Braids and Dynamics

Frontiers in Theory and Modelling with Scarce Data

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

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Taffy is a type of candy.

Needs to be pulled: this aerates it and makes it lighter and chewier.

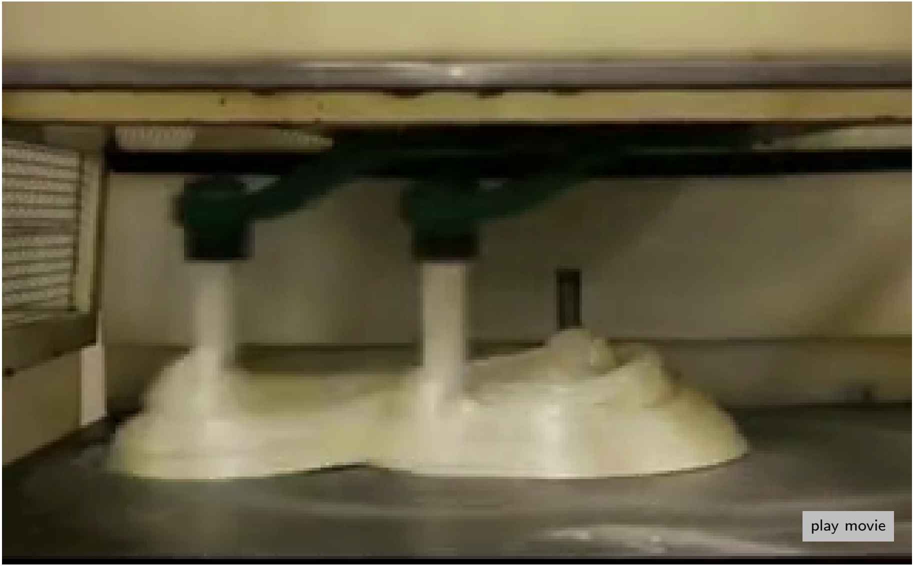
We can assign a growth: length multiplier per period.

(Here $(1+\sqrt{2})^2 \ldots$ more later.)

[movie by M. D. Finn]

play movie

making candy cane



play movie

[*Wired:* This Is How You Craft 16,000 Candy Canes in a Day]

four-pronged taffy puller

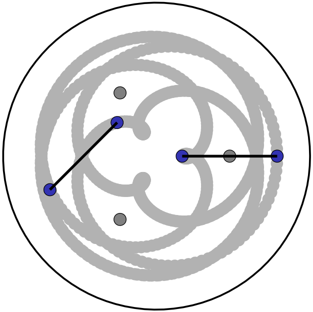
play movie

http://www.youtube.com/watch?v=Y7tlHDsquVM

[MacKay (2001); Halbert & Yorke (2014)]

the mixograph

Experimental device for kneading bread dough:



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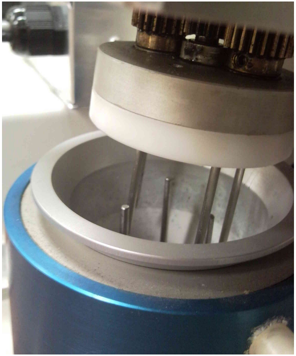
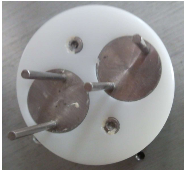
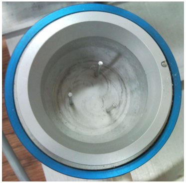






[Department of Food Science, University of Wisconsin. Photos by J-LT.]

Encode the topological information as a sequence of generators of the Artin braid group B_n.

Equivalent to the 7-braid

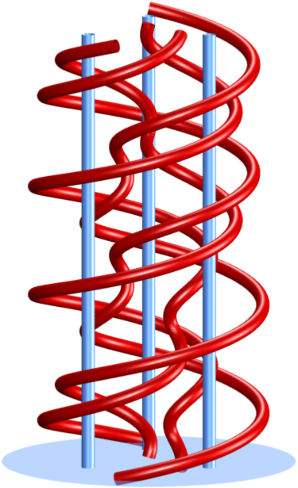
$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

The growth is the largest root of

$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1$$

$\boxed{\simeq 4.186}$

Compare to taffy pullers: $\boxed{5.828}$

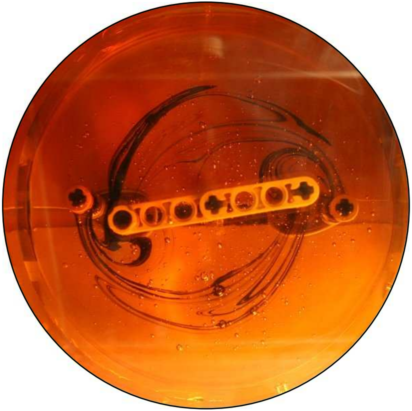


braids and rod-stirring

play movie

play movie

[Boyland, P. L., Aref, H., & Stremler, M. A. (2000). *J. Fluid Mech.* **403**, 277–304; Simulations by M. D. Finn, S. E. Tumasz, and J-LT.]

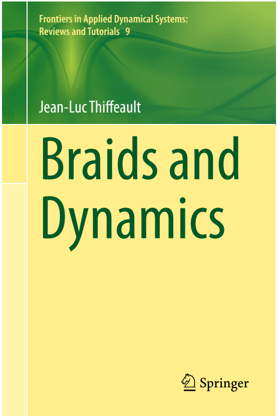


play movie

[Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743]

mathematical description

- There is an underlying topological description of the rod motions.
- Regard fluid motion as a map from the domain to itself.
- Map can be classified into finite-order, reducible, and pseudo-Anosov.
- Pseudo-Anosov maps have some inherent topological mixing, which is is a kind of chaotic behavior.
- Characterized by topological entropy, related to Lyapunov exponents.



[shameless plug for new book]

the topological program

- Consider a motion of stirring elements, such as rods.
- Determine if the motion is isotopic to a pseudo-Anosov mapping.
- Compute topological quantities, such as foliation, entropy, etc.
- Analyze and optimize.

insight: do we need the rods?

[Gouillart, E., Kuncio, N., Dauchot, O., Dubrulle, B., Roux, S., & Thiffeault, J.-L. (2007). *Phys. Rev. Lett.* **99**, 114501]

play movie

ghost rods ('tiges fantômes')

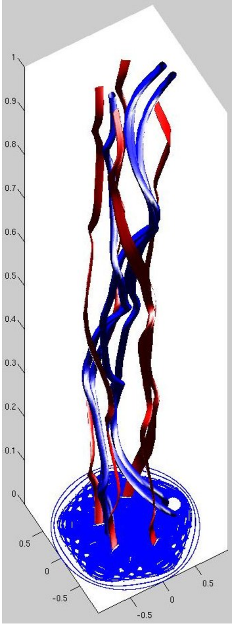
Topological analysis can be done on other objects than rods – for instance, islands or unstable periodic orbits.

We simply follow the islands and examine the braid they form, which gives us bounds on topological entropy.

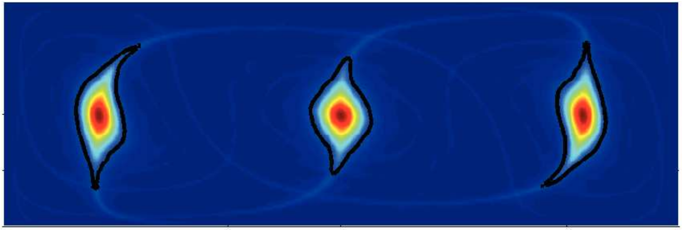
In this framework we call the islands ghost rods.

[Gouillart, E., Finn, M. D., & Thiffeault, J.-L. (2006). *Phys. Rev. E*, **73**, 036311]

[implemented by Stremler & Chen (2007); Thiffeault *et al.* (2009); Binder (2010); Stremler *et al.* (2011)]



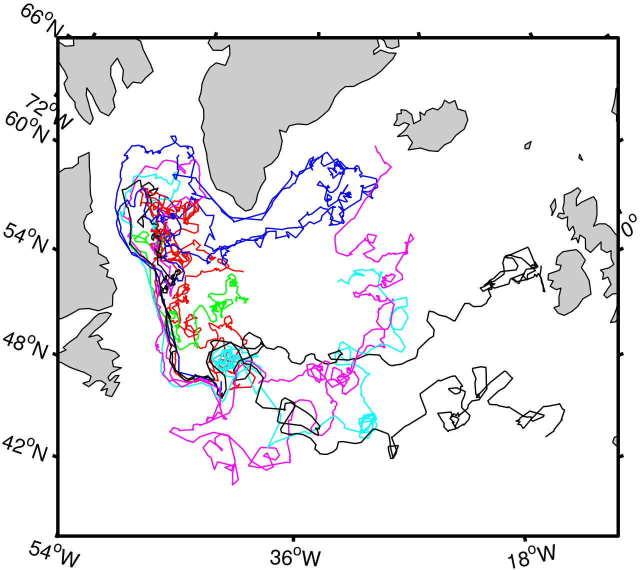
One of the best examples of ghost rods is from Stremler *et al.* (2011):



The islands are made to follow the $\sigma_2\sigma_1^{-1}$ stirring protocol by clever wall motions! (viscous Stokes flow)

[Stremler, M. A., Ross, S. D., Grover, P., & Kumar, P. (2011). *Phys. Rev. Lett.* **106**, 114101]

play movie


66°N
72°W
60°N
54°N
48°N
42°N
0°
54°W
36°W
18°W

oceanic floats: data analysis

What can we measure?

- single-particle dispersion (not a good use of all data)
- correlation functions (useful)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the braid group generators σ_i for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent, or to the 'growth' of taffy pullers).

iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

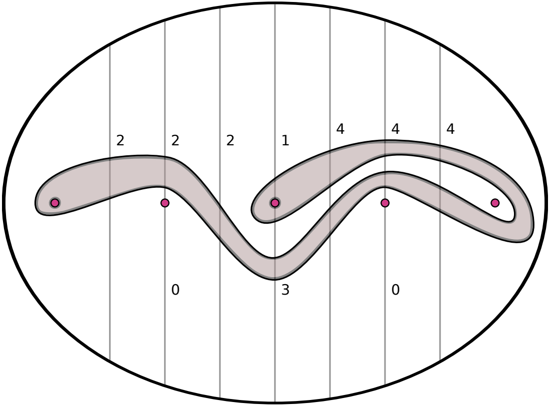
The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.

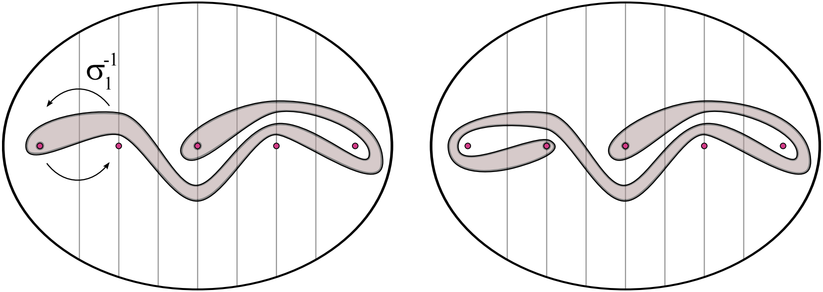
solution to problem 1: loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:



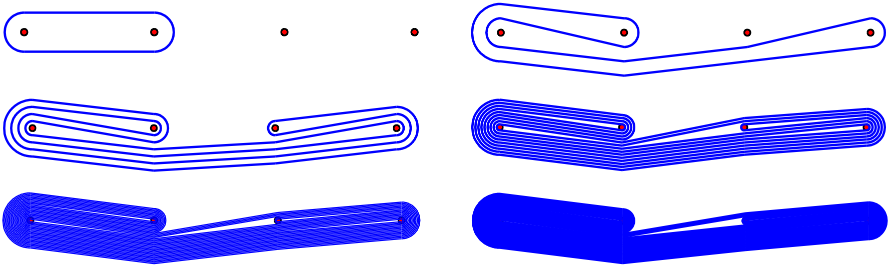
solution to problem 2: action on coordinates

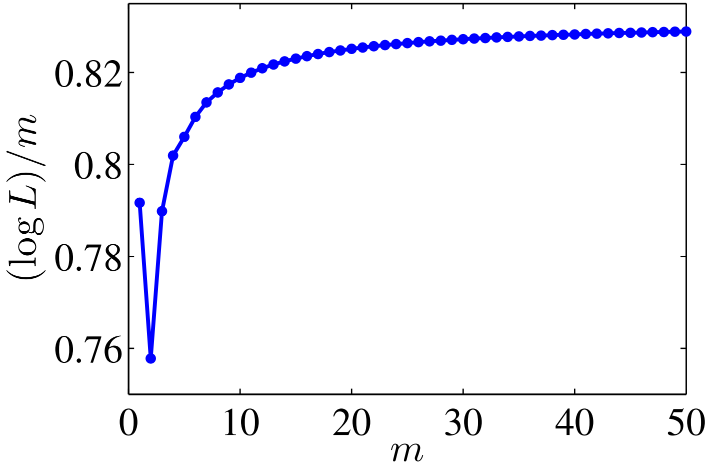
Moving the punctures according to a braid generator changes some crossing numbers:



There is an explicit formula for the change in the coordinates! [Dynnikov (2002); Moussafir (2006); Hall & Yurttaş (2009); Thiffeault (2010)]

For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of L and thus measure the entropy:



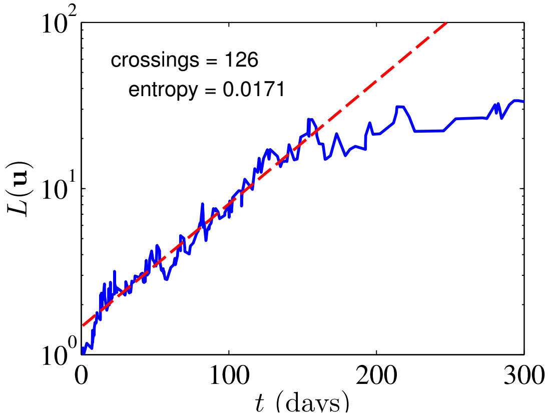


m is the number of times the braid acted on the initial loop.

[Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* **1** (1), 37–46]

oceanic floats: entropy

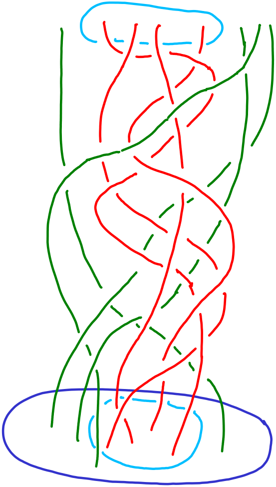
10 floats from Davis' Labrador sea data:



Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

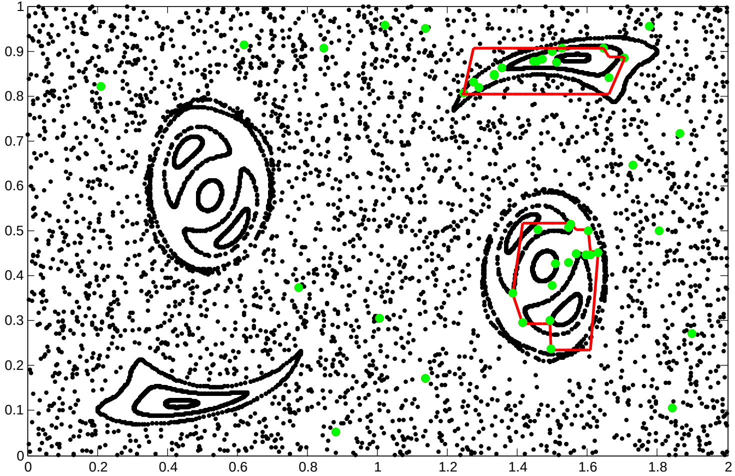
Source: WOCE subsurface float data assembly center (2004)

Lagrangian Coherent Structures



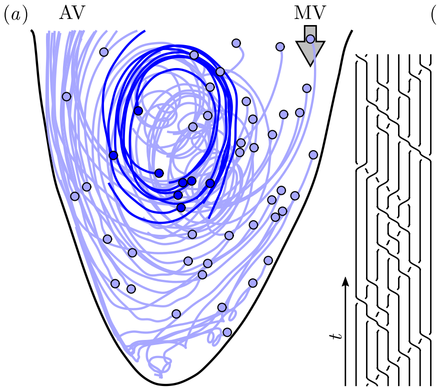
- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- [see Haller, G. & Beron-Vera, F. J. (2012). *Physica D*, **241** (20), 1680–1702.]
- Topological approach: [Allshouse & Thiffeault (2012); Filippi *et al.* (2020); Yeung *et al.* (2020)].

double-gyre coherent structures



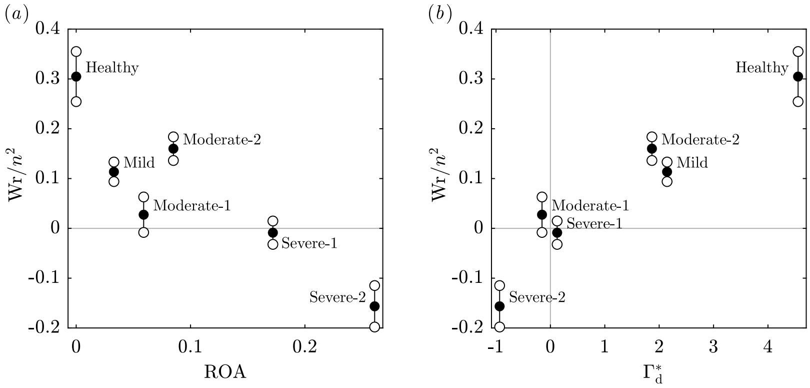
play movie [Allshouse, M. R. & Thiffeault, J.-L. (2012). *Physica D,* **241** (2), 95–105]

braids in the heart



[Di Labbio, G., Thiffeault, J.-L., & Kadem, L. (2022). *Flow*, **2**, E12]

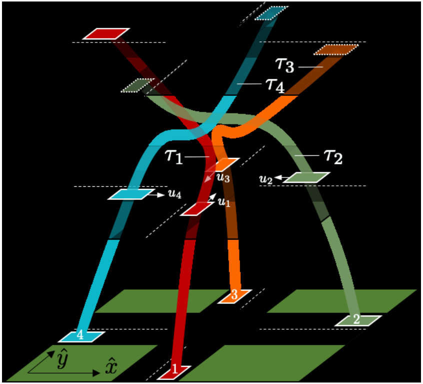
braids in the heart (cont'd)



Wr	writhe of braid
ROA	Regurgitant Orifice Area
Γ_d^*	circulation

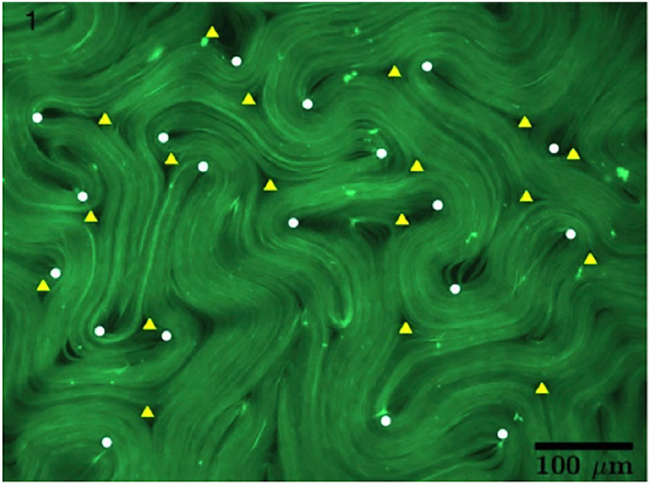
[Di Labbio, G., Thiffeault, J.-L., & Kadem, L. (2022). *Flow*, **2**, E12]

multiagent modeling: cars at an intersection



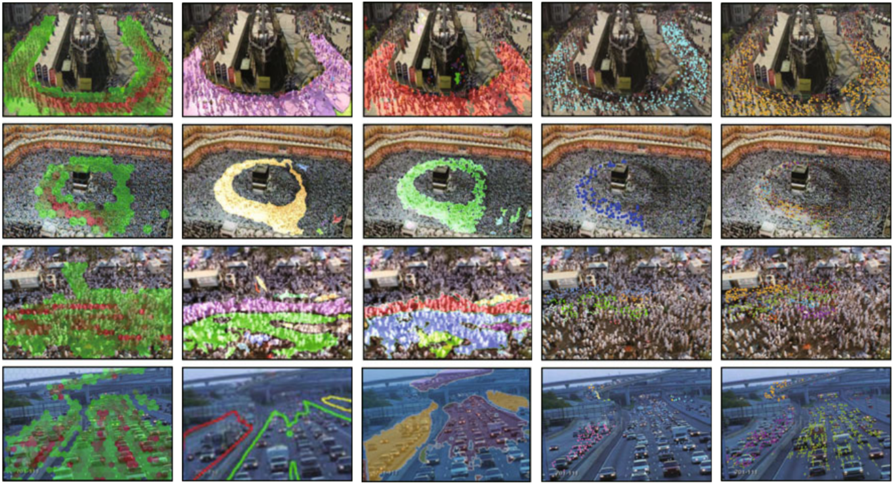
[Mavrogiannis, C., DeCastro, J., & Srinivasa, S. (2022). preprint]

braiding in active nematics



[Smith, S. A. & Gong, R. (2022). *Frontiers in Physics,* **10**]

complexity of crowd movement



[Akpulat, M. & Ekinci, M. (2019). *Frontiers of Information Technology & Electronic Engineering,* **20** (6), 849–861]

some research directions

- We don't have solid theory for aperodic or open braids.
- Computational methods for isotopy class (random entanglements of trajectories – LCS method, see Allshouse & Thiffeault (2012); Filippi *et al.* (2020); Yeung *et al.* (2020).
- 'Designing' for topological chaos (see Stremler & Chen (2007)).
- Combine with other measures, e.g., mix-norms (Mathew *et al.*, 2005; Lin *et al.*, 2011; Thiffeault, 2012).
- Matlab toolbox — `https://github.com/jeanluct/braidlab`.
- 3D?! (lots of missing theory; E-Tec approach shows promise [Roberts, E., Sindi, S., Smith, S. A., & Mitchell, K. A. (2019). *Chaos,* **29** (1), 013124]).

Akpulat, M. & Ekinci, M. (2019). *Frontiers of Information Technology & Electronic Engineering,* **20** (6), 849–861.

Allshouse, M. R. & Thiffeault, J.-L. (2012). *Physica D,* **241** (2), 95–105.

Bestvina, M. & Handel, M. (1995). *Topology,* **34** (1), 109–140.

Binder, B. J. (2010). *Phys. Lett. A,* **374**, 3483–3486.

Binder, B. J. & Cox, S. M. (2008). *Fluid Dyn. Res.* **40**, 34–44.

Bowen, R. (1978). In: *Structure of Attractors* volume 668 of *Lecture Notes in Math.* pp. 21–29, New York: Springer.

Boyland, P. L., Aref, H., & Stremler, M. A. (2000). *J. Fluid Mech.* **403**, 277–304.

Boyland, P. L., Stremler, M. A., & Aref, H. (2003). *Physica D,* **175**, 69–95.

D'Alessandro, D., Dahleh, M., & Mezić, I. (1999). *IEEE Transactions on Automatic Control,* **44** (10), 1852–1863.

Di Labbio, G., Thiffeault, J.-L., & Kadem, L. (2022). *Flow,* **2**, E12.

Dynnikov, I. A. (2002). *Russian Math. Surveys,* **57** (3), 592–594.

Filippi, M., Budišić, M., Allshouse, M. R., Atis, S., Thiffeault, J.-L., & Peacock, T. (2020). *Phys. Rev. Fluids,* **5**, 054504.

Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743.

Gouillart, E., Finn, M. D., & Thiffeault, J.-L. (2006). *Phys. Rev. E,* **73**, 036311.

references II

Gouillart, E., Kuncio, N., Dauchot, O., Dubrulle, B., Roux, S., & Thiffeault, J.-L. (2007). *Phys. Rev. Lett.* **99**, 114501.

Halbert, J. T. & Yorke, J. A. (2014). *Topology Proceedings,* **44**, 257–284.

Hall, T. & Yurttaş, S. Ö. (2009). *Topology Appl.* **156** (8), 1554–1564.

Haller, G. & Beron-Vera, F. J. (2012). *Physica D,* **241** (20), 1680–1702.

Handel, M. (1985). *Ergod. Th. Dynam. Sys.* **8**, 373–377.

Kobayashi, T. & Umeda, S. (2007). In: *Proceedings of the International Workshop on Knot Theory for Scientific Objects, Osaka, Japan* pp. 97–109, Osaka, Japan: Osaka Municipal Universities Press.

Lin, Z., Doering, C. R., & Thiffeault, J.-L. (2011). *J. Fluid Mech.* **675**, 465–476.

MacKay, R. S. (2001). *Philos. Trans. Royal Soc. Lond. A,* **359**, 1479–1496.

Mathew, G., Mezić, I., & Petzold, L. (2005). *Physica D,* **211** (1-2), 23–46.

Mavrogiannis, C., DeCastro, J., & Srinivasa, S. (2022). preprint.

Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* **1** (1), 37–46.

Roberts, E., Sindi, S., Smith, S. A., & Mitchell, K. A. (2019). *Chaos,* **29** (1), 013124.

Smith, S. A. & Gong, R. (2022). *Frontiers in Physics,* **10**.

Stremler, M. A. & Chen, J. (2007). *Phys. Fluids,* **19**, 103602.

references III

Stremler, M. A., Ross, S. D., Grover, P., & Kumar, P. (2011). *Phys. Rev. Lett.* **106**, 114101.

Thiffeault, J.-L. (2005). *Phys. Rev. Lett.* **94** (8), 084502.

Thiffeault, J.-L. (2010). *Chaos,* **20**, 017516.

Thiffeault, J.-L. (2012). *Nonlinearity,* **25** (2), R1–R44.

Thiffeault, J.-L. & Finn, M. D. (2006). *Philos. Trans. Royal Soc. Lond. A,* **364** (1849), 3251–3266.

Thiffeault, J.-L., Finn, M. D., Gouillart, E., & Hall, T. (2008). *Chaos,* **18**, 033123.

Thiffeault, J.-L., Gouillart, E., & Finn, M. D. (2009). In: *Analysis and Control of Mixing with Applications to Micro and Macro Flow Processes*, (Cortelezzi, L. & Mezić, I., eds) volume 510 of *CISM International Centre for Mechanical Sciences* pp. 339–350, Vienna: Springer.

Thurston, W. P. (1988). *Bull. Am. Math. Soc.* **19**, 417–431.

Yeung, M., Cohen-Steiner, D., & Desbrun, M. (2020). *Chaos,* **30**, 033122.