Why are all my power cords entangled?
A lecture on random entanglement

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1   Winding around a single point

This lecture is on simple models of entanglement. Here entanglement is interpreted broadly, and in particular it can apply to the two-dimensional trajectories of some particles, such as ocean floats or a crowd of people. Figure 1(a) shows two such particles moving around each other on the plane.
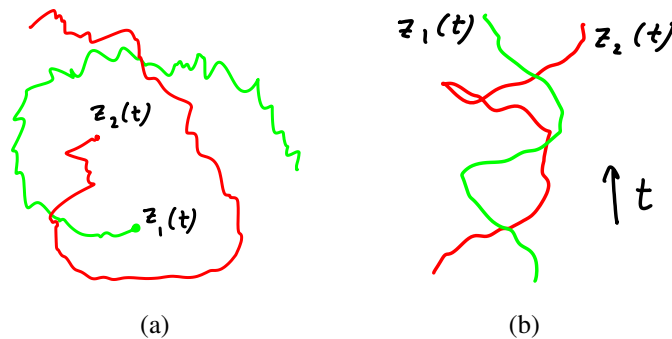

(a)   (b)

Figure 1: Two particles moving in the plane.

If we plot time vertically, as in Fig. 1(b), we obtain a *braid* of two strands.

The first question we want to ask is, how do the particles wind around each other? Let us assume that both particles are undergoing *Brownian motion* with some diffusivity D.
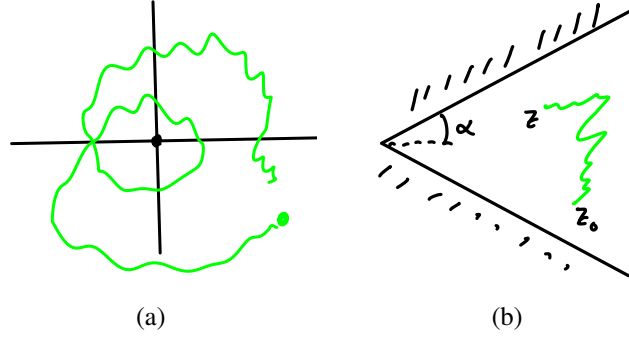
Figure 2:

We consider the vector $z(t) = z_1(t) - z_2(t)$, which behaves like a Brownian particle of diffusivity $2D$ (though we will drop the factor of two). Now the situation looks as in Fig. 2(a), with a single Brownian particle. We define $\theta \in (-\infty, \infty)$ to be the total winding angle of $z(t)$ around the origin. What we are looking for is $P(\theta, t)$, the winding angle distribution of a particle about the origin after a time t.

This problem was solved by Spitzer (1958). We shall solve it in a slightly different way. Assume for now that our Brownian particle lives in a wedge of half-angle α, as in Fig. 2(b). The Green's function $P(\boldsymbol{x}, t; \boldsymbol{x}_0, 0)$ gives the probability of finding the particle at $\boldsymbol{x}$ at time t, given that it started at $\boldsymbol{x}_0$ at time 0. It satisfies

$$\frac{\partial P}{\partial t} = D\,\nabla^2 P, \qquad P(\boldsymbol{x}, 0; \boldsymbol{x}_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{1}$$

In the wedge region 2(b), we take reflecting boundary conditions at the two infinite walls:

$$\hat{\boldsymbol{n}} \cdot \nabla P = 0 \quad \text{at each wall}, \tag{2}$$

where $\hat{\boldsymbol{n}}$ is the normal to the wall.

Solving 1 is a standard problem and can be done with separation of variables. There is a certain rough beauty to the detailed solution of such problems, but in the interest of brevity here is the final answer:

$$P(r, \theta, t; r_0, \theta_0, 0) = \frac{1}{4\alpha Dt}\,\mathrm{e}^{-(r^2+r_0^2)/4Dt}\left\{ I_0\left(\frac{rr_0}{2Dt}\right)\right.$$

$$\left. + 2\left(\sum_{k>0\text{ even}} \cos(\nu_k\theta)\cos(\nu_k\theta_0) + \sum_{k>0\text{ odd}} \sin(\nu_k\theta)\sin(\nu_k\theta_0) \right) I_{\nu_k}\left(\frac{rr_0}{2Dt}\right) \right\} \tag{3}$$

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where $\nu_k = \pi k/2\alpha$ and I_ν is a modified Bessel function of the first kind. Here (r, θ) and (r_0, θ_0) are respectively the polar coordinates of $\boldsymbol{x}$ and $\boldsymbol{x}_0$. The solution (3) is taken from Carslaw & Jaeger (1959, p. 379, Eq. (8)), after integrating their solution in z and centering the wedge so $\theta = 0$ is its bisector, as in Fig. 2(b).

The solution (3) is certainly valid for $0 < \alpha < \pi$, corresponding to a very thin wedge interior ($\alpha \to 0$) or a very thin wedge exterior ($\alpha \to \pi$). What about $\alpha = \pi$? Well, that solution is still valid and does *not* reduce to the Green's function on the plane: there is still an infinitely thin plate along the negative real axis which reflects the Brownian particle.

But surely $\alpha > \pi$ is nonsense? Physically, yes. But mathematically, let's see what happens if we take the limit $\alpha \to \infty$ (!), an infinite wedge angle. In (3) we have two sums, one over even integers and the other over odd ones. In either sum, since $\nu_k = \pi k/2\alpha$, the difference between two successive ν_k is

$$d\nu_k = \nu_{k+2} - \nu_k = \pi/\alpha, \tag{4}$$

which is of course small as $\alpha \to \infty$. Thus, in the limit as $\alpha \to \infty$, we can replace the sums by integrals using (4) and obtain

$$P = \frac{1}{2\pi Dt} e^{-(r^2+r_0^2)/4Dt} \int_0^\infty (\sin(\nu\theta)\sin(\nu\theta_0) + \cos(\nu\theta)\cos(\nu\theta_0)) \, I_\nu\!\left(\frac{rr_0}{2Dt}\right) d\nu \tag{5}$$

or

$$P(r, \theta, t; r_0, \theta_0, 0) = \frac{1}{2\pi Dt} e^{-(r^2+r_0^2)/4Dt} \int_0^\infty \cos\nu(\theta - \theta_0) \, I_\nu\!\left(\frac{rr_0}{2Dt}\right) d\nu. \tag{6}$$

Note that the I_0 term in (3) vanishes in the limit, since it is divided by α with no compensating sum in the numerator. We conclude that taking the limit $\alpha \to \infty$ has caused no catastrophe: Eq. (6) suffers from no obvious pathologies.

Let's rewrite the Green's function (6), which is a probability density function, in terms of the scaled variables $x = r/2\sqrt{Dt}$, $y = r_0/2\sqrt{Dt}$, and write θ for $\theta - \theta_0$ without loss of generality:

$$P(x, y, \theta) = \frac{2}{\pi} e^{-(x^2+y^2)} \int_0^\infty \cos\nu\theta \, I_\nu(2xy) \, d\nu. \tag{7}$$

For large t, the argument $2xy$ of the Bessel function is small, so we can use the asymptotic form

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} (x/2)^\nu, \qquad x \ll 1. \tag{8}$$

We also use the integral formula

$$\int_0^\infty \cos\nu\theta \, \xi^\nu \, d\nu = -\frac{\log\xi}{\theta^2 + \log^2\xi}, \qquad 0 < \xi < 1, \tag{9}$$
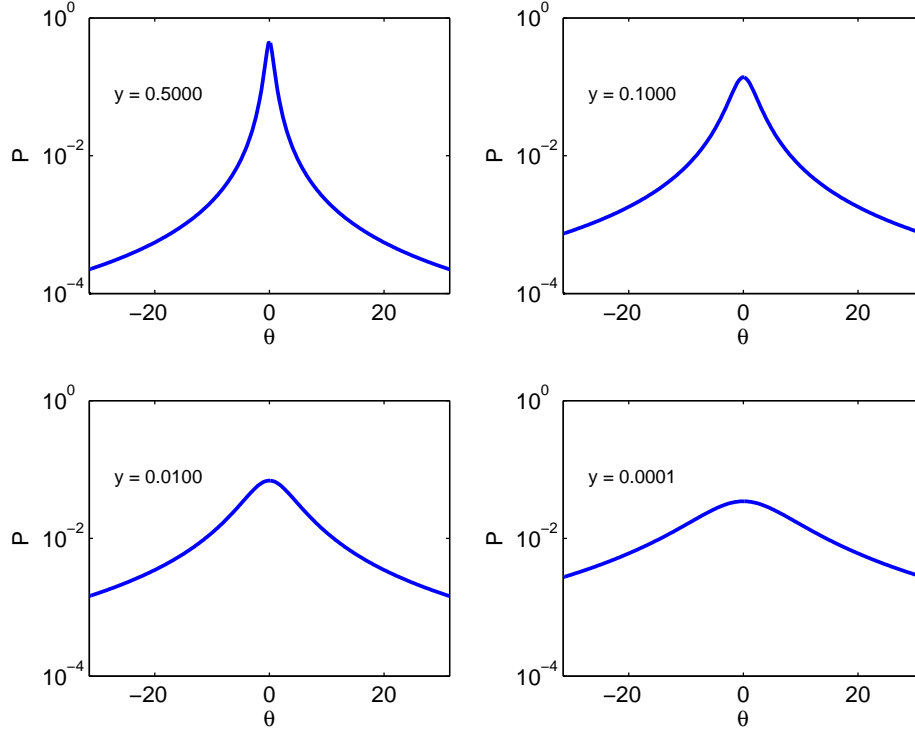
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Figure 3: The Cauchy–Lorentz distribution (11) for various y values.

to evaluate the integral in (7), and find

$$P(x,y,\theta) \simeq -\frac{2}{\pi}\frac{\log(xy)}{\theta^2 + \log^2(xy)}\,\mathrm{e}^{-(x^2+y^2)}. \tag{10}$$

The most interesting thing to us is the total probability of reaching a certain angle θ for any x, so we integrate over x using $\log x \ll \log y$, since y is very small whereas x varies from 0 to ∞. Thus,

$$\int_0^\infty P(x,y,\theta)\,\mathrm{d}x \approx \frac{1}{\pi}\frac{-\log y}{\theta^2 + \log^2 y}, \qquad y = \frac{r_0}{2\sqrt{Dt}} \ll 1, \tag{11}$$

where we also approximated $\mathrm{e}^{-y^2} \simeq 1$, to obtain a consistent normalization in θ.

The probability density (11) is a Cauchy–Lorentz distribution in θ (see Fig. 3). This is the probability of finding the particle at θ at time t, given that it started at $\theta = 0$ and radius r_0. Our taking the limit of infinite wedge angle has solved the problem: we now know how many windings the particle makes around the origin, because it crosses a 'branch cut'
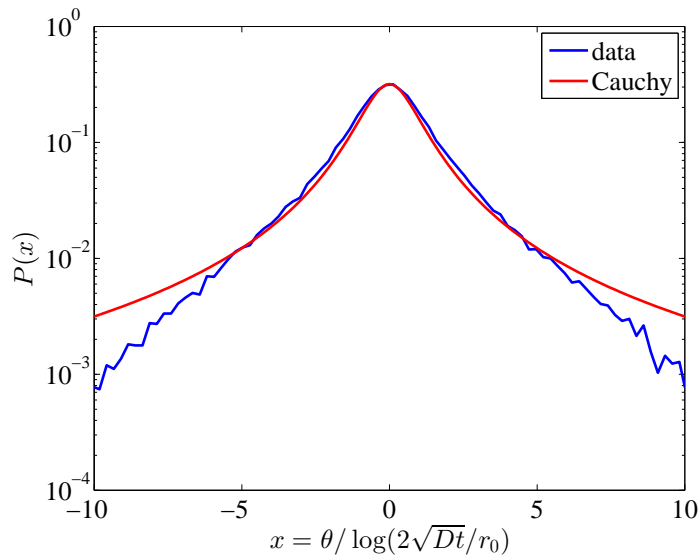
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Figure 4: Cauchy distribution (red) compared to numerical simulations of a Brownian process. We used 10,000 realizations of the Brownian motion, with $Dt/r_0^2 = 10$.

each time it goes around, and ends up on a different Riemann sheet. The distribution is singular at $y = 0$: if the particle starts at the origin, then it's impossible to define its Riemann sheet.

The form (11) compares favorably with numerical simulations (Fig. 4), though there are problems with the tails, as is obvious in Fig. 4. This is due to the scale-free nature of a Brownian process: since the trajectory is rough on all scales, if the particle comes near the origin it can wind an infinite number of times, causing the heavy tails of the Cauchy distribution. But this cannot be simulated on a computer: the steps of the Brownian process are necessarily discrete. The numerical distribution in Fig. 4 is closer to a hyperbolic secant, with exponential tails, which is the distribution predicted for a fixed-stepsize random walk rather than a Brownian process (Bélisle, 1989; Bélisle & Faraway, 1991).

2 Winding around two points

After considering the winding of a Brownian motion around one point in the plane, a natural extension is to consider the winding of a Brownian motion around *two* points, as depicted in Fig. 5(a). The history of a path can be encoded by a symbolic sequence of *generators* a and b, which are defined in Fig. 5(b). The generator a corresponds to looping
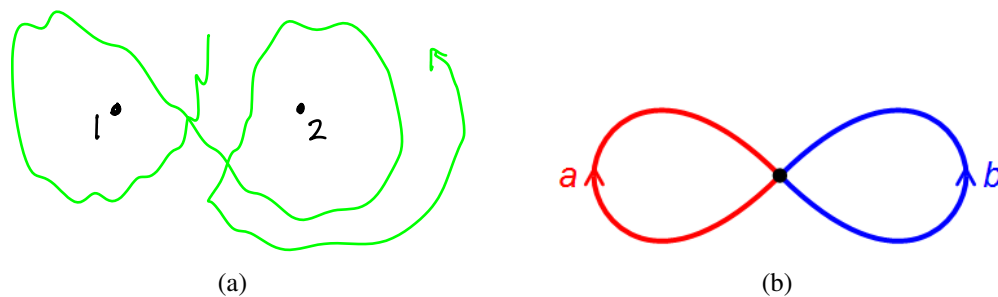
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Figure 5: (a) A Brownian motion winding around two points. (b) The generators a and b. (Source: Wikipedia.)

clockwise around the left point, and b counterclockwise around the right point. The inverse of the generators correspond to circling against the arrow direction. Hence, after a long time, we can write a sequence such as $abba^{-1}bab^{-1}\ldots$ that records the history of the path. (We cancel out any adjacent inverse letters, such as aa^{-1}.)

The set $\{a, b, a^{-1}, b^{-1}\}$ generate the *free group on two generators*, F_2. This group is best represented as a tree, as in Fig. 6. The center is the identity element (the untangled initial state, also called the root of the tree), and then from this initial configuration there are four directions to go. Afterward there are always three directions that take us deeper into the tree, and one which takes us back towards the root. Another standard way of depicting a rooted tree is as in Fig. 7. From the top root node, there are N branches leading down. Afterward, there are $N - 1$ branches leading down from each node. The level of a node is defined as the shortest distance from the root node.

Now consider a random walk in this tree. At any step, the random walker chooses a branch at random, with equal probability among the branches at that node, and jumps along the chosen branch to an adjacent node. Question: what is the probability that the walk ever returns to the root node, as a function of its starting depth?

This is a simple problem in probability. At a given depth, all the nodes are identical, so we only care about the depth of the walk. Thus we write the depth of the walk as a random walk $(X_n)_{n\geq 0}$ on $\{0, 1, 2, \ldots\}$. Here $X_n = i$ means we are at depth i at time n. If the walker is at depth 0, then the *transition probability* of going from depth 0 to depth 1 is

$$p_{0,1} = 1 \tag{12}$$

since the walker has no other place to go! If the walker is at depth $i > 0$, there are $N - 1$
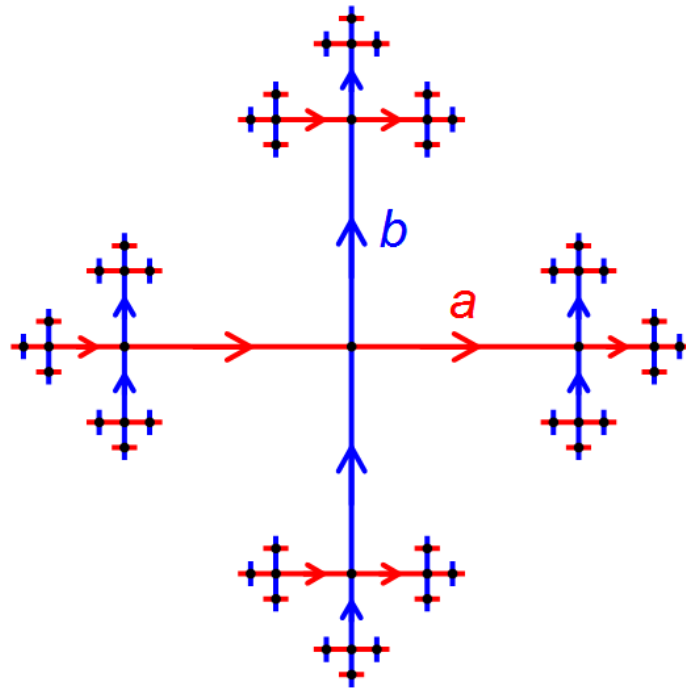
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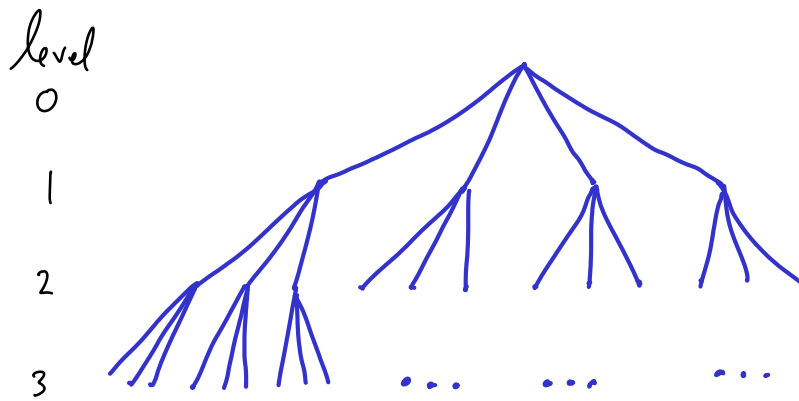Figure 6: Cayley graph for a free group with two generators. (Source: Wikipedia.)

Figure 7: A tree with $N - 1 = 3$ branches from each node, after level 0.

branches leading down and 1 branch leading up, so the transition probabilities are

$$p_{i,i-1} = \frac{1}{N} =: p \qquad \text{(prob. of going up);} \qquad (13a)$$

$$p_{i,i+1} = 1 - \frac{1}{N} =: q \qquad \text{(prob. of going down).} \qquad (13b)$$

Recall that a *conditional probability* is defined as

$$\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B) / \mathbb{P}(B). \qquad (14)$$

This is read as 'the probability of event A occurring given that event B has occurred.' $\mathbb{P}(A \cap B)$ is the probability of both A and B occurring together.

For some integer M, let $T_{0,M}$ be the first time that the walk $(X_n)_{n \geq 0}$ hits 0 or M. For $0 \leq i \leq M$, let

$$f_M(i) = \mathbb{P}(X_{T_{0,M}} = 0 \mid X_0 = i), \qquad (15)$$

that is, the probability that the walk hits 0 first, given that it started at i.

The following relation holds:

$$f_M(i) = p_{i,i-1}\, f_M(i-1) + p_{i,i+1}\, f_M(i+1). \qquad (16)$$

In words: if we start from i, then two things can happen: we either move to $i-1$ (with probability $p_{i,i-1}$) or to $i+1$ (with probability $p_{i,i+1}$). Afterward, though, our probability of hitting 0 first must be dictated by $f_M(i-1)$ or $f_M(i+1)$, respectively. The relation (16) is valid for $1 \leq i \leq M-1$, since we know that

$$f_M(0) = 1, \qquad \text{(we hit 0 immediately);} \qquad (17a)$$
$$f_M(M) = 0, \qquad \text{(oops\ldots we've hit M first).} \qquad (17b)$$

Let's rewrite (16) as

$$p\, f_M(i-1) - f_M(i) + q\, f_M(i+1) = 0. \qquad (18)$$

Take the case $N = 2$, in which case our tree has only two branches, so it looks like the integer line $\mathbb{Z}$, with the left branch giving negative numbers! Then $p = q = 1/2$, and (18) becomes

$$f_M(i-1) - 2f_M(i) + f_M(i+1) = 0 \qquad (19)$$

after multiplying by 2. Anybody who's ever done even a little numerics involving finite differences will recognize (19) as the one-dimensional discretization of the Laplacian, with

unit grid size! Functions whose Laplacian is zero are called *harmonic*. Thus, equation (18) says that the function $f_M(i)$ is harmonic with respect to the *graph Laplacian* defined on our tree. The equations (17) act as boundary conditions.

Let us solve the recurrence relation (18) with boundary conditions (17). Much as for ODEs, this is often done by guessing the correct form. Put

$$f_M(i) = \lambda^i \tag{20}$$

and insert in (18) to obtain

$$p\,\lambda^{i-1} - \lambda^i + q\,\lambda^{i+1} = 0. \tag{21}$$

Multiplying by λ^{1-i}, we have

$$p - \lambda + q\,\lambda^2 = 0, \tag{22}$$

which means (since $p + q = 1$) that $\lambda = 1$ or $\lambda = p/q$. Thus, the general solution is

$$f_M(i) = A_M + B_M\,(p/q)^i. \tag{23}$$

This holds for $p \neq q$, so not for $N = 2$. For $N > 2$ we have $p < q$.

We apply the boundary conditions (17) to fix the constants:

$$A_M + B_M = 1, \qquad A_M + B_M(p/q)^M = 0, \tag{24}$$

so that

$$A_M = -\frac{(p/q)^M}{1 - (p/q)^M}, \qquad B_M = \frac{1}{1 - (p/q)^M}. \tag{25}$$

Finally, for $p < q$ ($N > 2$) we take the limit of M going to infinity:

$$\lim_{M \to \infty} A_M = 0, \qquad \lim_{M \to \infty} B_M = 1. \tag{26}$$

We are left with

$$\boxed{f_\infty(i) = (p/q)^i.} \tag{27}$$

A walk starting at 0 has $f_\infty(i) = 1$, since it is already at 0. It must jump to level 1, so the probability of returning to zero once, starting from 0, is $f_\infty(1) = (p/q) < 1$.

What does this mean? It gets exponentially more difficult to return to 0 as the depth i is increased. This is important for our entanglement picture: recall that 0 stands for the 'disentangled' state. It can be shown that the form (27) implies that *the walk will almost surely return to zero only a finite number of times*. This is plausible: a walk starting at 0 has a fair chance (p/q) or returning to 0, but over time it becomes geometrically less and less likely that it will do so over and over gain. Hence, the ultimate state is almost surely escape to ∞, or complete entanglement.

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Note that I am not claiming that the random walk on the tree we just described is an accurate representation of the winding process in Fig. 5(a). There are many reasons for this: for one, the process in Fig. 5(a) will likely lead to correlations between successive steps in the tree. It will also have a variable time between steps whose distribution would have to be taken into account. The branching process, as expressed by the tree structure, is a cartoon that explains why spontaneous entangling is much easier than spontaneous detangling, but the details should not be taken too seriously. For a more detailed treatment see the book by Nechaev (1996).

Acknowledgments

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Side note: Lazy random walk

In the model presented earlier the walker must jump. But we can include a nonzero probability that the walker stay put. Such a walker is often called 'lazy.' (Lazy random walks are used to destroy some parity properties: for instance, a random walk on the integers always alternates between even and odd integers.)

If the walker is allowed to stay in place, then we must include this in the recurrence relation (16):

$$f_M(i) = p_{i,i-1}\, f_M(i-1) + p_{i,i+1}\, f_M(i+1) + p_{i,i}\, f_M(i) \tag{28}$$

or

$$p\, f_M(i-1) - (1-r) f_M(i) + q\, f_M(i+1) = 0 \tag{29}$$

with $p + q + r = 1$. The characteristic equation is now

$$p - (1-r)\lambda + q\, \lambda^2 = 0. \tag{30}$$

but the two solutions are again $\lambda = 1$ and $\lambda = p/q$! This only depends on r through the fact that $p + q + r = 1$.