Hamiltonian Dynamics from Lie–Poisson Brackets

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12 February 2002

Overview

- Many physical systems have a Hamiltonian formulation in terms of Lie–Poisson brackets obtained from Lie algebra extensions.
- This is true for finite-dimensional (e.g., heavy top, Kida vortex) and infinite-dimensional (Euler's equation with an advected scalar, reduced magnetohydrodynamics) systems.
- We the classify low-order extensions, thus showing that there are only a small number of independent normal forms. We make use of Lie algebra cohomology to achieve this.
- We give a simple example of a mechanical system with nontrivial Lie–Poisson structure, the Twisted Top.

Noncanonical Hamiltonian Formulation

A system of equations has a noncanonical Hamiltonian formulation if it can be written in the form

$$\boxed{\dot{\xi}^\lambda = \{\xi^\lambda\,, H\}}$$

where H is a Hamiltonian functional, and $\xi^\lambda(\mathbf{x}, t)$ or $\xi^\lambda_i(t)$ represents a vector of state variables.

(angular momentum, vorticity, temperature, . . .)

The Poisson bracket $\{\,,\}$ is antisymmetric and satisfies the Jacobi identity,

$$\{F\,, \{G\,, H\}\} + \{G\,, \{H\,, F\}\} + \{H\,, \{F\,, G\}\} = 0.$$

Jacobi tells us that there exist local canonical coordinates.

The Lie–Poisson Bracket

We define the Lie–Poisson bracket for one field variable ξ as

$$\boxed{\{F\,,G\} := \left\langle \xi\,, \left[\frac{\delta F}{\delta \xi}\,, \frac{\delta G}{\delta \xi}\right]\right\rangle}$$

In infinite dimensions, the pairing $\langle\cdot\rangle$ is an integral over a 2-D spatial domain, and the inner bracket is the 2-D Jacobian,

$$[\,a\,,b\,] = \frac{\partial a}{\partial x}\,\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\,\frac{\partial a}{\partial y}.$$

For finite-dimensional systems, the inner bracket we use is

$$[\,a\,,b\,] = a \times b,$$

where our state variables a and b are now 3-vectors; the pairing is a dot product of vectors.

Example: The 2-D Euler Equation

Consider the Hamiltonian

$$H[\omega] = \tfrac{1}{2}\int_\Omega |\nabla\phi(\mathbf{x},t)|^2\,\mathrm{d}^2x, \qquad \frac{\delta H}{\delta\omega} = -\phi,$$

where ϕ is the streamfunction and $\omega = \nabla^2\phi$ is the vorticity.

Inserting this into the Lie–Poisson bracket, we have

$$\dot\omega(\mathbf{x},t) = \{\omega\,,H\} = \int_\Omega \omega(\mathbf{x}',t)\left[\,\frac{\delta\omega(\mathbf{x},t)}{\delta\omega(\mathbf{x}',t)}\,,\frac{\delta H}{\delta\omega(\mathbf{x}',t)}\,\right]\,\mathrm{d}^2x'$$

$$= \int_\Omega \omega(\mathbf{x}',t)\,[\,\delta(\mathbf{x}-\mathbf{x}')\,,-\phi(\mathbf{x}',t)\,]\,\mathrm{d}^2x' \;\boxed{= [\,\omega(\mathbf{x},t)\,,\phi(\mathbf{x},t)\,]}$$

which is Euler's equation for the 2-D ideal fluid.

Similarly, the finite-dimensional Lie–Poisson bracket generates Euler's equation for the dynamics of the rigid body. The dynamical variable ξ is then the angular momentum vector $\boldsymbol{\ell}$.

Lie–Poisson Bracket Extensions

These "Euler equation" brackets act as our building block.

We wish to describe a Lie–Poisson system consisting of several state variables ξ^λ. The most general linear combination of one-variable brackets is

$$\{F\,,G\} = W_\lambda{}^{\mu\nu}\left\langle \xi^\lambda\,,\left[\frac{\delta F}{\delta \xi^\mu}\,,\frac{\delta G}{\delta \xi^\nu}\right]\right\rangle$$

where repeated indices are summed from 0 to n. W is constant and transforms like a 3-tensor under linear transformations of ξ; it determines the structure of the bracket.

We call this structure an extension of the one-variable bracket.

Properties of W

In order for the extension to be a good Poisson bracket, it must satisfy

1. Antisymmetry: Since the inner bracket [,] is already antisymmetric, W must be symmetric in its upper indices:

$$W_\lambda{}^{\mu\nu} = W_\lambda{}^{\nu\mu}$$

2. Jacobi identity: assuming the inner bracket [,] satisfies Jacobi, it is easy to show that W must satisfy

$$W_\lambda{}^{\sigma\mu} \, W_\sigma{}^{\tau\nu} = W_\lambda{}^{\sigma\nu} \, W_\sigma{}^{\tau\mu}$$

If we look at W as a collection of matrices $W^{(\mu)}$, then this means that these matrices commute.

Example: Compressible Reduced MHD

The four-field model derived by Hazeltine *et al.* (1987) for 2-D compressible reduced MHD (CRMHD) has a Lie–Poisson structure.

$$\begin{aligned}
\dot\omega &= [\,\omega\,,\phi\,] + [\,\psi\,,J\,] + 2\,[\,p\,,x\,] \\
\dot v &= [\,v\,,\phi\,] + [\,\psi\,,p\,] + 2\beta_{\rm e}\,[\,x\,,\psi\,] \\
\dot p &= [\,p\,,\phi\,] + \beta_{\rm e}\,[\,\psi\,,v\,] \\
\dot\psi &= [\,\psi\,,\phi\,]\,,
\end{aligned}$$

The Hamiltonian functional is the total energy,

$$H[\omega,v,p,\psi] = \frac{1}{2}\int_\Omega \left(|\nabla\phi|^2 + v^2 + \frac{(p-2\beta_{\rm e}\,x)^2}{\beta_{\rm e}} + |\nabla\psi|^2\right)\,{\rm d}^2x.$$

$$(\xi^0,\xi^1,\xi^2,\xi^3) = (\omega,v,p,\psi)$$

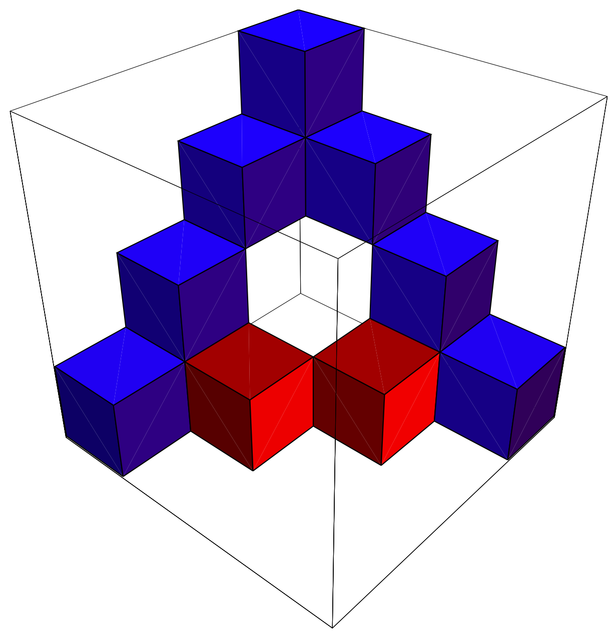
The W tensor for CRMHD

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_{\mathrm{e}} & 0 \end{pmatrix},$$

$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\beta_{\mathrm{e}} & 0 & 0 \end{pmatrix}, \qquad W^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily verified that these commute, so that the Jacobi identity holds. (Note the lower-triangular structure.)

Since W is a 3-tensor, we can represent it as a cube:



The vertical axis is the lower index of $W_\lambda{}^{\mu\nu}$, with the origin at the top rear. The two horizontal axes are the symmetric upper indices. The red cubes are $-\beta_{\mathrm{e}}$ terms, the blue cubes are unity.

(Symmetric when viewed from the top.)

Classification of Extensions

How many independent extensions are there?

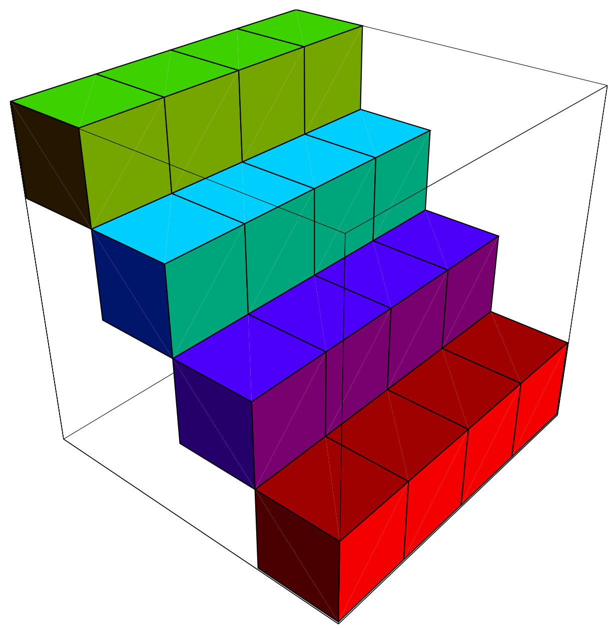
The answer amounts to finding normal forms for W, independent under coordinate transformations.

Threefold process:

1. Decomposition into a direct sum.
2. Transforming the matrices $W^{(\mu)}$ to lower-triangular form.
3. Finally, the hard part is to use Lie algebra cohomology to achieve the classification (almost).

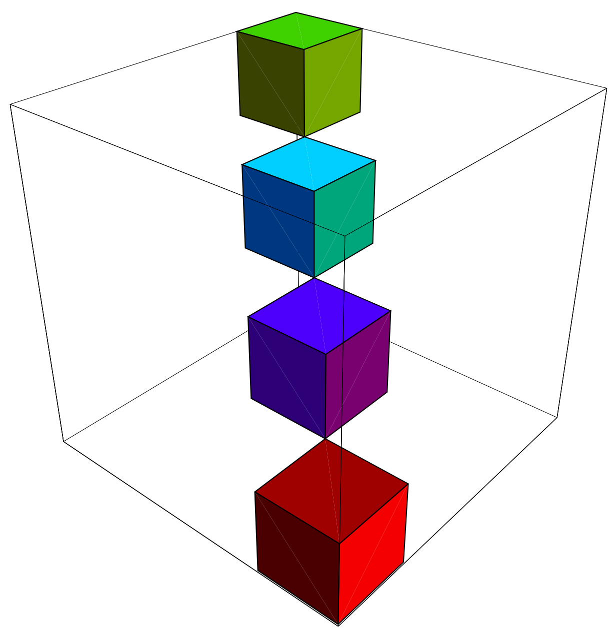
Classification 1: Direct Sum Structure

A set of commuting matrices, by a coordinate transformation, can always be put in block-diagonal form. The 3-tensor W then looks like:



Each “step” corresponds to a degenerate eigenvalue of the $W^{(\mu)}$.

Then, the symmetry of the upper indices of W implies the following structure:



We can focus on each block independently.

Classification 2: Lower-triangular Form

We focus on a single block, and thus assume that the $W^{(\mu)}$ have $(n+1)$-fold degenerate eigenvalues.

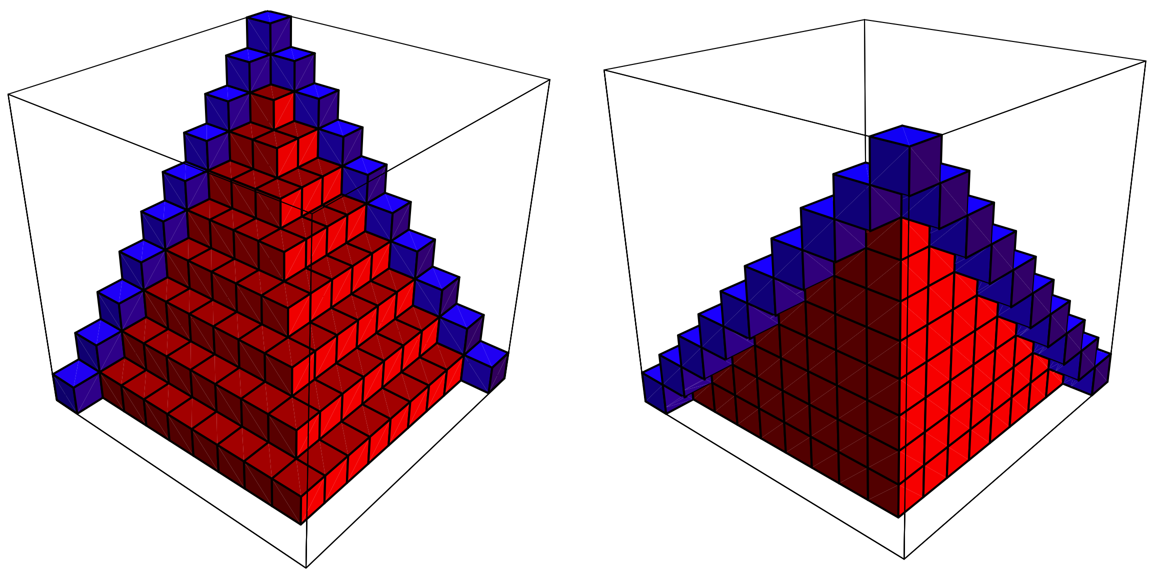
A set of commuting matrices can always be put into lower-triangular form by a coordinate transformation.

Once we do this, by the symmetry of the upper indices of W it is easy to show that

only the eigenvalue of $W^{(0)}$ can be nonzero.

Furthermore, if it is nonzero it can be rescaled to unity. We assume this is the case.

The most general form of W for an extension is thus



The red cubes form a solvable subalgebra, and are constrained by the commutation requirement. The blue cubes represent unit elements.

Classification 3: Cohomology

The problem of classifying extensions is reduced to classifying the solvable (red) part of the extension. This is achieved by the techniques of Lie algebra cohomology.

Cohomology gives us a class of linear transformations that preserve the lower-triangular structure of the extensions.

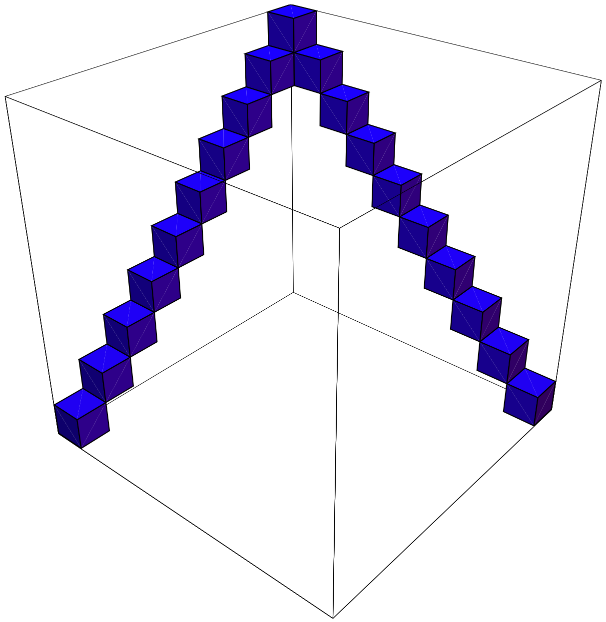
The parts of the extension that can be removed (i.e., made to vanish) by such transformations are called coboundaries.

What is left are nontrivial cocycles.

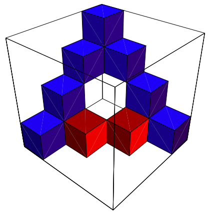
(Cohomology does not quite get it all. . .)

Pure Semidirect Sum

A common form for the bracket is the semidirect sum (SDS), for which the solvable part of W vanishes:

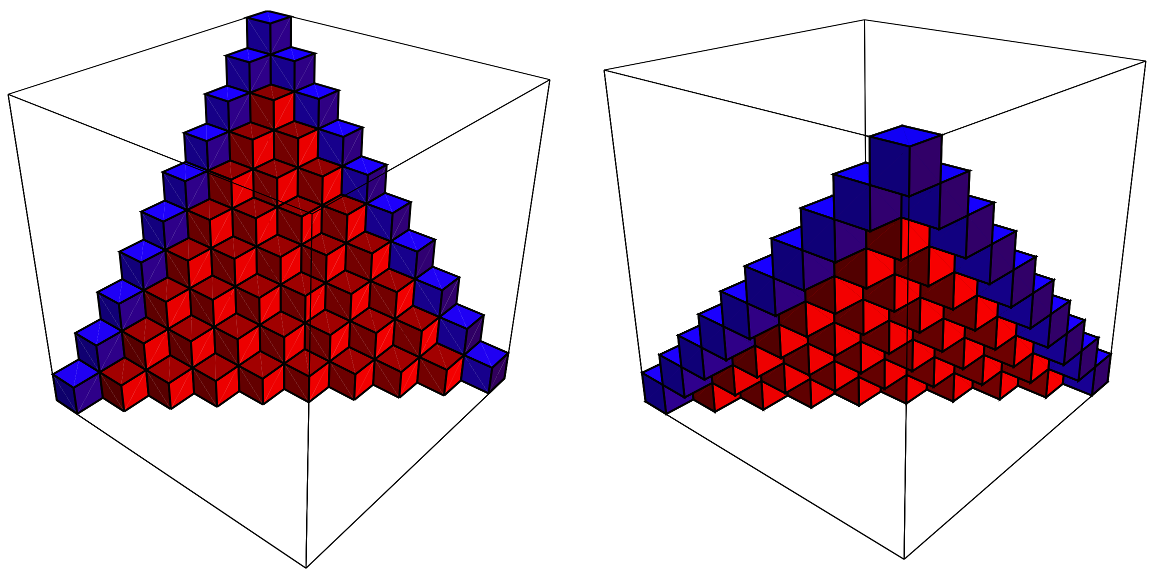


Note that CRMHD does not have a semidirect sum structure because of its extra nonzero blocks equal to $-\beta_{\rm e}$ (a cocycle).



Leibniz Extension

The opposite extreme to the pure semidirect sum is the case for which none of the $W^{(\mu)}$ vanishes. Then W must have the structure



This is called the Leibniz extension. All the cubes, red and blue, are equal to unity.

In between these two extreme cases, there are other possible extensions, including the CRMHD bracket.

Order	Number of extensions
1	1
2	1
3	2
4	4
5	9

None of these normal forms contains any free parameter!

(Do not expect this to be true at order 6 and beyond.)

[Thiffeault and Morrison, Physica D 136, 205 (2000)]

The Twisted Top

The charged heavy top has a semidirect structure. The simplest nonsemidirect extension we can make corresponds to the Leibniz extension:

$$H(\boldsymbol{\ell},\boldsymbol{\alpha},\boldsymbol{\beta}) = \tfrac{1}{2}\boldsymbol{\ell}\cdot\boldsymbol{\omega} + \boldsymbol{\alpha}\cdot\mathbf{a} + \boldsymbol{\beta}\cdot\mathbf{b}$$

$$\begin{aligned}
\dot{\boldsymbol{\ell}} &= \boldsymbol{\ell}\times\boldsymbol{\omega} + \boldsymbol{\alpha}\times\mathbf{a} + \boldsymbol{\beta}\times\mathbf{b},\\
\dot{\boldsymbol{\alpha}} &= \boldsymbol{\alpha}\times\boldsymbol{\omega} + \varepsilon\,\boldsymbol{\beta}\times\mathbf{a},\\
\dot{\boldsymbol{\beta}} &= \boldsymbol{\beta}\times\boldsymbol{\omega}.
\end{aligned}$$

Same Hamiltonian as charged heavy top, but different Lie–Poisson bracket → cocycle.

[Thiffeault and Morrison, Physics Letters A 283, 335 (2001)]

Invariants

The twisted top has three Casimir invariants,

$$C_1 = \|\boldsymbol{\alpha}\|^2 + 2\varepsilon\,\boldsymbol{\ell}\cdot\boldsymbol{\beta}, \quad C_2 = \boldsymbol{\alpha}\cdot\boldsymbol{\beta}, \quad C_3 = \|\boldsymbol{\beta}\|^2 \,.$$

The number of degrees of freedom is thus $(9-3)/2 = 3$.

For the Lagrange (symmetric) case,

$$I_1 = I_2, \qquad \boldsymbol{\alpha} = (0, 0, \alpha_3), \qquad \boldsymbol{\beta} = 0,$$

in addition to the energy there are two more invariants,

$$\boldsymbol{\ell}_3 \qquad \text{and} \qquad P = \boldsymbol{\ell}\cdot\boldsymbol{\alpha} + \varepsilon\, I_1 a_3\, \beta_3\,,$$

so that the symmetric case is integrable, as for the heavy top.

This dynamical system remains largely unexplored:

- Equilibria, periodic orbits, stability...
- Bifurcations of Poincaré sections, à la Dullin *et al.* (Canonical coordinates)
- Kovalevskaya case? Not obviously, but maybe nearby... (Painlevé analysis)
- Physical relevance? (Underwater vehicles?)
- Is there a signature of the cocycle in the dynamics?
- Twisted fluid?

Poincaré Section