Topological optimization of rod-stirring devices

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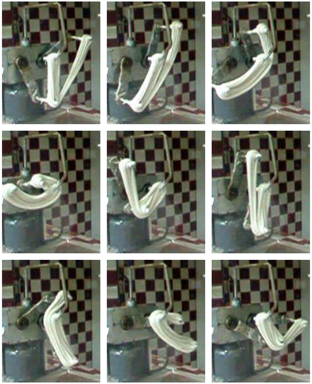
The taffy puller

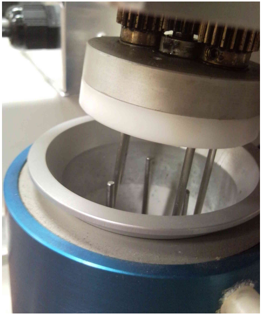
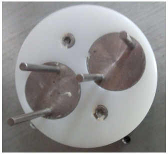
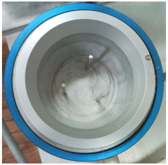


[Photo and movie by M. D. Finn.]

[movie 1]

The mixograph

Model experiment for kneading bread dough:

[Department of Food Science, University of Wisconsin. Photos by J-LT.]

Planetary mixers

In food processing, rods are often used for stirring.

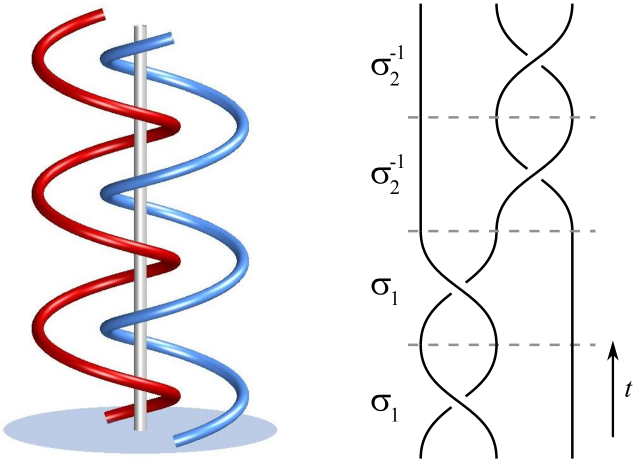
[movie 2]

Experiment of Boyland, Aref & Stremler

[movie 3] [movie 4]

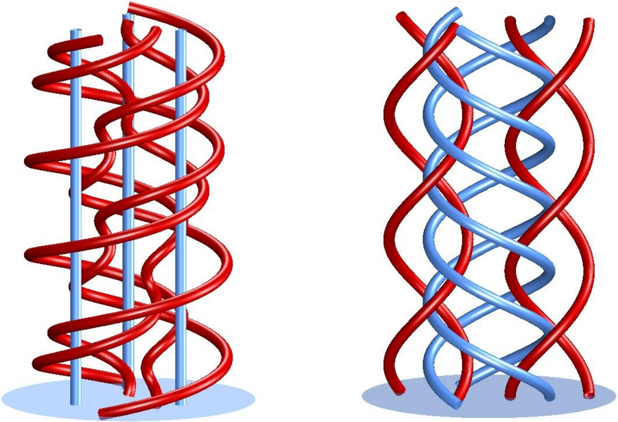
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Braid description of taffy puller



The three rods of the taffy puller in a space-time diagram. Defines a braid on $n = 3$ strands, $\sigma_1^2\sigma_2^{-2}$ (three periods shown).

Braid description of mixograph



$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

braid on B_7, the braid group on 7 strands.

Topological entropy of a braid

Burau representation for 3-braids:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$[\sigma_1^{-1}\,\sigma_2] = [\sigma_1^{-1}] \cdot [\sigma_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

This matrix has spectral radius $(3 + \sqrt{5})/2$ (Golden Ratio^2), and hence the topological entropy is $\log[(3 + \sqrt{5})/2]$.

This is the growth rate of a ‘rubber band’ caught on the rods.

This matrix trick only works for 3-braids, unfortunately.

Optimizing over generators

- Entropy can grow without bound as the length of a braid increases;
- A proper definition of optimal entropy requires a cost associated with the braid.
- Divide the entropy by the smallest number of generators required to write the braid word.
- For example, the braid $\sigma_1^{-1}\,\sigma_2$ has entropy $\log[(3+\sqrt{5})/2]$ and consists of two generators.
- Its Topological Entropy Per Generator (TEPG) is thus $\frac{1}{2}\log[(3+\sqrt{5})/2] = \log[\text{Golden Ratio}]$.
- Assume all the generators are used (stronger: irreducible).

Optimal braid

- In B_3 and B_4, the optimal TEPG is log[Golden Ratio].
- Realized by $\sigma_1^{-1}\sigma_2$ and $\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$, respectively.
- In B_n, $n > 4$, the optimal TEPG is $<$ log[Golden Ratio].

Why? Recall Burau representation:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

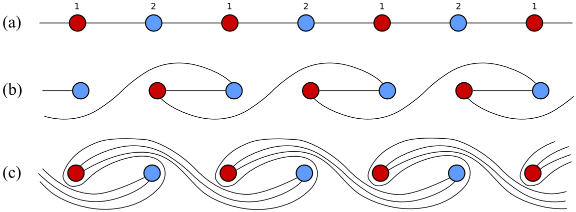
Its spectral radius provides a lower bound on entropy. However,

$$|[\sigma_1]| = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad |[\sigma_2]| = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

provides an upper bound! Need to find Joint Spectral Radius.

Periodic array of rods

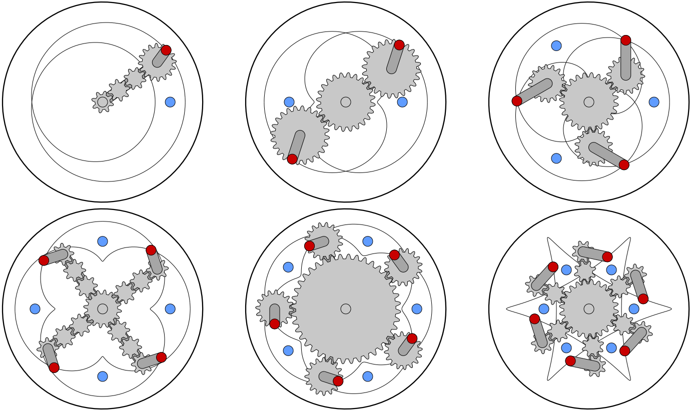
- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).



- The entropy per 'switch' is $\log(1 + \sqrt{2})$, the Silver Ratio!
- This is optimal for a periodic lattice of two rods (follows from D'Alessandro et al. (1999)).
- Also optimal if we assign cost by simultaneous operation.

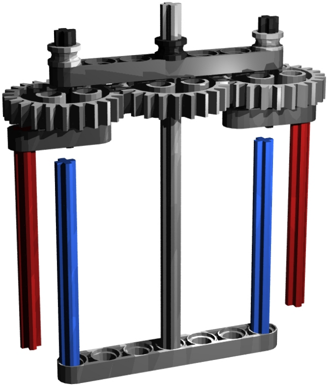
Silver mixers

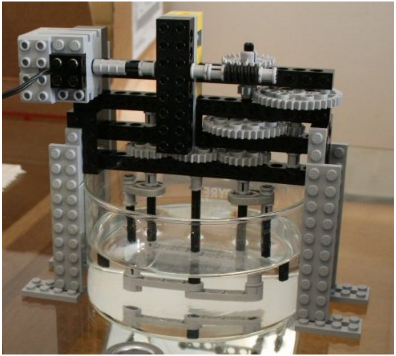
- The designs with entropy given by the Silver Ratio can be realized with simple gears.
- All the rods move at once: very efficient.



[movie 5]

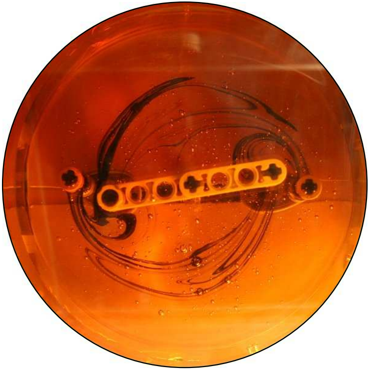
Build it!



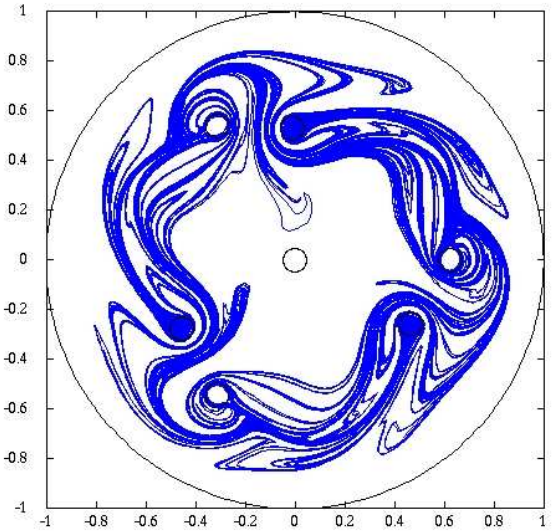


[movie 6] [movie 7]

Experiment: Silver mixer with four rods



Silver mixer with six rods



[movie 8]

Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amound of entropy (stretching of material lines).
- Can optimize to find the best rod motions, but depends on choice of ‘cost function.’
- For two natural cost functions, the Golden Ratio and Silver Ratio pop up!

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