Grilling Session

Canonical—NOT!

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The Hamiltonian Perspective

1. Unified approach to various systems.

2. Studying stability.

3. Methods for taking advantage of symmetries.

Canonical Variables

Hamilton's equations:

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}, \quad \alpha = 1, \ldots N.$$

Poisson bracket:

$$[f\,, g] \equiv \frac{\partial f}{\partial q^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q^\alpha}$$

$$\dot{q}^\alpha = [q_\alpha\,, H]\,, \quad \dot{p}_\alpha = [p^\alpha\,, H]\,.$$

Phase space volume is conserved:

$$\frac{\partial \dot{q}^\alpha}{\partial q^\alpha} + \frac{\partial \dot{p}_\alpha}{\partial p_\alpha} = \frac{\partial^2 H}{\partial q^\alpha \partial p_\alpha} - \frac{\partial^2 H}{\partial p_\alpha \partial q^\alpha} \equiv 0.$$

Form $2N$–tuplets:

$$z^i = (q^\alpha, p_\alpha), \quad i = 1, \dots 2N.$$

$$[f\,,g] = \frac{\partial f}{\partial z^i}\, J_c^{ij}\, \frac{\partial g}{\partial z^j}$$

$$\dot{z}^i = \left[z^i\,, H\right] = J_c^{ij}\, \frac{\partial H}{\partial z^j}$$

$$J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

J_c is called the cosymplectic form.

Noncanonical Coordinates

Generalization:

$$[f\,,g] = \frac{\partial f}{\partial z^i}\,J^{ij}(z)\,\frac{\partial g}{\partial z^j}, \quad i = 1, \ldots M.$$

Want energy to be conserved:

$$\dot{H} = \frac{\partial H}{\partial z^i}\,\dot{z}^i = \frac{\partial H}{\partial z^i}\,J^{ij}(z)\,\frac{\partial H}{\partial z^j}$$

Hence, we require J^{ij} (and so $[f\,,g]$)
to be antisymmetric.

Is this enough for the system
to be "Hamiltonian"?

$J(z)$ is the Poisson-Bracket of a Hamiltonian system if there exists a coordinate transformation which brings J^{ij} to the cosymplectic form, J_c.

Darboux's Theorem

Conditions on J for this transformation to exist are:

- det $J \neq 0$.
- Antisymmetry: $[f\,,g] = -\,[g\,,f]$.
- Jacobi identity:

  $$[f\,,[g\,,h]] + [g\,,[h\,,f]] + [h\,,[f\,,g]] = 0.$$

More general theorem:

If det $J = 0$, can always find a transformation to bring J to the form

$$\bar{J} = \begin{pmatrix} 0_N & I_N & \\ -I_N & 0_N & \\ & & 0_{M-2N} \end{pmatrix}$$

where the rank of J is $2N$.

$M - 2N$ independent null eigenvectors:

$$J^{ij} \frac{\partial C^{(\alpha)}}{\partial z^j} = 0.$$

The $C^{(\alpha)}$'s are called *Casimir Invariants*.

Casimirs are conserved:

$$\dot{C}^{(\alpha)} = \frac{\partial C^{(\alpha)}}{\partial z^i} J^{ij} \frac{\partial H}{\partial z^j} = 0,$$

regardless of H.

Free Rigid Body

$$\begin{aligned}
\dot{\ell}_1 &= \ell_2 \ell_3 \left(\frac{1}{I_3} - \frac{1}{I_2} \right), \\
\dot{\ell}_2 &= \ell_3 \ell_1 \left(\frac{1}{I_1} - \frac{1}{I_3} \right), \\
\dot{\ell}_3 &= \ell_1 \ell_2 \left(\frac{1}{I_2} - \frac{1}{I_1} \right).
\end{aligned}$$

$$\text{Hamiltonian :} \quad H = \frac{1}{2} \sum_{i=1}^{3} \frac{\ell_i^2}{I_i}.$$

$$\text{Bracket :} \quad [f\,, g] = -\epsilon_{ijk} \ell_k \frac{\partial f}{\partial \ell_i} \frac{\partial g}{\partial \ell_j}.$$

$$\text{Casimir :} \quad C = \frac{1}{2} \sum_{i=1}^{3} \ell_i^2.$$

The Continuous Case

Sums $\rightarrow$ Integrals

For 2–D Euler: $\mathbf{v} = (-\partial_y \psi, \partial_x \psi), \quad \omega = \nabla^2 \psi,$

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = 0.$$

$$\text{Hamiltonian :} \quad H[\omega] = \frac{1}{2} \int_D v^2 \, d^2 r$$

$$\text{Bracket :} \quad \{F[\omega]\,, G[\omega]\} = \int_D \omega \left[\frac{\delta F}{\delta \omega}\,, \frac{\delta G}{\delta \omega} \right] \, d^2 r.$$

$$\text{with} \;\; [A\,, B] \equiv \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}.$$

Now suppose we have two fields: (ω, T),

Could build a bracket of the form:

$$\{F[\omega\ , T], G[\omega, T]\} = \left\langle \omega, \left[\frac{\delta F}{\delta \omega}\, , \frac{\delta G}{\delta \omega}\right]\right\rangle + \left\langle T, \left[\frac{\delta F}{\delta T}\, , \frac{\delta G}{\delta T}\right]\right\rangle ,$$

$\rightarrow$ **Direct Product**

More interesting choice is the **Semidirect Product**:

$$\begin{aligned} \{F[\omega\ , T], G[\omega, T]\} &= \left\langle \omega, \left[\frac{\delta F}{\delta \omega}\, , \frac{\delta G}{\delta \omega}\right]\right\rangle \\ &+ \left\langle T, \left(\left[\frac{\delta F}{\delta \omega}\, , \frac{\delta G}{\delta T}\right] + \left[\frac{\delta F}{\delta T}\, , \frac{\delta G}{\delta \omega}\right] \right) \right\rangle , \end{aligned}$$

If we consider the Hamiltonian:

$$H = \int \left(\frac{1}{2}v^2 - yT\right) d^2r,$$

we get the equations of motion:

$$\frac{\partial \omega}{\partial t} + [\psi\,,\omega] = \frac{\partial T}{\partial x},$$

$$\frac{\partial T}{\partial t} + [\psi\,,T] = 0.$$

If we let $T = \widetilde{T} - y$, we see that what we obtained is a dissipationless version of the Boussinesq equations for Rayleigh–Bénard convection:

$$\frac{\partial \omega}{\partial t} + [\psi\,,\omega] = \frac{\partial \widetilde{T}}{\partial x} + \mu\nabla^2\omega,$$

$$\frac{\partial T}{\partial t} + \left[\psi\,,\widetilde{T}\right] = \frac{\partial \psi}{\partial x} + \kappa\nabla^2\widetilde{T}.$$

In Fourier space, this bracket looks like:

$$
\begin{aligned}
\{F, G\} &= \sum_{\mathbf{k},\mathbf{l}} \frac{\partial F}{\partial \omega_{\mathbf{k}}} J^{\omega\omega}_{\mathbf{kl}} \frac{\partial G}{\partial \omega_{\mathbf{l}}} \\
&\quad + \sum_{\mathbf{k},\mathbf{l}} \frac{\partial F}{\partial \omega_{\mathbf{k}}} J^{\omega\widetilde{T}}_{\mathbf{kl}} \frac{\partial G}{\partial \widetilde{T}} - \sum_{\mathbf{k},\mathbf{l}} \frac{\partial F}{\partial \widetilde{T}_{\mathbf{k}}} J^{\widetilde{T}\omega}_{\mathbf{kl}} \frac{\partial G}{\partial \omega_{\mathbf{l}}}
\end{aligned}
$$

with

$$
\begin{aligned}
J^{\omega\omega}_{\mathbf{kl}} &= \omega_{\mathbf{k}+\mathbf{l}} (\mathbf{l} \times \mathbf{k}) \cdot \hat{\mathbf{z}}, \\
J^{\omega\widetilde{T}}_{\mathbf{kl}} &= \widetilde{T}_{\mathbf{k}+\mathbf{l}} (\mathbf{l} \times \mathbf{k}) \cdot \hat{\mathbf{z}} - i(-1)^{k_y + l_y} \delta_{k_x, -l_x} = -J^{\widetilde{T}\omega}_{\mathbf{lk}},
\end{aligned}
$$

and Hamiltonian

$$
H = \frac{1}{2} \sum_{\mathbf{k}} \frac{|\omega_{\mathbf{k}}|^2}{\mathbf{k}^2} + \sum_{k_y} i \frac{(-1)^{k_y}}{k_y} \widetilde{T}_{0,k_y}.
$$

Other systems that admit a noncanonical representation:

- The **Korteweg–deVries** equation, $\partial_t u + u\partial_x u + \partial_x^3 u = 0$.

- 3–D ideal fluid.

- Compressible Reduced Magnetohydrodynamics.

Reduction

Casimirs reduce the dimension of the phase space accessible to the system.

- Free rigid body: $(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}) \rightarrow (\ell_1, \ell_2, \ell_3)$.

- Lagrangian variables $\rightarrow$ Euler variables. Hamiltonian independent of "identity" of fluid particles.

Summary

- Many problems, especially in fluid dynamics, admit a noncanonical representation.

- Reduction is a method for taking advantage of symmetries. Reduces size of phase space.

- Do all noncanonical representations in Nature come from a reduction?