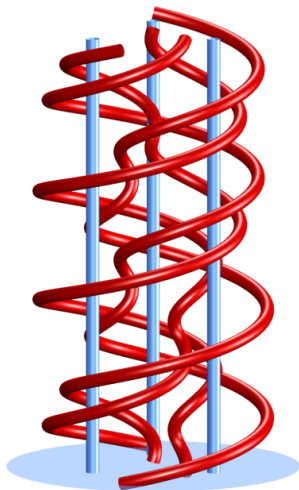
$$\sigma_3 \sigma_2 \sigma_3 \sigma_5 \sigma_6^{-1} \sigma_2 \sigma_3 \sigma_4 \sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_5$$

The **growth** is the largest root of

$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1$$

$$\simeq 4.186$$

Compare to taffy pullers: $\boxed{5.828}$





play movie



play movie

[Boyland, P. L., Aref, H., & Stremler, M. A. (2000). *J. Fluid Mech.* **403**, 277–304; Simulations by M. D. Finn, S. E. Tumas, and J-LT.]



Periodic stirring protocols in two dimensions can be described by a **homeomorphism** $\varphi : \mathcal{S} \rightarrow \mathcal{S}$, where \mathcal{S} is a surface.

For instance, in a closed circular container,

- φ describes the mapping of fluid elements after one full period of stirring, obtained by solving the Stokes equation;
- \mathcal{S} is the **disc** with holes in it, corresponding to the stirring rods.

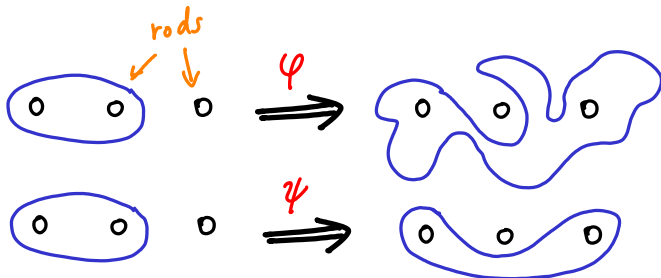
Goal: **Topological characterization of φ .**

[The theory extends to handlebodies, but not as relevant for applications. . .]

φ and ψ are **isotopic** if ψ can be continuously 'reached' from φ without moving the rods. Write $\varphi \simeq \psi$.

(Defines **isotopy classes**.)

Convenient to think of isotopy in terms of material loops. Isotopic maps act the same way on loops (up to continuous deformation).



(Loops will always mean **essential** loops.)



Theorem

φ is isotopic to a homeomorphism ψ , where ψ is in one of the following three categories:

finite-order for some integer $k > 0$, $\psi^k \simeq \text{identity}$;

reducible ψ leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;

pseudo-Anosov ψ leaves invariant a pair of transverse measured **singular foliations**, \mathcal{F}^u and \mathcal{F}^s , such that $\psi(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u)$ and $\psi(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1} \mu^s)$, for **dilatation** $\lambda > 1$.

The three categories characterize the **isotopy class** of φ .

We want **pseudo-Anosov** for good mixing.



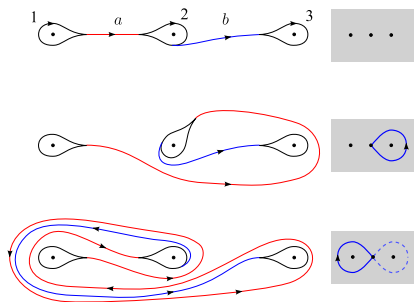
- Consider a **motion of stirring elements**, such as rods.
- Determine if the motion is **isotopic to a pseudo-Anosov mapping**.
- **Compute** topological quantities, such as foliation, entropy, etc.
- **Analyze** and **optimize**.

train tracks: computing entropy and foliations



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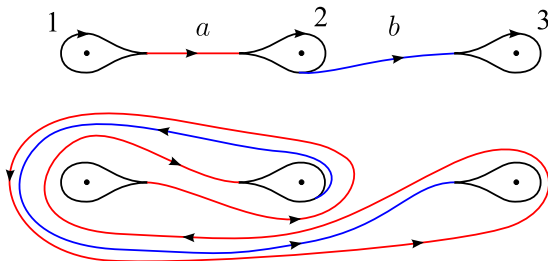
'Figure-8' motion: $\sigma_2^{-2}\sigma_1^2$



[Guillart *et al.* (2007)]

Thurston introduced **train tracks** as a way of characterizing the measured foliation. The name stems from the 'cusps' that look like train switches.

train track map for figure-eight



$$a \mapsto a\bar{2}\bar{a}\bar{1}ab\bar{3}\bar{b}\bar{a}1a, \quad b \mapsto \bar{2}\bar{a}\bar{1}ab$$

Easy to show that this map is **efficient**: under repeated iteration, cancellations of the type $a\bar{a}$ or $b\bar{b}$ never occur.

[There are algorithms, such as Bestvina & Handel (1995), to find efficient train tracks. (Toby Hall has an implementation in C++.)]



As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the **topological entropy**, $\log \lambda$. This is a lower bound on the **minimal length of a material line** caught on the rods.

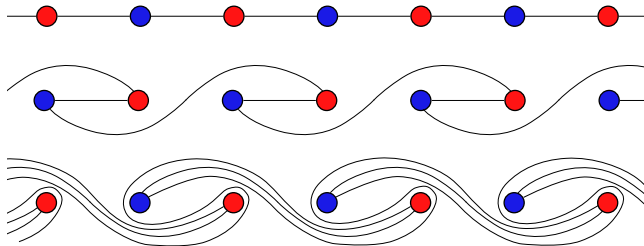
Find from the TT map by **Abelianizing**: count the number of occurrences of a and b , and write as matrix:

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The largest eigenvalue of the matrix is $\lambda = (1 + \sqrt{2})^2 \simeq 5.83$. Hence, asymptotically, the length of the 'blob' is multiplied by 5.83 for each full stirring period.

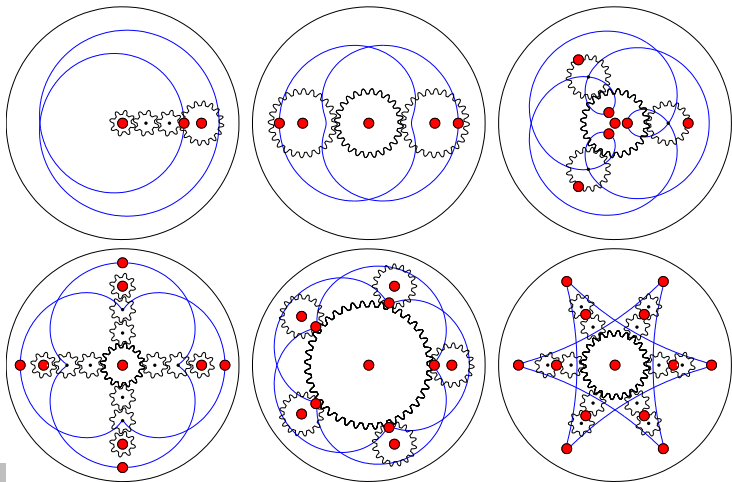
[This is the growth for the 3 and 4-pronged taffy pullers.]

- Consider periodic lattice of rods.
- Move all the rods such that they execute the Boyland *et al.* (2000) rod motion (Thiffeault & Finn, 2006; Finn & Thiffeault, 2011).



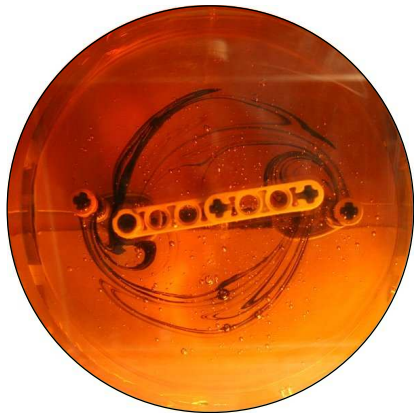
- The dilatation per period is χ^2 , where $\chi = 1 + \sqrt{2}$ is the **Silver Ratio!**
- This is **optimal** for a periodic lattice of two rods (Follows from D'Alessandro *et al.* (1999)).

- The designs with dilatation given by the silver ratio can be realized with simple gears.
- All the rods move at once: very efficient.





play movie

[play movie](#)

[See Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743 for proofs, heavily influenced by work on π_1 -stirrers of Boyland, P. L. & Harrington, J. (2011). *Algeb. Geom. Topology*, **11** (4), 2265–2296.]

ghost rods ('tiges fantômes')



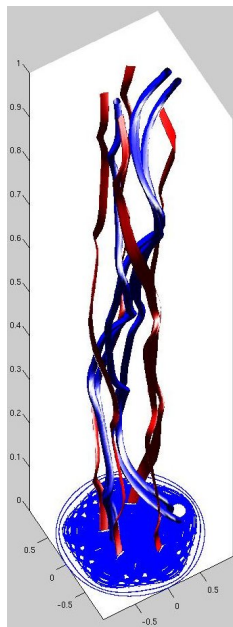
Topological analysis can be done on other objects than rods – for instance, **islands** or **unstable periodic orbits**.

We simply follow the islands and examine the braid they form, which gives us bounds on **topological entropy**.

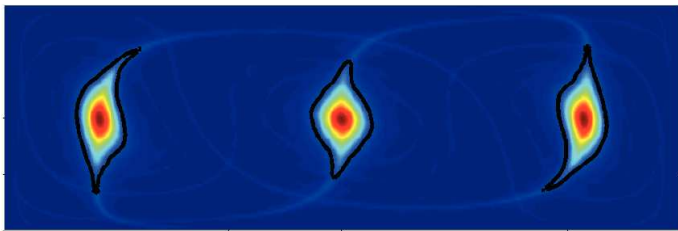
In this framework we call the islands **ghost rods**.

[Guillart, E., Finn, M. D., & Thiffeault, J.-L. (2006). *Phys. Rev. E*, **73**, 036311]

[implemented by Stremler & Chen (2007); Thiffeault *et al.* (2009); Binder (2010); Stremler *et al.* (2011)]



One of the best examples of ghost rods is from Stremler *et al.* (2011):

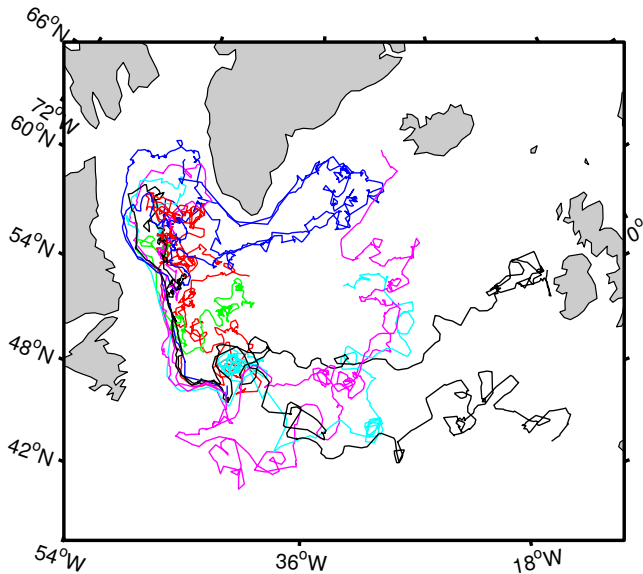


The islands are made to follow the $\sigma_2\sigma_1^{-1}$ stirring protocol by clever wall motions! (viscous Stokes flow)

[Stremler, M. A., Ross, S. D., Grover, P., & Kumar, P. (2011). *Phys. Rev. Lett.* **106**, 114101]

play movie

oceanic float trajectories





What can we measure?

- single-particle dispersion (not a good use of all data)
- correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the **braid group generators** σ_i for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a **topological entropy** for the motion (similar to Lyapunov exponent).



It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

The problem is twofold:

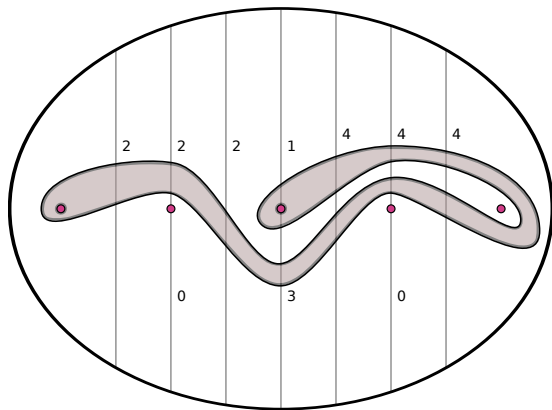
- 1 Need to keep track of the loop, since its length is growing exponentially;
- 2 Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them **topologically** with very few numbers.

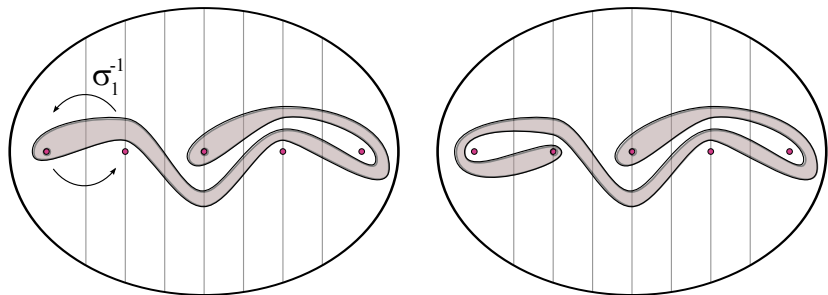
solution to problem 1: loop coordinates



What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the [Dybnikov coordinates](#) involve intersections with vertical lines:

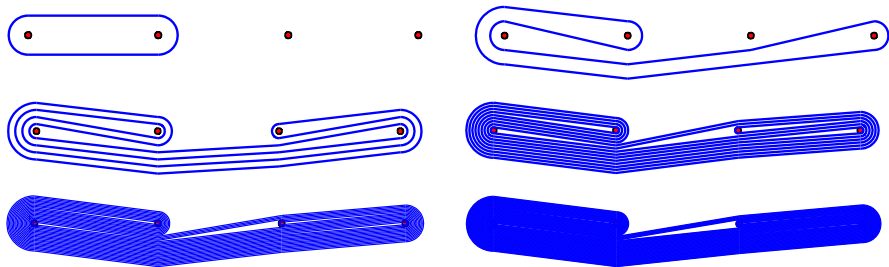


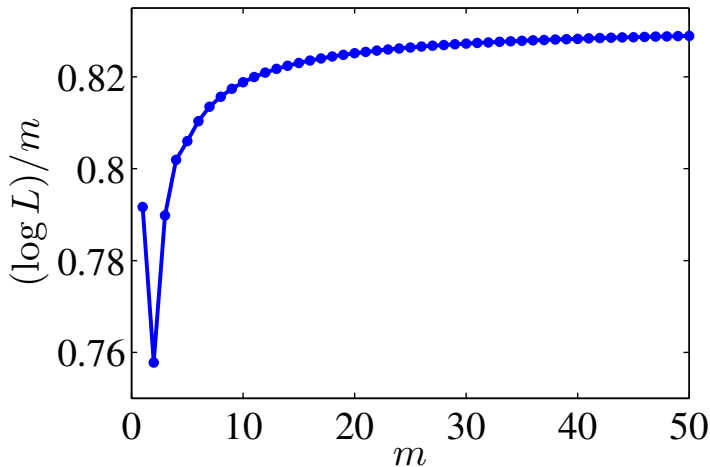
Moving the punctures according to a braid generator changes some crossing numbers:



There is an explicit formula for the change in the coordinates! [Dyannikov (2002); Moussafir (2006); Hall & Yurttas (2009); Thiffeault (2010)]

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:

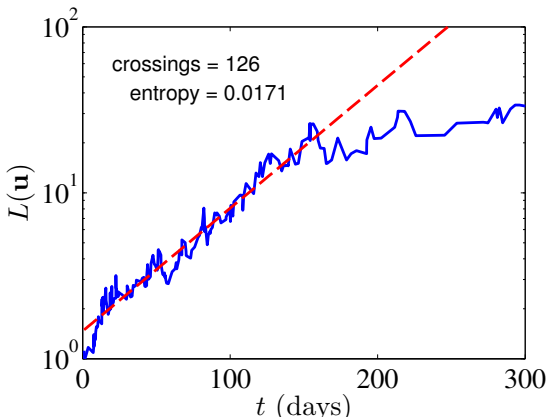




m is the number of times the braid acted on the initial loop.

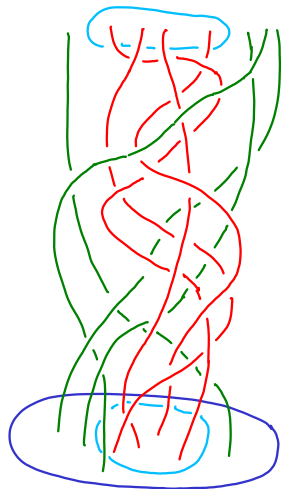
[Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* 1 (1), 37–46]

10 floats from Davis' Labrador sea data:



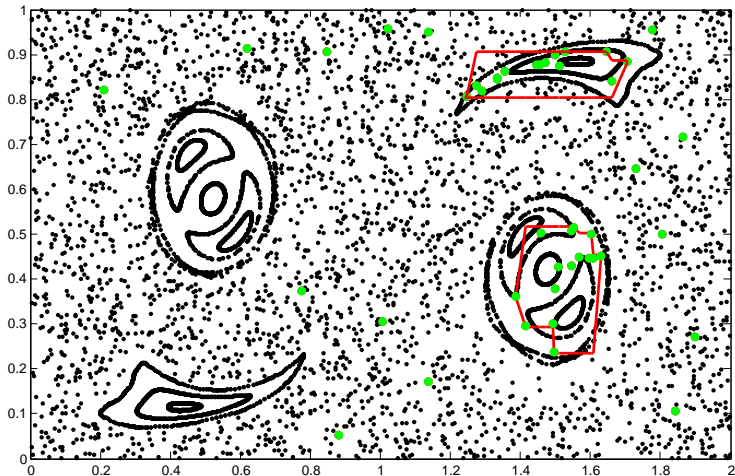
Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)



- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an **isolated region** in the flow that does not interact with the rest, bounded by **Lagrangian coherent structures (LCS)**;
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- [see Haller, G. & Beron-Vera, F. J. (2012). *Physica D*, **241** (20), 1680–1702.]

double-gyre coherent structures



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[Allhouse, M. R. & Thiffeault, J.-L. (2012). *Physica D*, 241 (2), 95–105]



- The **nature of the isotopy** between the pA and real system.
- **Sharpness** of the entropy bound (Tumasz & Thiffeault, 2013).
- **Computational methods** for isotopy class (random entanglements of trajectories – LCS method, see Allshouse & Thiffeault (2012), ongoing work also with Marko Budisic, Margaux Filippi, and Tom Peacock).
- **'Designing'** for topological chaos (see Stremler & Chen (2007)).
- Combine with **other measures**, e.g., **mix-norms** (Mathew *et al.*, 2005; Lin *et al.*, 2011; Thiffeault, 2012).
- We're developing a Matlab toolbox — **braidlab**.
- **3D?! (lots of missing theory)**



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