how to make mathematical candy

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the taffy puller

Taffy is a type of candy.

Needs to be pulled: this aerates it and makes it lighter and chewier.

[movie by M. D. Finn]

play movie

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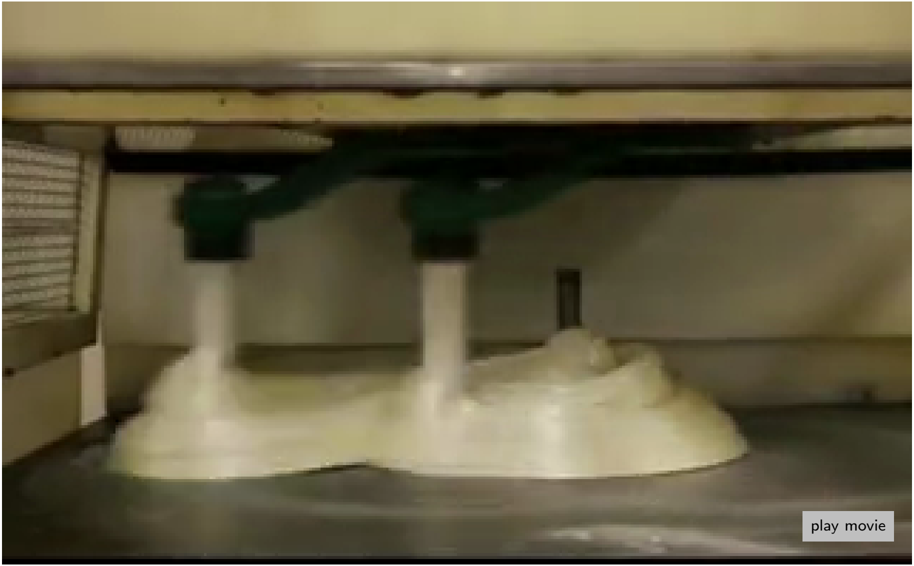
We can assign a growth: length multiplier per period.

[movie by M. D. Finn]

play movie

making candy cane



[*Wired:* This Is How You Craft 16,000 Candy Canes in a Day]

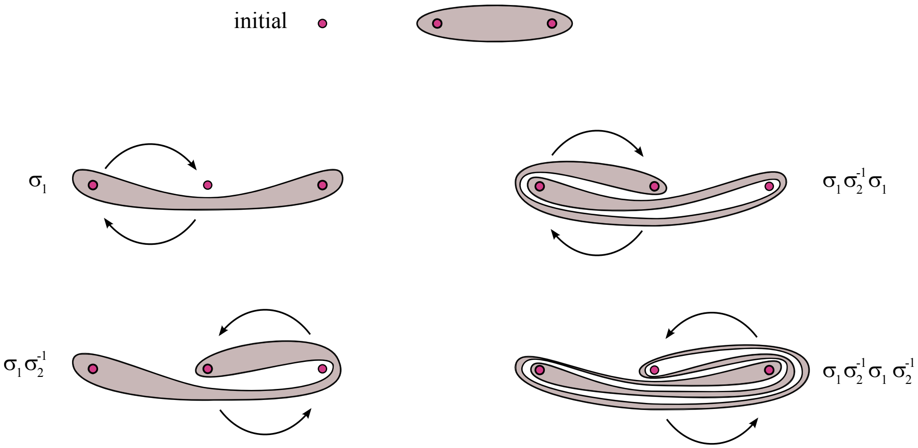
four-pronged taffy puller

play movie

http://www.youtube.com/watch?v=Y7tlHDsquVM

[MacKay (2001); Halbert & Yorke (2014)]

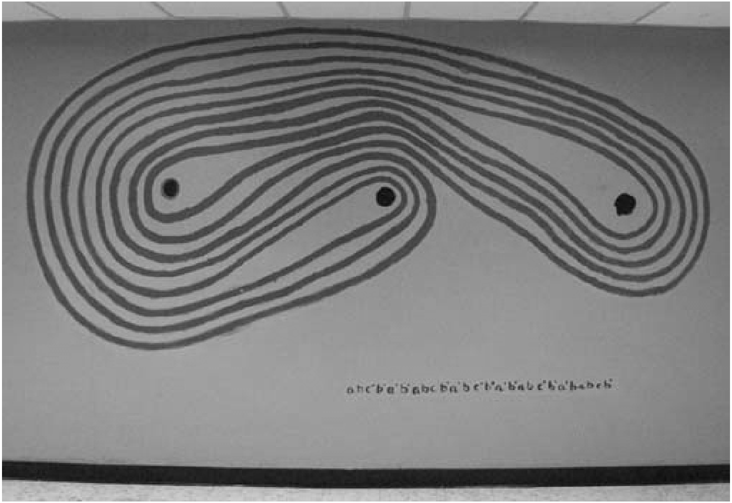
a simple taffy puller



[Remark for later: each rod moves in a ‘figure-eight’ shape.]

the famous mural

This is the same action as in the famous mural painted at Berkeley by Thurston and Sullivan in the Fall of 1971:



number of folds

[Matlab: `demo1`]

Let's count alternating left/right folds.

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$$\#\text{folds} = 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \ldots$$

What is the rule?

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$$\#\text{folds}_n = \#\text{folds}_{n-1} + \#\text{folds}_{n-2}$$

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This is the famous Fibonacci sequence, F_n.

It is well-known that for large n,

$$\frac{F_n}{F_{n-1}} \quad \to \quad \phi = \frac{1+\sqrt{5}}{2} = 1.6180\ldots$$

where ϕ is the Golden Ratio, also called the Golden Mean.

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So the ratio of lengths of the taffy between two successive steps is ϕ^2, where the squared is due to the left/right alternation.

how fast does the taffy grow?

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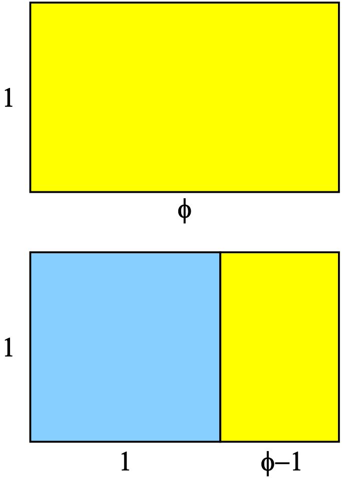
So the ratio of lengths of the taffy between two successive steps is ϕ^2, where the squared is due to the left/right alternation.

Hence, the growth factor for this taffy puller is

$$\boxed{\phi^2 = \phi + 1 = 2.6180\ldots.}$$

the Golden Ratio, ϕ



A rectangle has the proportions of the Golden Ratio if, after taking out a square, the remaining rectangle has the same proportions as the original:

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

a slightly more complex taffy puller

[Matlab: `demo2`]

Now let's swap our prongs twice each time.

number of folds

We get for the number of left/right folds

$$\#\text{folds} = 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408\ldots$$

number of folds

We get for the number of left/right folds

$$\#\text{folds} = 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408\ldots$$

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$$\#\text{folds}_n = 2\#\text{folds}_{n-1} + \#\text{folds}_{n-2}$$

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Hence, the growth factor for this taffy puller is

$$\boxed{\chi^2 = 2\chi + 1 = 5.8284\ldots.}$$

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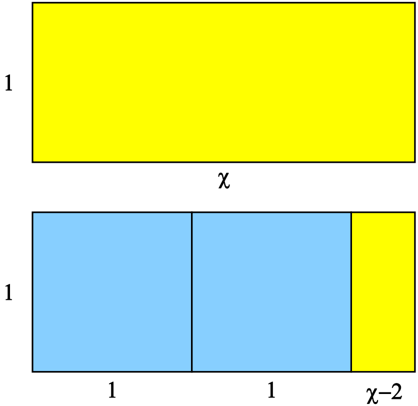

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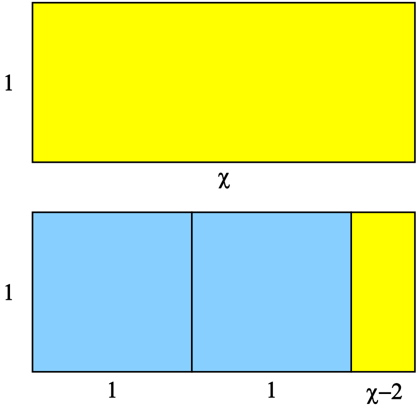

$$\frac{\chi}{1} = \frac{1}{\chi - 2}$$

Both major taffy puller designs (3- and 4-rod) have growth χ^2.

These quadratic numbers are reminiscent of invertible linear maps on the torus T^2, such as Arnold's Cat Map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1, \quad x, y \in [0,1]^2$$

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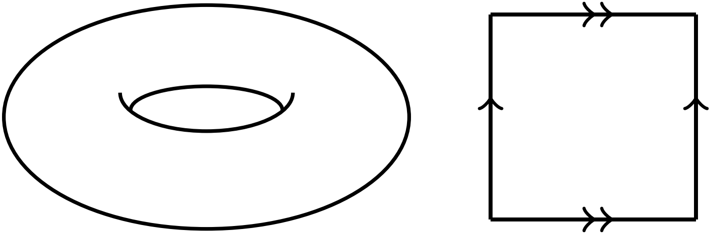
What's the connection between taffy pullers and these maps?

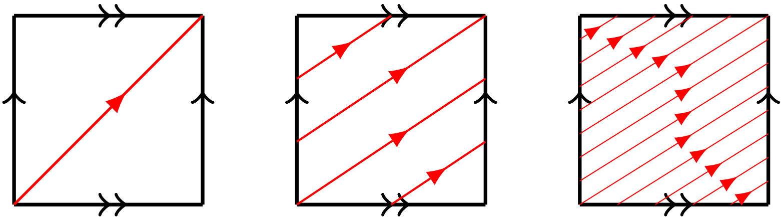
the torus

The ‘standard model’ for the torus is the biperiodic unit square:



action of map on the torus

The Cat Map stretches loops exponentially:



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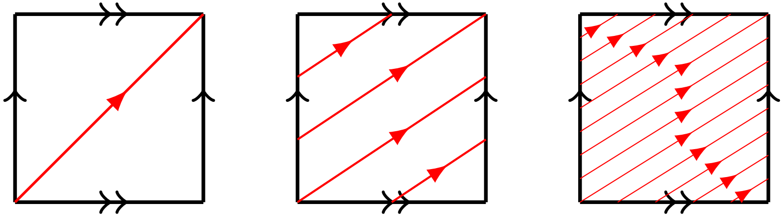

This loop will stand in for a piece of taffy.

hyperelliptic involution

Consider the linear map $\iota(x) = -x \bmod 1$. This map is called the hyperelliptic involution ($\iota^2 = \mathrm{id}$).

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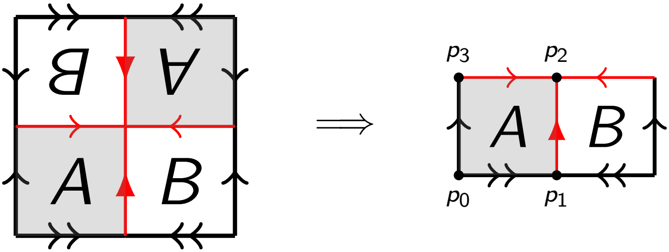

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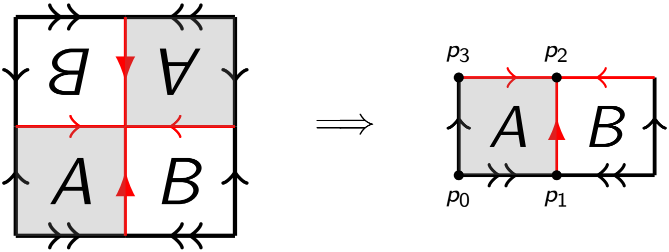
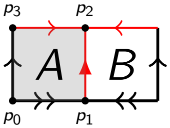


Claim: the surface S (right) is a sphere with four punctures!

Here's how we see that S is a sphere:



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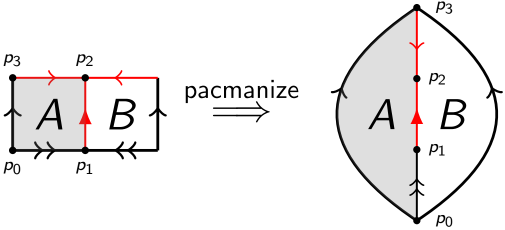

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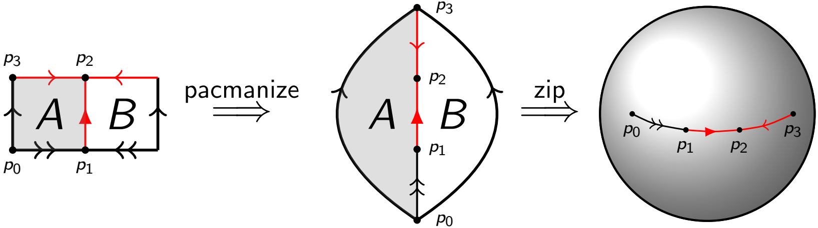

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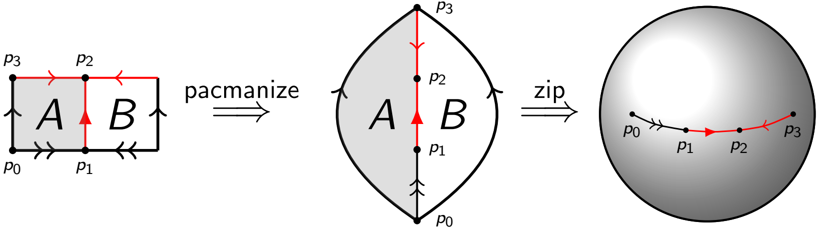

The punctures $p_{1,2,3}$ are the rods of our taffy puller. The puncture p_0 is like a point at infinity on the plane.

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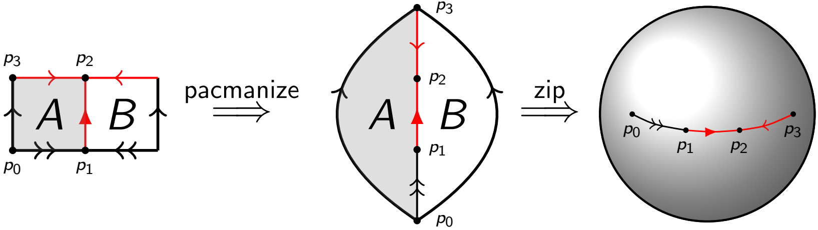

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Linear maps commute with ι, so all linear torus maps 'descend' to taffy puller motions.

Dehn twists

Any taffy puller motion can be represented as a product of

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and their inverses, known as Dehn twists.

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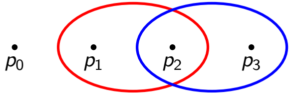

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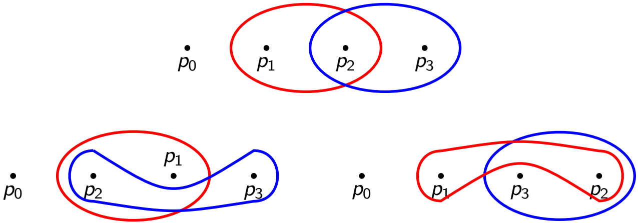

action of σ_1 action of σ_2

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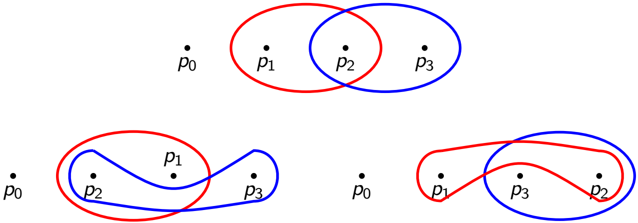

action of σ_1　　action of σ_2

These can also be view as generators for the braid group for 3 strings.

the history of taffy pullers

This past Summer I started wondering about who invented the well-known designs for taffy pullers.

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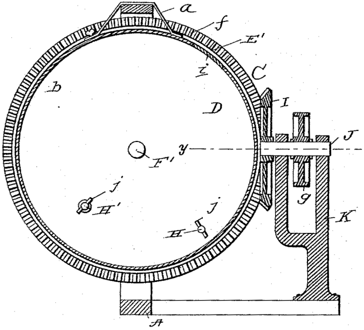
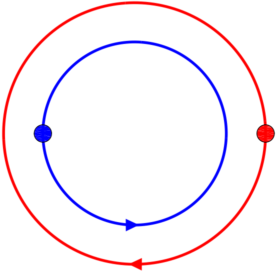
The very first: Firchau (1893)





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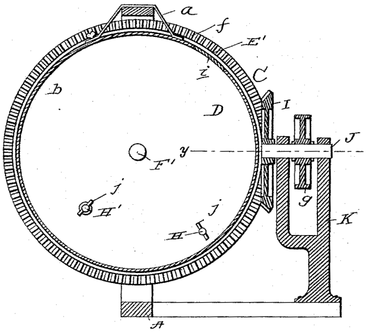
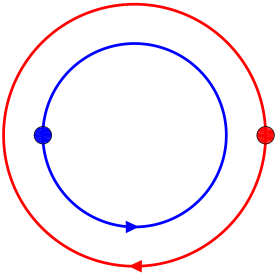


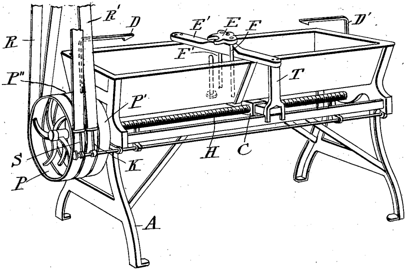


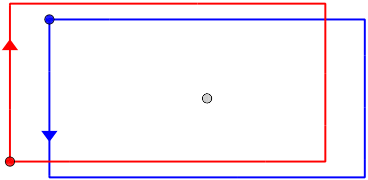
This is a terrible taffy puller. It was likely never built, but plays an important role in the looming. . .

taffy patent wars

The first true taffy puller

I think Dickinson (1906, but filed in 1901) deserves the title of inventor of the first taffy puller:





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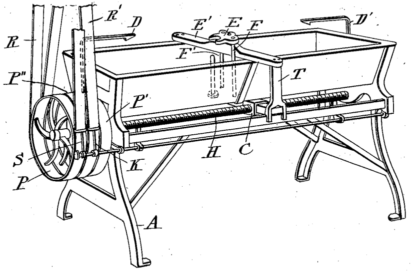

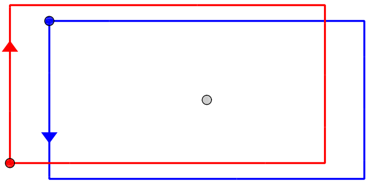

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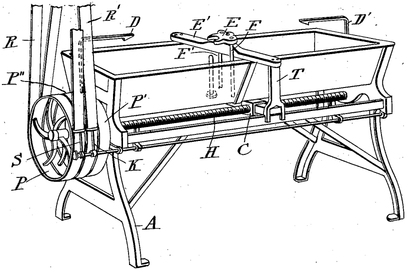
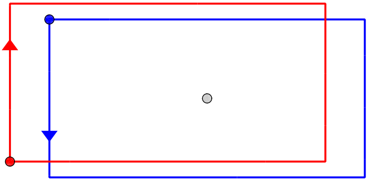


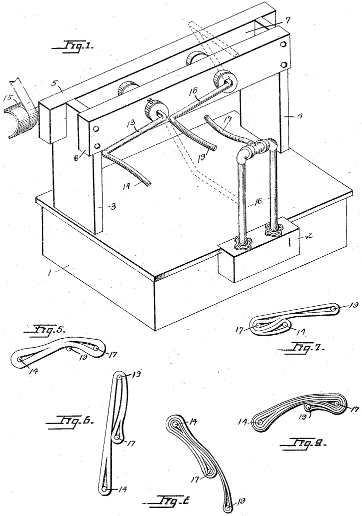


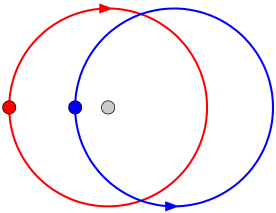
Awkward design: the moving rods get 'tripped' each cycle. But it is topologically the same as the 3-rod device still in use today.

There seem to be questions as to whether it ever worked, or if it really pulled taffy rather than mixing candy.

The modern 3-rod design

Robinson & Deiter (1908) greatly simplified this design to one still in use today.

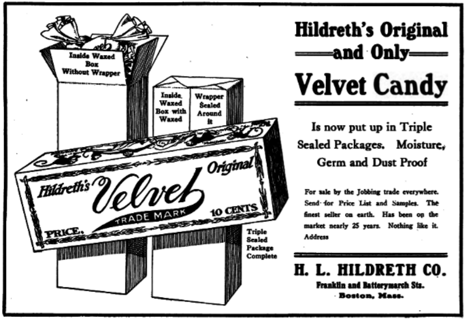




Herbert L. Hildreth

The uncontested taffy magnate of the early 20th century was Herbert L. Hildreth of Maine.

The Hotel Velvet in Old Orchard, Maine



The Confectioners Gazette (1914)

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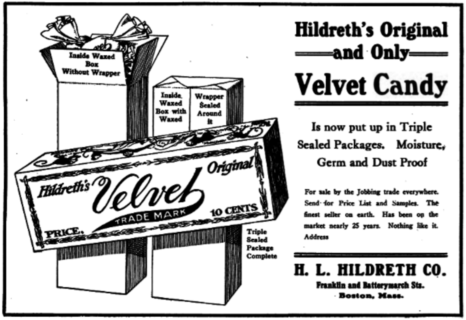

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His hotel was on the beach, and taffy was popular at such resorts. He sold it wholesale as well.

the 4-rod design

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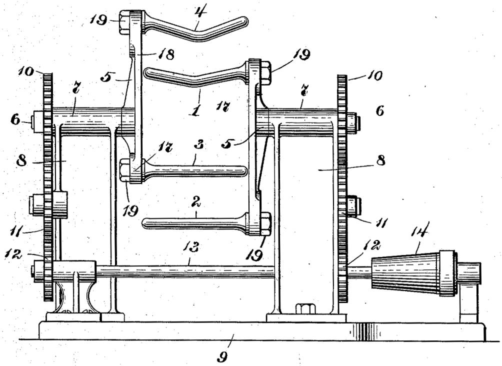

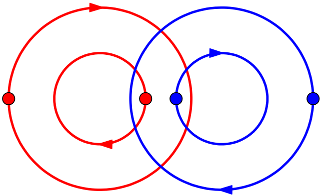

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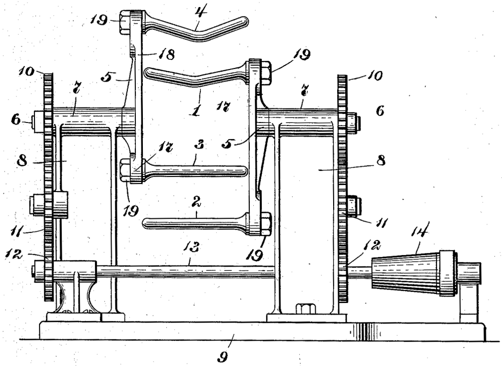
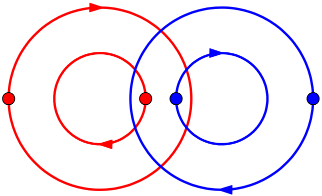
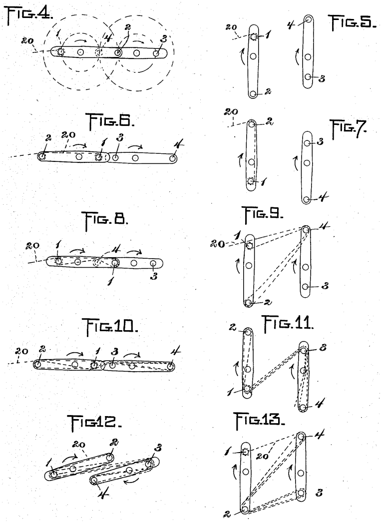




Hildreth was not pleased by this but bought the patent for $75,000 (about two million of today's dollars).

the best patents have beautiful diagrams



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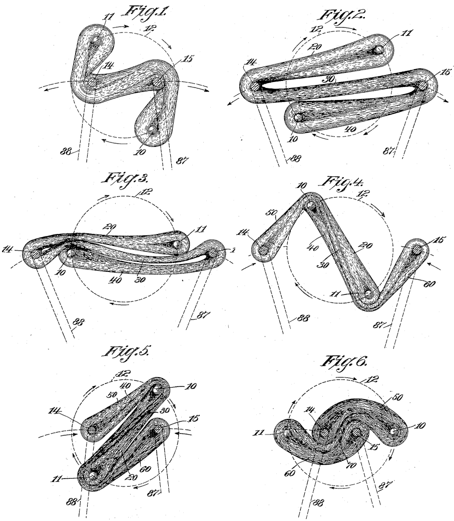

Richards (1905)

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> *The machine shown in the Firchau patent [has two pins that] pass each other twice during each revolution [. . .] and move in concentric circles, but do not have the relative in-and-out motion or Figure 8 movement of the Dickinson machine. With only two hooks there could be no lapping of the candy, because there was no third pin to re-engage the candy while it was held between the other two pins.*

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The Supreme Court opinion displays the fundamental insight that at least three rods are required to produce some sort of rapid growth.

the quest for the Golden ratio

Is it possible to build a device that realizes the simplest taffy puller, with growth ϕ^2?

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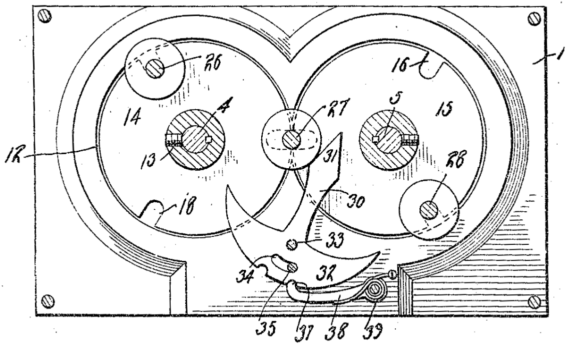
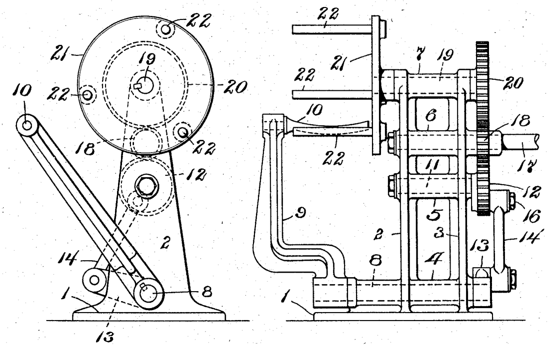
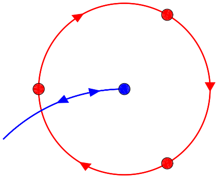
After some digging, found the patent of Nitz (1918):



the quest for the Golden ratio (2)

There is actually an earlier 4-rod design by Thibodeau (1904) which has Golden ratio growth:





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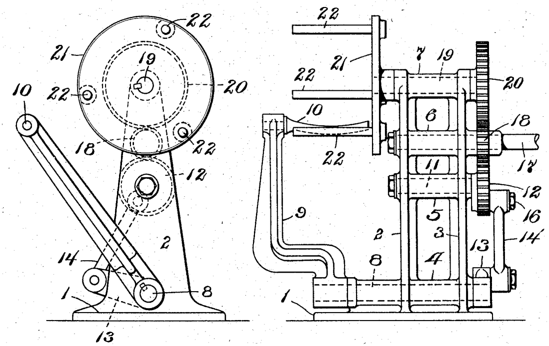
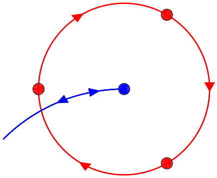
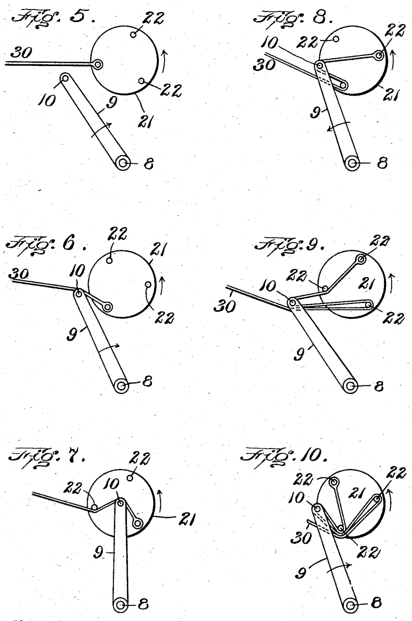




Since it uses four rods to get a quadratic growth, the map must involve a branched cover of the torus by a theorem of Franks & Rykken (1999). (The same happens for the 4-rod vs 3-rod 'standard' taffy pullers.)

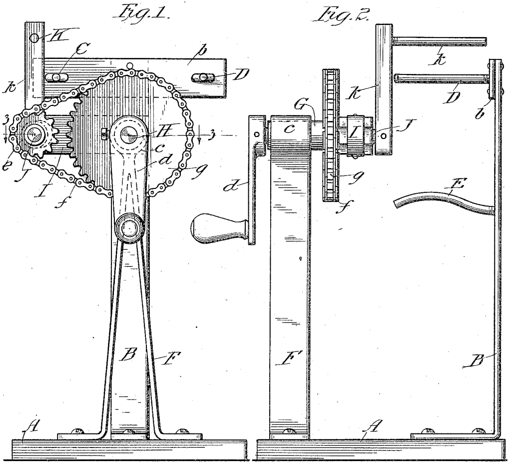
the quest for the Golden ratio (3)

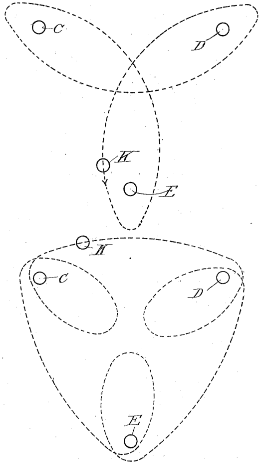


Thibodeau (1904) once again gives very nice diagrams for the action of his taffy puller.

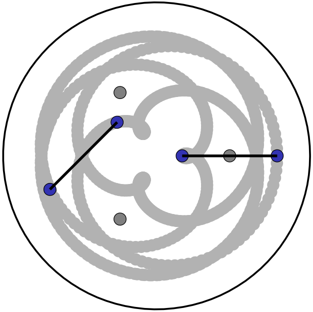
(He has at least 5 patents for taffy pullers.)

A few designs are based on 'planetary' gears, such as McCarthy (1916):

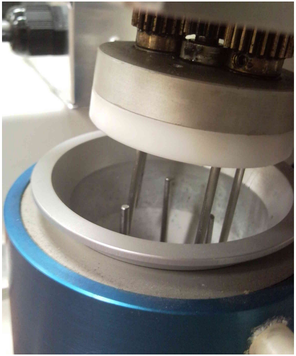




A modern planetary design is the mixograph, a device for measuring the properties of dough:

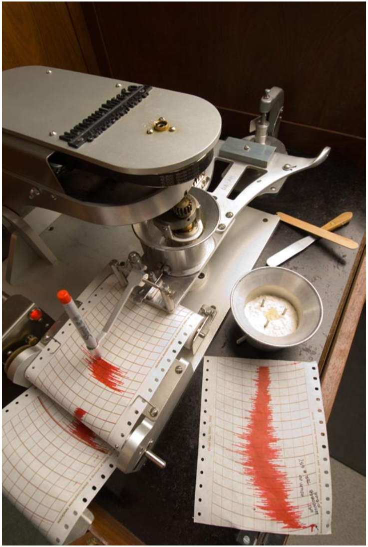


play movie



[Department of Food Science, University of Wisconsin. Photos by J-LT.]

the mixograph (2)



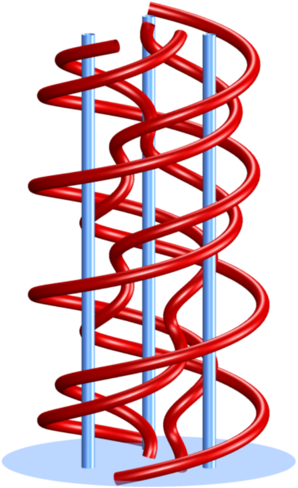
The mixograph measures the resistance of the dough to the pin motion.

This is graphed to determine properties of the dough, such as water absorption and 'peak time.'

Wheat and Flour Testing Methods: A Guide to Understanding Wheat and Flour Quality

the mixograph as a braid

Encode the topological information as a sequence of generators of the Artin braid group B_n.



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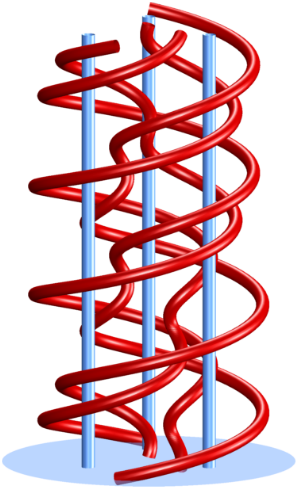
Equivalent to the 7-braid

$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$



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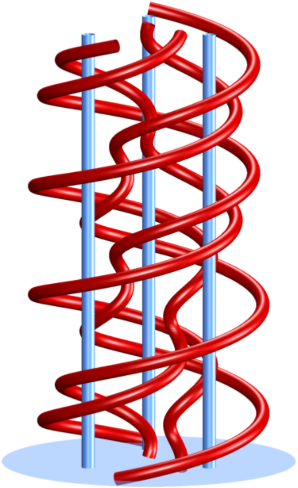
We feed this braid to the Bestvina–Handel algorithm, which determines the Thurston type of the braid (pseudo-Anosov) and finds the growth as the largest root of

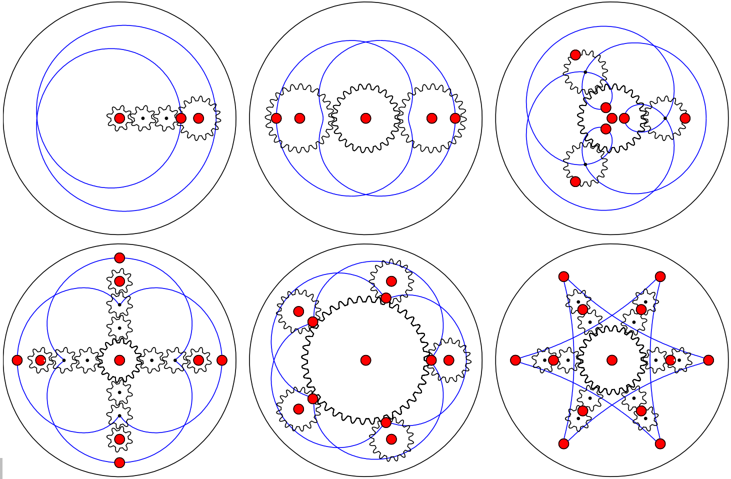
$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1$$

$\boxed{\simeq 4.186}$



silver mixers

As part of an optimization procedure, we (Finn & Thiffeault, 2011) designed a family of planetary mixers with silver ratio expansion:

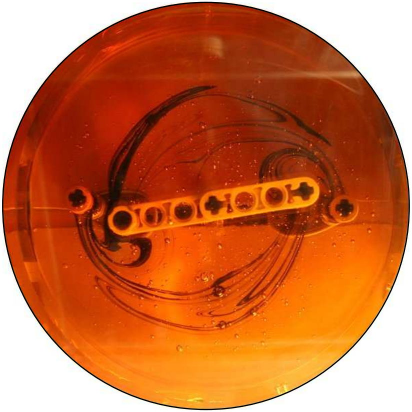


play movie

silver mixers: building one out of Legos

play movie

silver mixer in action



play movie

[See Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743 for proofs, heavily influenced by work on π_1-stirrers of Boyland, P. L. & Harrington, J. (2011). *Algeb. Geom. Topology,* **11** (4), 2265–2296.]

exotic designs

There remains many patents that I call 'exotic' which use nonstandard motions: such as Jenner (1905):

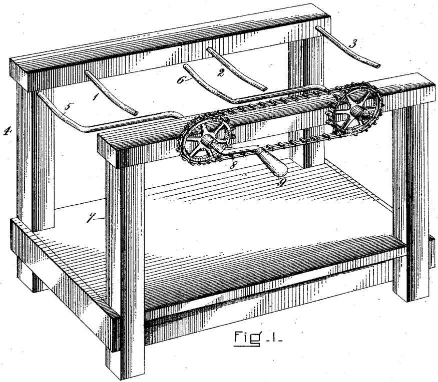
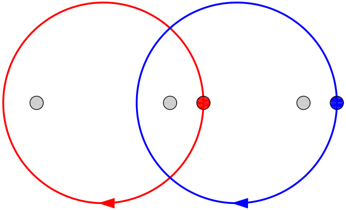
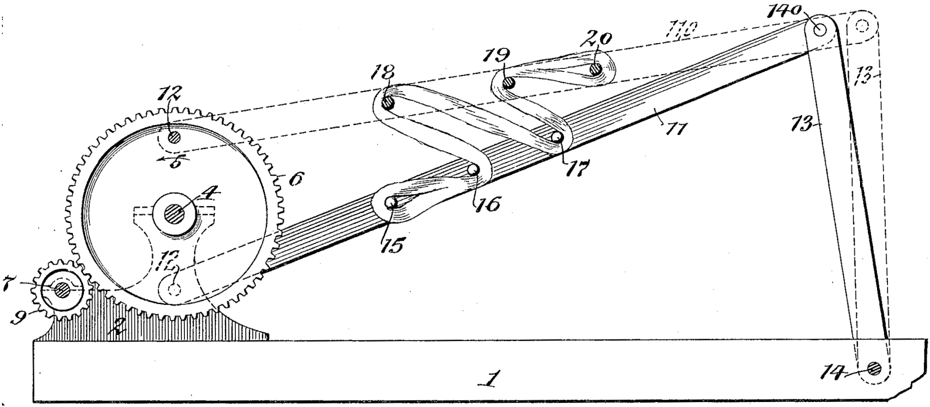


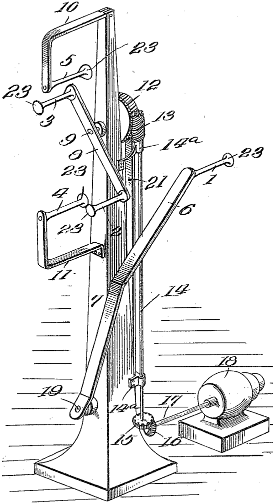
Fig. 1.

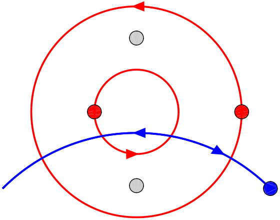


Shean & Schmelz (1914):



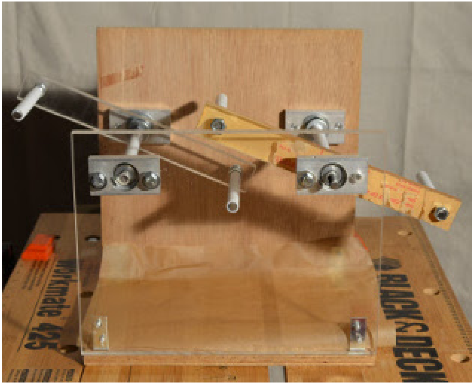
My personal favorite, McCarthy & Wilson (1915):

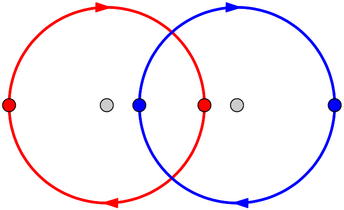




let's try our hand at this

Six-rod design with Alex Flanagan:

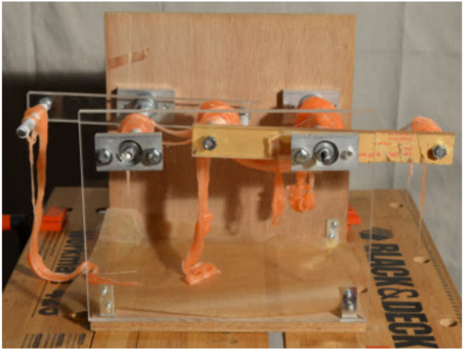




The software tools allow us to rapidly try designs. This one is simple and has huge growth (13.9 vs 5.8 for the standard pullers).

making taffy is hard

Early efforts yielded mixed results:



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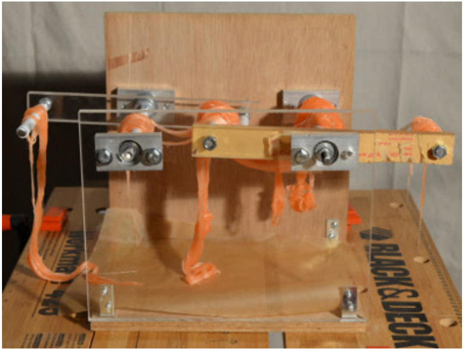
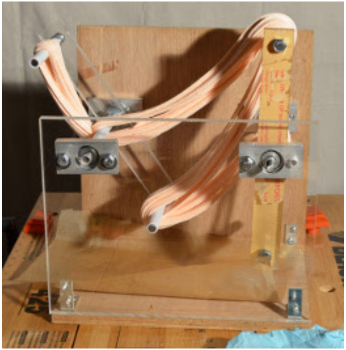


. . . but eventually we got better at it



play movie

(BTW: The physics of candy making is fascinating. . .)

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- pseudo-Anosov maps themselves are still the subject of intense study).

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