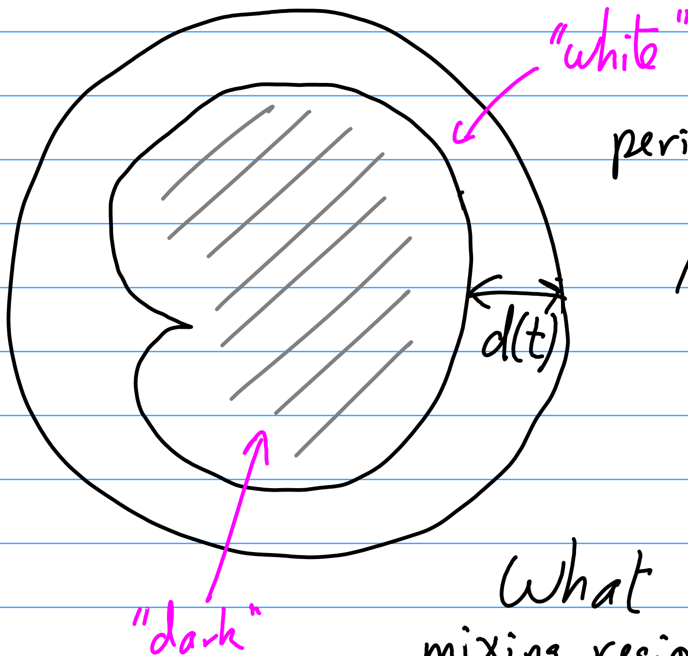
Mixing hits a wall

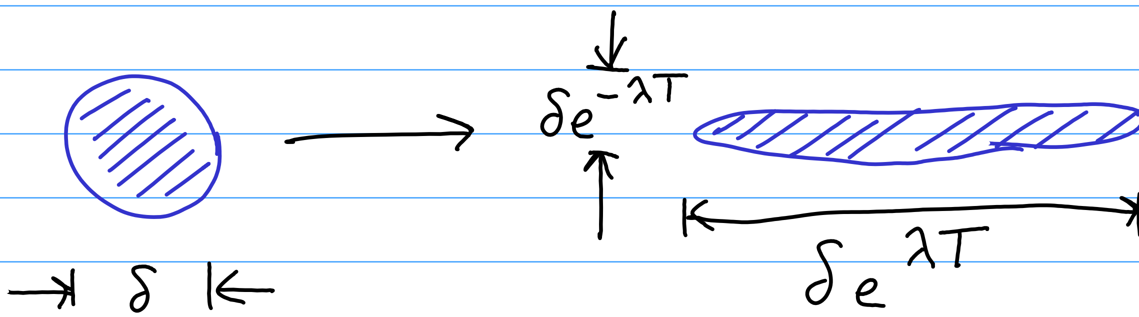
GFD, 6/30/2008



The mixing pattern is caused by some periodic stirring mechanism.

At every period T, some white fluid is shaved from the edge, decreasing the distance $d(t)$.

What happens inside the mixing region? Assume a very simple chaotic mixer, meaning that fluid elements are stretched at rate λ: (on average)



The maximum width a dye filament can have is given by the Batchelor length,

$$\ell = \sqrt{\frac{\kappa}{\lambda}}$$

κ = molecular diffusivity

(Balance between diffusion and strain)

The width of a white filament injected at time t is

$$\Delta(t) = d(t) - d(t+T) \simeq -T\dot{d}(t)$$

since d changes little at each period.

Now, if a white filament is injected at time $\tau < t$, how long does it last?

$$\underbrace{\Delta(\tau)}_{\substack{\text{initial}\\\text{width}}} \underbrace{e^{-\lambda \overbrace{(t-\tau)}^{\text{"age" of filament}}}}_{\substack{\text{compression}\\ t>\tau}} = \underbrace{\ell}_{\text{Batchelor length}}$$

In other words, solving for $\tau(t)$ in the above gives us the injection time of the oldest filaments still visible at time t. Filaments injected before $\tau(t)$ are below the Batchelor length, so they are no longer white.

At some point the injected filament width will equal the Batchelor length:

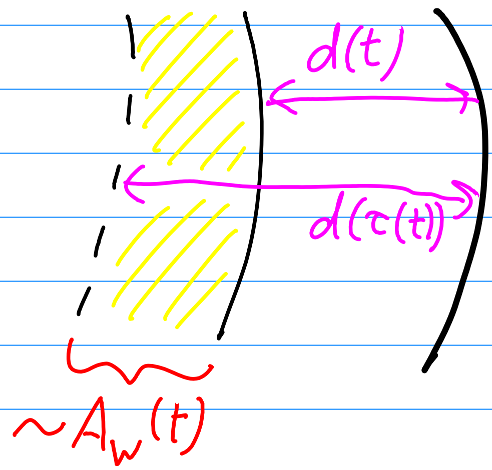
$$\Delta(t_B) = \ell$$

so that $\tau(t_B) = t_B$.

It makes no sense to speak of those filaments as "white" since they are dominated by diffusion. Hence, assume $t < t_B$.

Now, in the experiments we measure the number of "white pixels" in the central mixing region. This is proportional to the area of white material injected,

$$A_W(t) \sim d(\tau(t)) - d(t)$$



We've gone as far as we can without introducing some form for $d(t)$.

First assume $d(t) = d_0 e^{-\mu t}$

$$\Delta(t) = -T\dot{d} = d_0 \mu T e^{-\mu t} = \Delta_0 e^{-\mu t}$$

$$\Delta(t_B) = \ell \Rightarrow t_B = \frac{1}{\mu} \log\left(\frac{d_0 \mu T}{\ell}\right)$$

$(\underbrace{d_0 \mu T}_{\Delta_0} > \ell)$

Solve for $\tau(t)$:

$$\Delta_0 e^{-\mu \tau} e^{-\lambda(t-\tau)} = \ell$$

$$e^{(\lambda-\mu)\tau} = (\ell/\Delta_0) e^{\lambda t}$$

$$\tau = \frac{1}{\lambda-\mu}\left[\log(\ell/\Delta_0) + \lambda t\right]$$

$$= \frac{1}{\lambda-\mu}\left(-\mu t_B + \lambda t\right)$$

$$\boxed{\tau(t) = \frac{\lambda t - \mu t_B}{\lambda - \mu}}$$

$$\tau - t = \frac{\cancel{\lambda t} - \mu t_B - (\cancel{\lambda} - \mu)t}{\lambda - \mu} = \frac{\mu (t - t_B)}{\lambda - \mu}$$

Now, by assumption: $\tau < t < t_B$:

$$\underbrace{t - \tau}_{>0} = \frac{\mu}{\lambda - \mu} \underbrace{(t_B - t)}_{>0}$$

Hence, for consistency we require $\underline{\underline{\mu < \lambda}}$.

i.e., the rate of approach to the wall is slower than the rate of stretching in the mixing region. Otherwise the wall has no effect: the wall region is depleted faster than filaments can reach the Batchelor scale.

If $\mu < \lambda$, we have

$$A_W(t) \sim d(\tau(t)) - d(t)$$

$$\sim d_0 \left(e^{-\mu\tau} - e^{-\mu t} \right)$$

$$\sim d_0 e^{-\mu t} \left(e^{(t-\tau)\mu} - 1 \right)$$

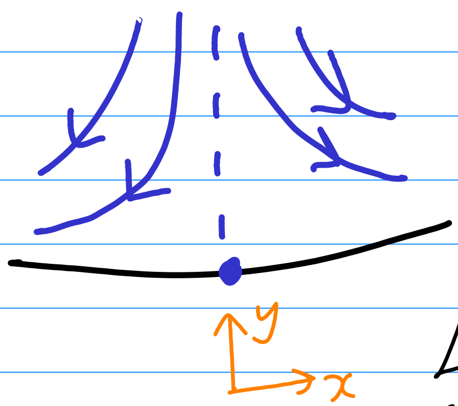
$$\sim d_0 e^{-\mu t} \left(\exp\left(\frac{\mu^2}{\lambda - \mu} (t_B - t) \right) - 1 \right)$$

We see that for $t \ll t_B$, $\boxed{A_W(t) \sim e^{-\mu t}.}$

So the decay rate of the "white" area is completely dominated by the walls. The central mixing process is efficient ($\lambda > \mu$), but it is "starved" by the boundary.

So what happens in practice?

The "figure-8" protocol shown has a reattachment point:



Let's model it as a steady flow (we can take the time-dependence into account by using a map rather than a flow).

Let x be the "along wall" coordinate.

Taylor in y:

$$u(x,y) = \underbrace{\cancel{u_0(x)}}_{\text{no slip}} + u_1(x)y + \cdots$$

$$v(x,y) = \underbrace{\cancel{v_0(x)}}_{\text{no throughflow}} + v_1(x)y + v_2(x)y^2 + \cdots$$

Now, $\nabla \cdot \underline{u} = 0 \Rightarrow u_1'(x)y + v_1(x) + v_2(x)2y$

$$= 0$$

Equating terms, we find $v_1(x) = 0$, $v_2(x) = -\frac{1}{2}u_1'(x)$.

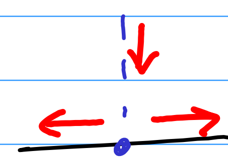
Letting $u_1(x) = A(x)$, get

$$u(x,y) = A(x)y + O(y^2)$$

$$v(x,y) = -\frac{1}{2}A'(x)y^2 + O(y^3)$$

This is a "boundary layer" form.

A reattachment point corresponds to $A(x)$ changing sign, say at $x=0$, with $A'(0) > 0$.



Hence, along the separatrix ¦, we can solve for the motion of a fluid particle,

$$\dot{y} = v(0,y) = -\frac{1}{2}A'(0)y^2$$

This has solution

$$\boxed{y(t) = \frac{y_0}{1 + A'(0)t\,y_0}} \sim \frac{1}{A'(0)t}, \quad t \text{ large.}$$

Note that asymptotically a particle "forgets" its initial condition y_0. This explains why material lines "bunch up" against each other faster than they approach the wall.

There is a visible "front".

Now we have our asymptotic form for $d(t)$:

$$d(t) \sim \frac{1}{A'(0)\,t}, \qquad t \gg 1.$$

Hence, $\Delta(t) = -T\dot{d}(t) \sim \dfrac{T}{A'(0)\,t^2}$.

We have $A_W(t) = d(\tau(t)) - d(t)$

with $\Delta(\tau)\, e^{-\lambda(t-\tau)} = \ell$.

$$A_W = \frac{1}{A'(0)\tau} - \frac{1}{A'(0)t} = \frac{1}{A'(0)} \frac{(t-\tau)}{\tau t}$$

But also, $\log \Delta - \lambda(t-\tau) = \log \ell$

or $$t - \tau = \frac{1}{\lambda} \log\left(\Delta(\tau)/\ell\right)$$

$\Delta(\tau)$ is algebraic, so for large time the RHS is not large. Hence, for large time we must have

$$t/\tau \simeq 1, \qquad t \gg 1.$$

We conclude: $$\boxed{A_W \sim \frac{1}{A'(0)} \frac{\log\left(T/A'(0)\ell t^2\right)}{\lambda t^2}}$$

$$t \gg 1 \qquad (\text{but } t < t_B)$$

This power-law decay is consistent with the data.

In a simple "parabolic baker's map" model, reproduces data extremely well for long times.

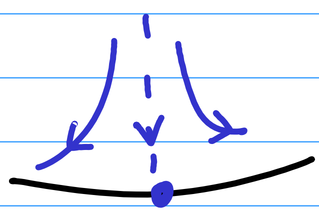
In experiments, determining λ is the most difficult part.

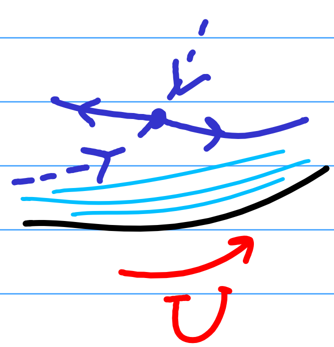
To improve the mixing rate, spin the wall!

$$u(x,y) = U + A(x)y + O(y^2)$$

$$v(x,y) = -\frac{1}{2}A'(x)y^2 + O(y^3)$$

The moving wall destroys the reattachment point and replaces it with a hyperbolic fixed point





The price to pay is that there is now an unmixed layer near the wall.

[There are other means of destroying the reattachment points.]