Stirring, mixing, and swimming at GFD

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- In '99 I was a new postdoc at Columbia.
- My project was on mixing, about which I knew little.
- Stroke of luck! Bill Young lectured on that very topic that summer.
- These lectures were very influential on me.

The Volume

Claudia Pasquero and I edited the '99 volume. Bill put an enormous amount of work in polishing these notes. He even invited me to San Diego after GFD to continue editing.

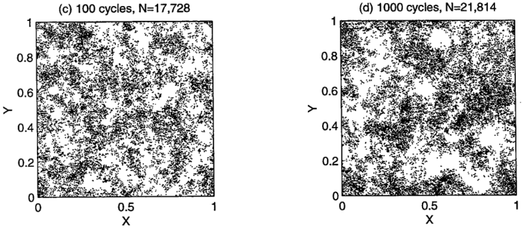


Figure 8: (a) The initial condition is $N = 20,000$ randomly located bugs in the unit square. Panels (b), (c) and (d) then show the development of patches after 10, 100 and 1000 cycles of random displacement followed by random birth/death. As the panel titles indicate, there are random fluctuations in the total size, N, of the population. The RMS step length of the underlying random walk is $\langle \delta x_k^2 \rangle^{1/2} = \langle \delta y_k^2 \rangle^{1/2} = 0.005$.

The volume contains many remarkable toy models that help understand subtle effects, including patchiness created by a random birth-death process.

The Volume (cont'd)

Bill's writing is admirably lively:

> *Marbled endpapers in old books were produced by floating coloured inks on water, stirring the surface, and then capturing the swirls by carefuly lowering a sheet of paper onto the inky film. This technique, probably originating in Persia in the 1400s, presses hydrodynamic correlations into the service of art. Fortunately for printers, and distressingly for statisticians, a single realization does not resemble the blurry diffusion equation.*

and more:

> *I have hinted darkly at problems associated with spatial filters. These issues are largely ignored by optimistic scientists. The hope is that scale separation justifies the application of diffusive closures to the coarse-grained version of a single realization*

2002: Bounds on turbulent transport

- Charlie Doering, Fritz Busse, Colm Caulfield, Rich Kerswell, and Peter Constantin were lecturers.
- Bounding turned out to be a very rich source of Fellow's projects.
- Personally: this began a long collaboration with Charlie. I take some credit for turning his mind to mixing.
- Charlie and the introduction of mix-norms around that time have brought the applied analysts into the game.

Mix-norms

Introduced by Mathew *et al.* (2003, 2005), with some later generalizations by Doering & Thiffeault (2006).

For a concentration $c(\boldsymbol{x}, t)$, the H^{-1} norm of c is

$$\|c\|_{H^{-1}}^2 = \int_V c\,(-\nabla^2)^{-1}\,c\,\mathrm{d}V.$$

The inverse $(-\nabla^2)^{-1}$ can be defined in Fourier space. Mix-norms essentially smooth out the wrinkles in the concentration.

Mix-norms have the advantage that they nicely connect to mixing in the sense of dynamical systems, and will decay even in the absence of diffusion. They are very convenient for rigorous work, but also for optimization (Lin *et al.*, 2011a; Foures *et al.*, 2014).

[Review: Thiffeault, J.-L. (2012). *Nonlinearity,* **25** (2), R1–R44]

Sample recent result

As an example of the types of theorems being proved, here's one by Bedrossian *et al.* (2019):

They show that Navier–Stokes with random forcing will mix exponentially:

$$\|c\|_{H^{-s}}(t) \leq D\,\mathrm{e}^{-\gamma t}\,\|c_0\|_{H^s}\,,$$

that is, the norm of the concentration field decays at least exponentially fast (no scalar diffusion here). This may seem obvious from the physicist's point of view, but is a hard challenge mathematically.

Proving something like this in the non-random setting? As the authors have said: 'maybe in a few hundred years.'

[Bedrossian, J., Blumenthal, A., & Punshon-Smith, S. (2019). https://arxiv.org/abs/1905.03869]

Biology at GFD

The first summer where GFD tackled biology was 1994, with Biophysical Models of Oceanic Population Dynamics. Donald Olson, Glenn Flierl, Daniel Grunbaum, and Simon Levin were lecturers.

In 2010, we had a summer on Swirling and Swimming in Turbulence with Glenn, Antonello Provenzale, and myself as lecturers.

More biology at GFD

In 2015 there was a summer on Fluid-Structure Interaction in the Living Environment, with Mike Shelley and Peko Hosoi as lecturers. Peko interpreted "living environment" loosely: she lectured about sports!

This is where we learned, for instance, why catchers do a forward-and-back dance when catching a foul ball: the lift from the backspin causes a ribbon trajectory.

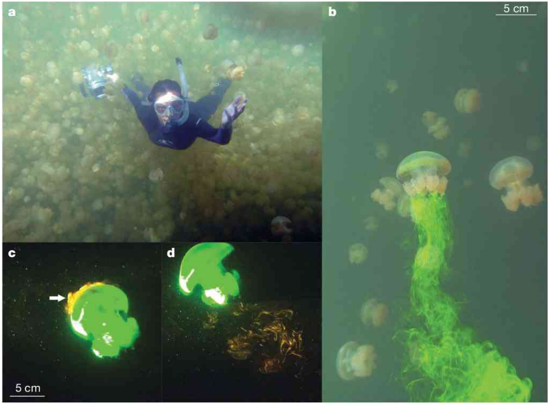
And more biology at GFD

In addition to the three summers, several of the Sears Lectures have had biological themes:

- Geoff Spedding (2009)
- James D. Murray (2010)
- L. Mahadevan (2011)
- Mimi Koehl (2016)
- John O. Dabiri (2018)
- Lydia Bourouiba (2019)

Biomixing: Stirring by swimming organisms

Katija & Dabiri (2009) looked at transport by jellyfish:



There was quite a stir at the time about biomixing and its possible role in the ocean.

The idea goes back to Walter Munk, who dismissed it. Revived by Bill Dewar and others in the 00's.

Since then I think the consensus is that the effect is negligible, in large part due to stratification (Visser, 2007; Wagner *et al.*, 2014).

Still could have important local impact, and is more relevant for micro-organisms.

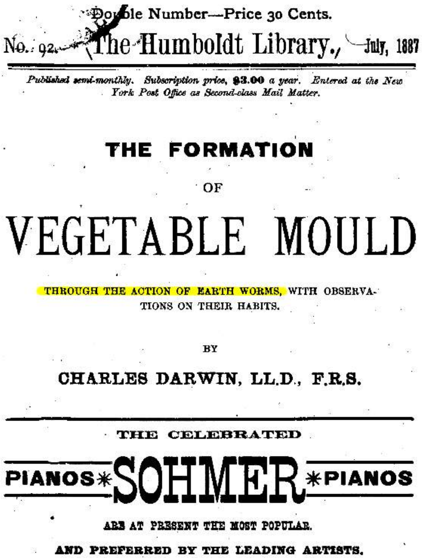
play movie

Bioturbation

The earliest case studied of animals ‘stirring’ their environment is the subject of Darwin’s last book.

This was suggested by his uncle and future father-in-law Josiah Wedgwood II, son of the famous potter.

“I was thus led to conclude that all the vegetable mould over the whole country has passed many times through, and will again pass many times through, the intestinal canals of worms.”



Double Number—Price 30 Cents.

No. 92. The Humboldt Library. July, 1887

Published semi-monthly. Subscription price, $3.00 a year. Entered at the New York Post Office as Second-class Mail Matter.

THE FORMATION

OF

VEGETABLE MOULD

THROUGH THE ACTION OF EARTH WORMS, WITH OBSERVATIONS ON THEIR HABITS.

BY

CHARLES DARWIN, LL.D., F.R.S.

Displacement by a moving body

With Steve Childress and George Lin, we set out to use drift trajectories to model mixing induced by swimmers.

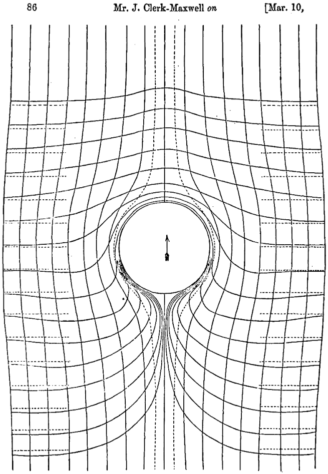


Fig. 1.
Fluid flowing past a fixed cylinder.

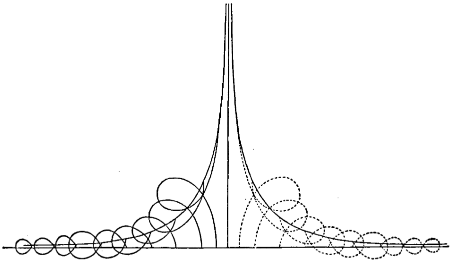


Fig. 2.
Paths of particles of the fluid when a cylinder moves through it.

Maxwell (1869); Darwin (1953)

A ‘gas’ of swimmers

Dilute theory: swimmers repeatedly ‘kick’ fluid particles.

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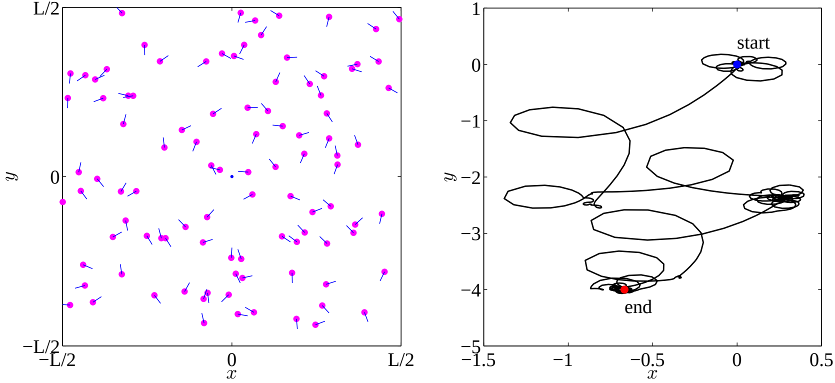
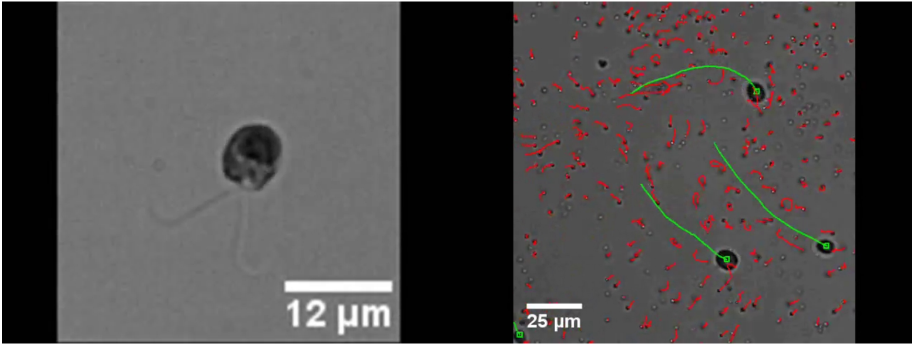


[Thiffeault, J.-L. & Childress, S. (2010). *Phys. Lett. A*, **374**, 3487–3490
Lin, Z., Thiffeault, J.-L., & Childress, S. (2011b). *J. Fluid Mech.* **669**, 167–177]

Lab experiments

Around the same time precise experiments were being made, most notably in the Gollub and the Goldstein groups:



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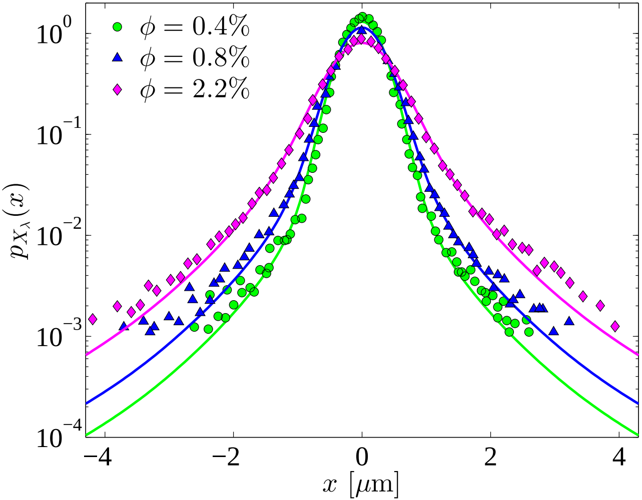
[Guasto, J. S., Johnson, K. A., & Gollub, J. P. (2010). *Phys. Rev. Lett.* **105**, 168102]

Comparing to experiments of Leptos *et al.* (2009)



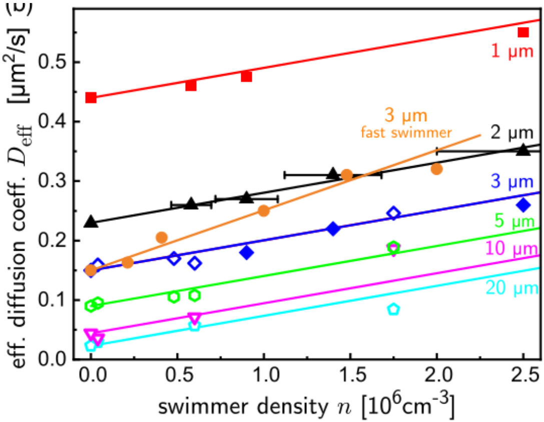
[Leptos, K. C., Guasto, J. S., Gollub, J. P., Pesci, A. I., & Goldstein, R. E. (2009). *Phys. Rev. Lett.* **103**, 198103

Thiffeault, J.-L. (2015). *Phys. Rev. E,* **92**, 023023]

More recent experiments of Ortlieb *et al.* (2019)

Formula for the effective diffusivity from Thiffeault (2015):

$$D_{\text{eff}} = D_0 + \left(0.266 + \tfrac{3}{4}\pi\beta\right) U\, n\, \ell^4$$



[Ortlieb, L., Rafaï, S., Peyla, P., Wagner, C., & John, T. (2019). *Phys. Rev. Lett.* **122**, 148101]

2015 was a summer on Stochastic Processes in Atmospheric and Oceanic Dynamics. Charlie Doering and Henk Dijkstra were lecturers.

I also learned a ton that summer from Oliver Buhler, who gave some extra lectures to help the fellows.

Mean Exit Time equation

There is an equation which I think is underused in the study of fluid transport:

The Mean Exit Time $\tau(\boldsymbol{x}_0, t_0)$ is the expected time for a fluid particle, starting at $(\boldsymbol{x}_0, t_0)$, to exit the system.

Remarkably, this satisfies a 'backward' advection-diffusion equation:

$$-\partial_{t_0}\tau - \boldsymbol{u}\cdot\nabla_0\tau = D\,\nabla_0^2\tau + 1$$

Time is a passive scalar!

Exits are determined by setting $\tau = 0$ on sections of the boundary.

Charlie D., Bill Y., and I did a project with Fellow Florence Marcotte to study optimization of heat exchangers, where the goal is to get a fluid particle to reach the boundary as fast as possible.

Active Brownian particles

Microswimmers and active particles are often modeled as Brownian particles with a propulsion, using an SDE such as

$$
\begin{aligned}
\mathrm{d}X &= U\,\mathrm{d}t + \sqrt{2D_X}\,\mathrm{d}W_1 \\
\mathrm{d}Y &= \sqrt{2D_Y}\,\mathrm{d}W_2 \\
\mathrm{d}\theta &= \sqrt{2D_\theta}\,\mathrm{d}W_3
\end{aligned}
$$

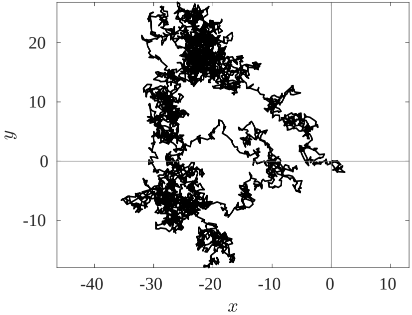
in its own rotating reference frame.

In terms of absolute x and y coordinates, this becomes

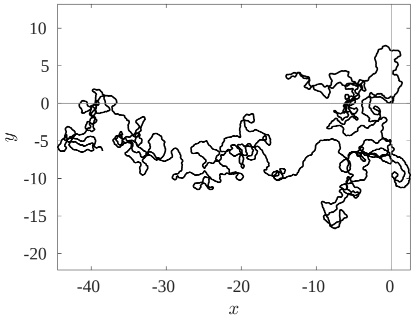
$$
\begin{aligned}
\mathrm{d}x &= \left(U\,\mathrm{d}t + \sqrt{2D_X}\,\mathrm{d}W_1\right)\cos\theta - \sin\theta\,\sqrt{2D_Y}\,\mathrm{d}W_2 \\
\mathrm{d}y &= \left(U\,\mathrm{d}t + \sqrt{2D_X}\,\mathrm{d}W_1\right)\sin\theta + \cos\theta\,\sqrt{2D_Y}\,\mathrm{d}W_2 \\
\mathrm{d}\theta &= \sqrt{2D_\theta}\,\mathrm{d}W_3\,.
\end{aligned}
$$

Sample paths

- Swimmer swims a distance U/D_θ in a time $1/D_\theta$.
- Swimmer diffuses a distance $\sqrt{D_X/D_\theta}$ in a time $1/D_\theta$.
- $\mathrm{Pe}_{\theta,X} := \frac{U}{D_\theta} \Big/ \sqrt{\frac{D_X}{D_\theta}} = \frac{U}{\sqrt{D_\theta D_X}}$ measures the smoothness of the path.



$\mathrm{Pe}_{\theta,X} \ll 1$



$\mathrm{Pe}_{\theta,X} \gg 1$

Fokker–Planck equation

The F–P equation for the probability density $p(x, y, \theta, t)$:

$$\partial_t p = -\nabla \cdot (\boldsymbol{u}\, p - \nabla \cdot \mathbb{D}\, p) + \partial_\theta^2 (D_\theta\, p)$$

where the drift vector and diffusion tensor are respectively

$$\boldsymbol{u} = \begin{pmatrix} U \cos\theta \\ U \sin\theta \end{pmatrix}$$

$$\mathbb{D} = \begin{pmatrix} D_X \cos^2\theta + D_Y \sin^2\theta & \frac{1}{2}(D_X - D_Y)\sin 2\theta \\ \frac{1}{2}(D_X - D_Y)\sin 2\theta & D_X \sin^2\theta + D_Y \cos^2\theta \end{pmatrix}.$$

Note that $\nabla := \hat{\boldsymbol{x}}\,\partial_x + \hat{\boldsymbol{y}}\,\partial_y$ (no θ).

Boundary condition

For any fixed volume V we have

$$\begin{aligned}\partial_t \int_V p \,\mathrm{d}V &= -\int_V \big(\nabla \cdot (\boldsymbol{u}\,p - \nabla \cdot (\mathbb{D}\,p)) - \partial_\theta^2 (D_\theta\,p)\big)\,\mathrm{d}V \\ &= -\int_{\partial V} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{S}\,,\end{aligned}$$

where ∂V is the boundary of V, and the flux vector is

$$\boldsymbol{f} = \boldsymbol{u}\,p - \nabla \cdot (\mathbb{D}\,p) - \hat{\boldsymbol{\theta}}\,\partial_\theta(D_\theta\,p).$$

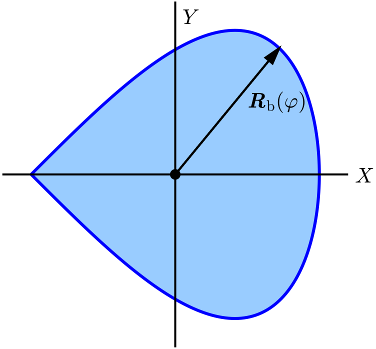
Thus, on the reflecting (impermeable) parts of the boundary we require the no-flux condition

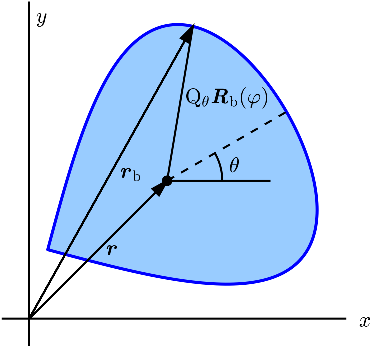
$$\boldsymbol{f} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \partial V_{\mathsf{refl}}$$

where $\boldsymbol{n}$ is normal to the boundary.

The shape of a 2D swimmer

In recent work with Hongfei Chen, we adapted this model to include the shape of a swimmer as it interacts with boundaries:





Convex swimmer in its frame (X, Y) and the fixed lab frame (x, y).

The swimming direction corresponds to $\varphi = 0$.

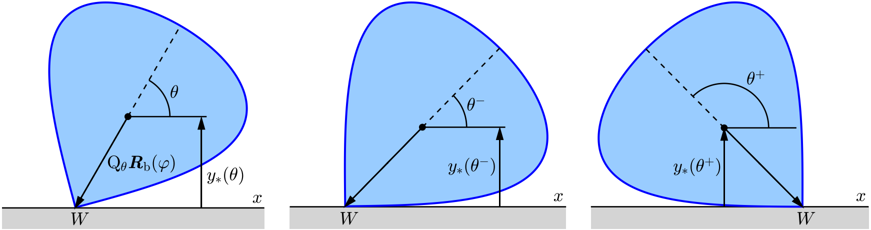
$\mathbb{Q}_\theta$ is a rotation matrix about a given center of rotation.

[Chen, H. & Thiffeault, J.-L. (2020). http://arxiv.org/abs/2006.07714]

Swimmer touching a wall at $y = 0$

Denote by $y_*(\theta)$ the vertical coordinate of a swimmer with orientation θ when it touches the wall.

Convex swimmer touching a horizontal wall at a corner point W:

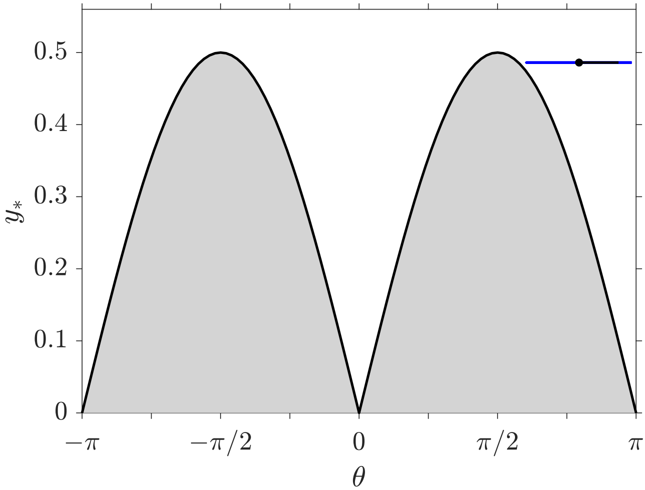


The angle θ can vary from the right-tangency angle θ^- to the left-tangency angle θ^+.

Range of y values:

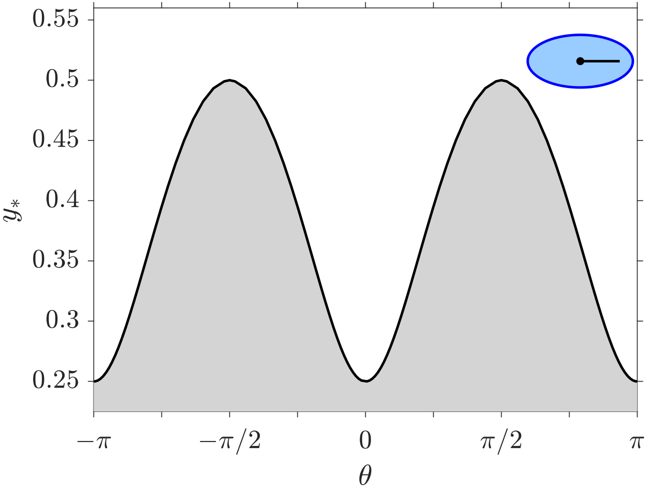
$$y_*(\theta) = -\sin\theta\, X(\varphi) - \cos\theta\, Y(\varphi), \qquad \theta^- \leq \theta \leq \theta^+.$$

Wall distance function $y_*(\theta)$: needle



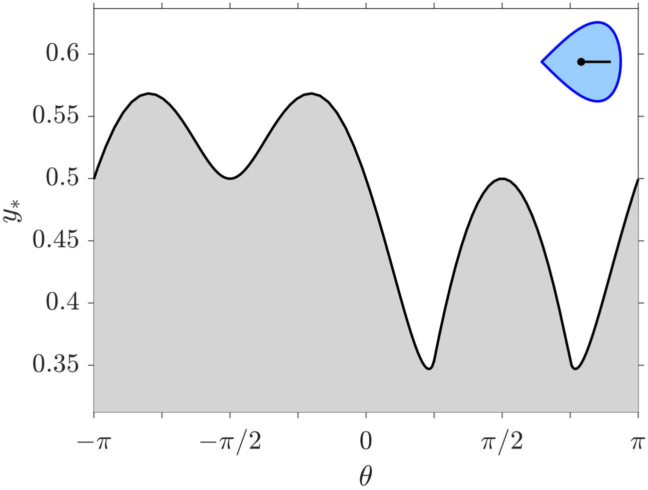
The needle has two corners; $y_*(\theta) = \frac{1}{2}\ell|\sin\theta|$

Wall distance function $y_*(\theta)$: ellipse



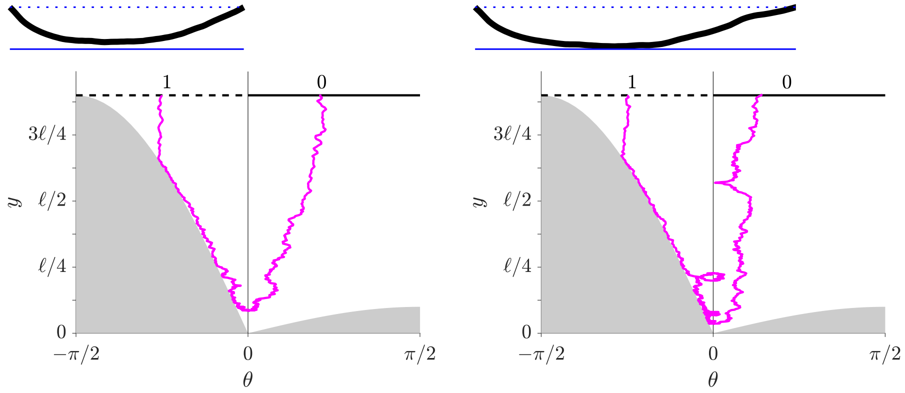
The ellipse has no corners; $y_*(\theta) = \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}$

Wall distance function $y_*(\theta)$: teardrop



The teardrop has a corner and a smooth boundary.

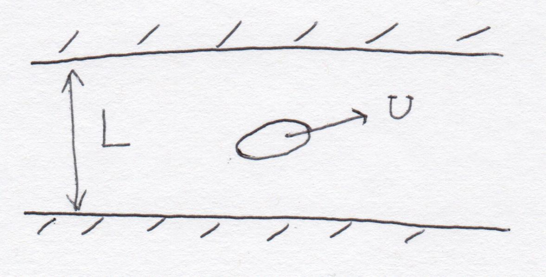
Drift is $U \sin\theta\, \hat{\boldsymbol{y}}$; no-flux condition forces swimmer to align with the wall.



Once the particle crosses $\theta = 0$ (parallel to wall), it is pushed upward by the drift.

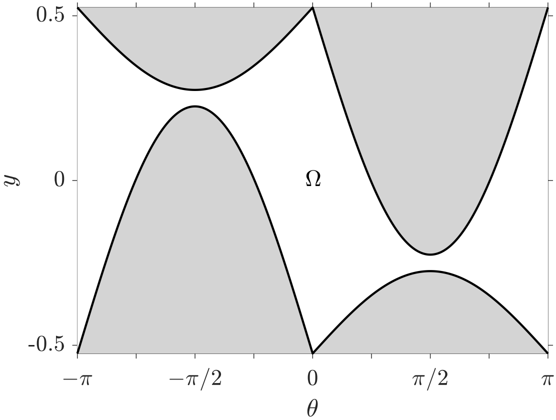
A Microswimmer in a Channel

For example, one application of this configuration space formalism is to the transport of microswimmers in narrow channels:



A swimmer will turn around once in a while, effectively undergoing a 1D random walk. What is the effective horizontal diffusion coefficient?

Channel configuration space

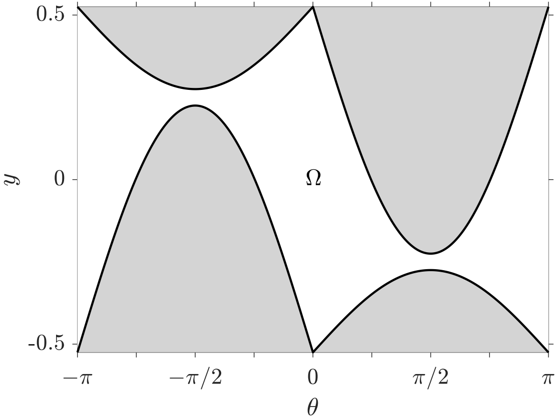


Configuration space for the needle in of length $\ell = 1$ in a channel of width $L = 1.05$. (x not shown.)

A point in this space specifies the position and orientation of the swimmer.

Reversal

Whenever the swimmer goes through one of the bottlenecks below, this corresponds to a reversal of swimming direction.



Mean Reversal Time

The mean reversal time τ_{rev} is

$$\boxed{\tau_{\mathrm{rev}} = \tfrac{1}{4}\int_0^{\pi} \frac{\mathrm{d}\vartheta}{\mathcal{P}(\vartheta)}}$$

where $\mathcal{P}(\theta)$ is the invariant probability density for the swimmer.

Intuitively, small $\mathcal{P}$ corresponds to "bottlenecks" that dominate the reversal time.

For the needle swimmer,

$$\tau_{\mathrm{rev}} \approx \frac{\pi}{2\beta}\,\mathrm{e}^{\beta}, \qquad \beta = U\ell/4D_Y\,.$$

From this we get an effective diffusivity

$$D_{\mathsf{eff}} \approx \tfrac{1}{2}\tau_{\mathrm{rev}}\,U^2$$

Conclusions

- Long tradition of studying stirring and mixing at the GFD program, even when the summer was not specifically on this topic.
- Some important innovations were pioneered at the program.
- Transport and mixing of and due to microswimmers is an active area of study.
- The interaction of microswimmers with boundaries is a huge topic, and I apologize for not doing justice to the literature today, for lack of time.
- Our focus is on modeling interactions using the rich concept of configuration space, where we work on an extended phase space involving all the degrees of freedom of the swimmer.
- Those degrees of freedom are limited by boudaries.
- Missing lots of effects! (Most notably, hydrodynamics!)
- Apologies to Fellows I worked with but whose work I didn't cover: Anshuman Roy, Michael Allshouse, Kiori Obuse, Amanda O'Rourke, Tiffany Shaw,

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