Mixing with Ghost Rods

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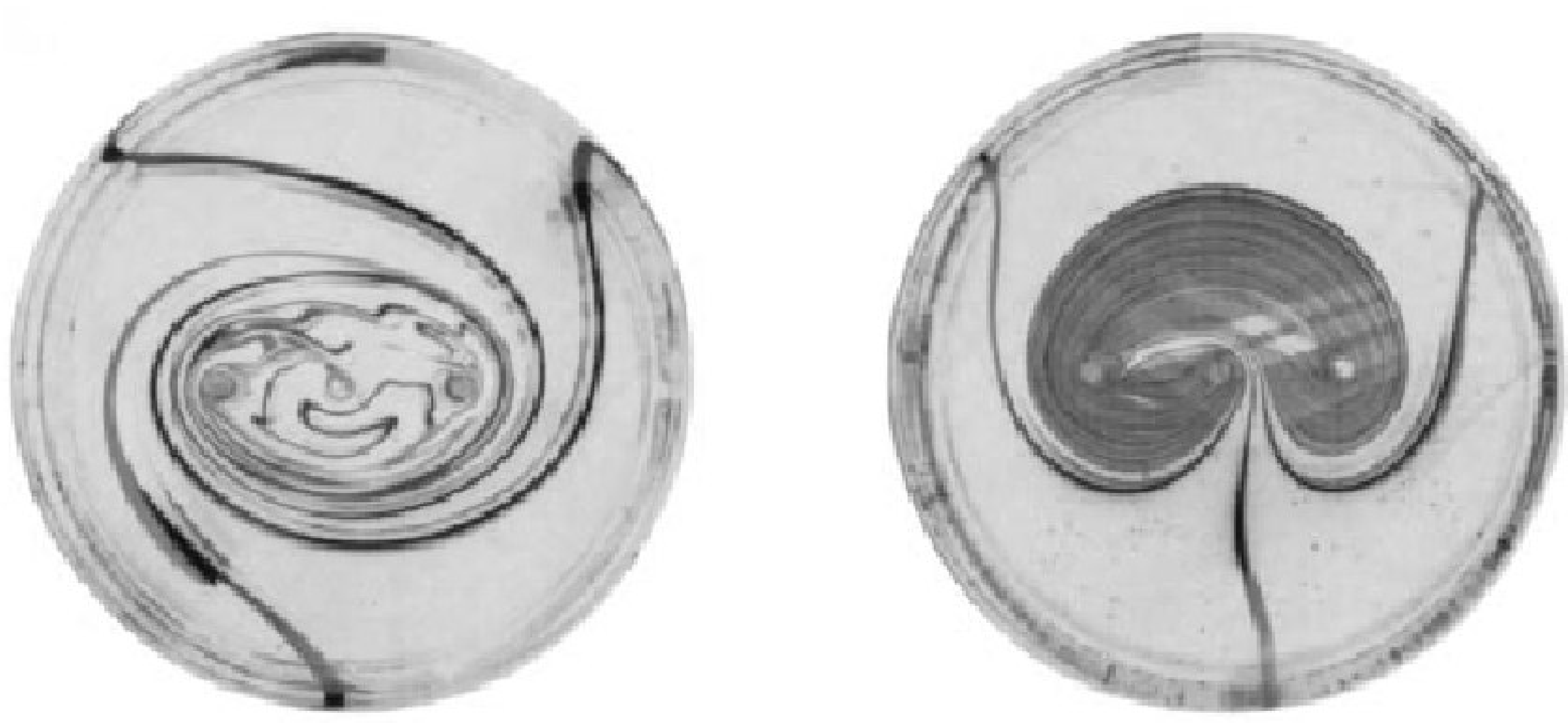
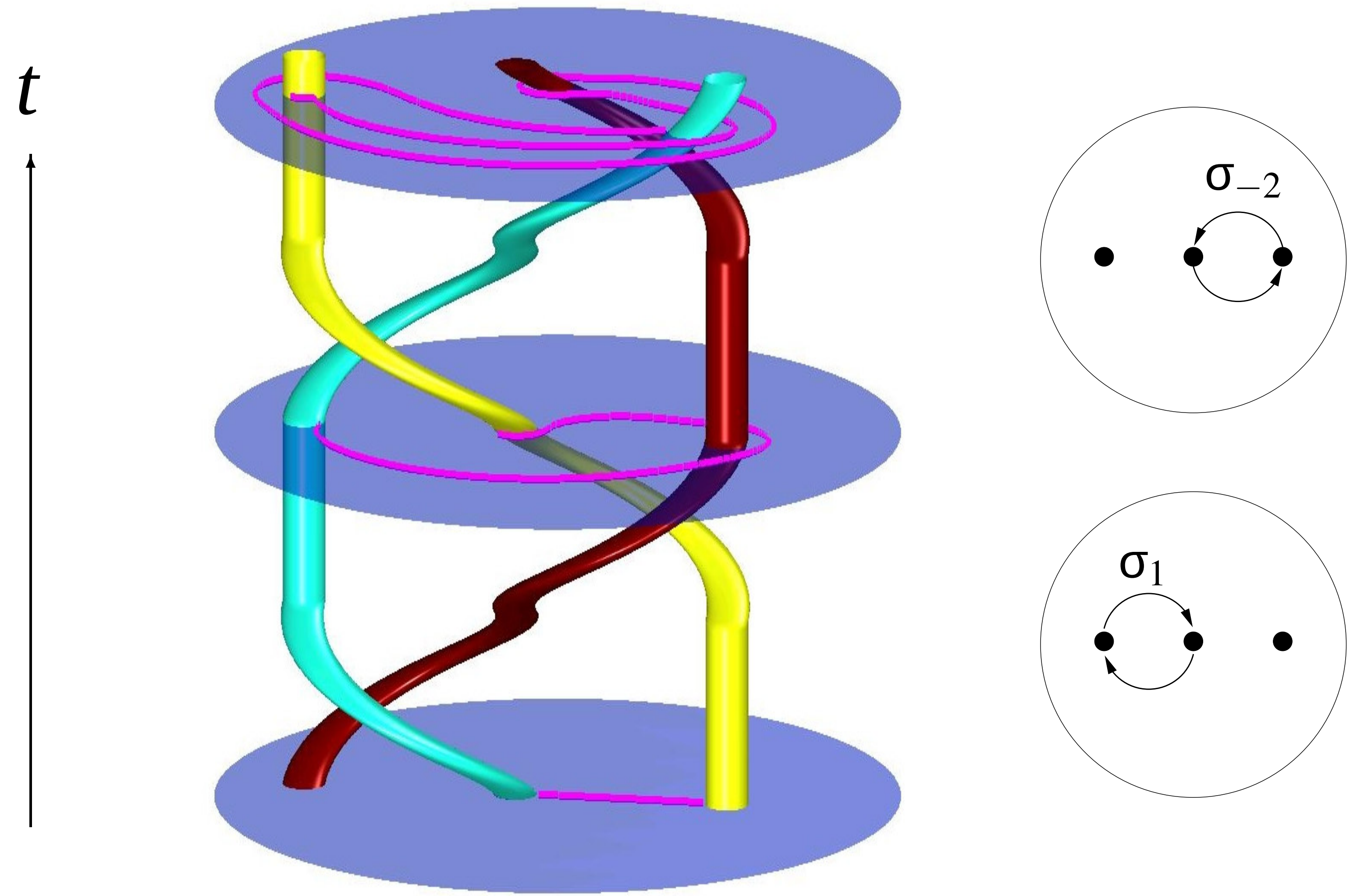
Experiment of Boyland, Aref, & Stremler



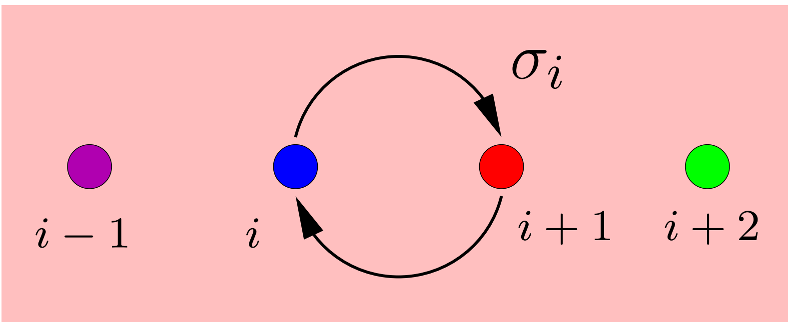
[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

[P. L. Boyland, M. A. Stremler, and H. Aref, *Physica D* **175**, 69 (2003)]

The Connection with Braiding



Generators of the n-Braid Group

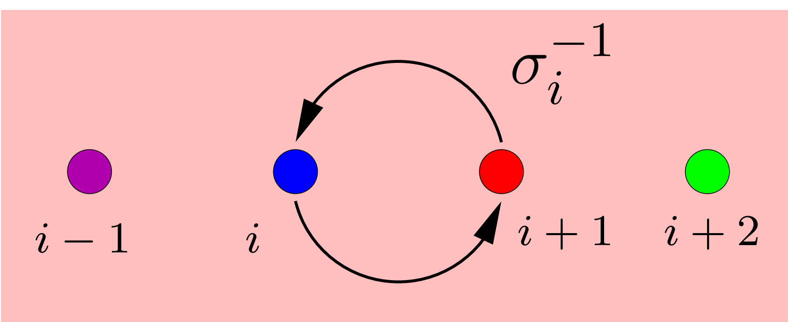


A generator

$$\sigma_i\,, \quad i = 1, \ldots, n-1$$

is the clockwise interchange of the i th and $(i+1)$th rod.

The generators obey the presentation



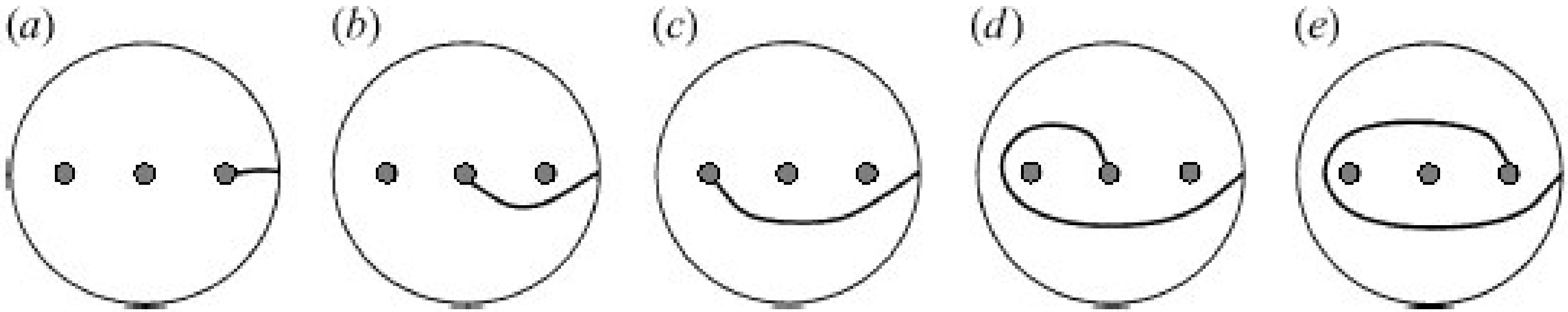
$$\sigma_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\sigma_i$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| > 1$$

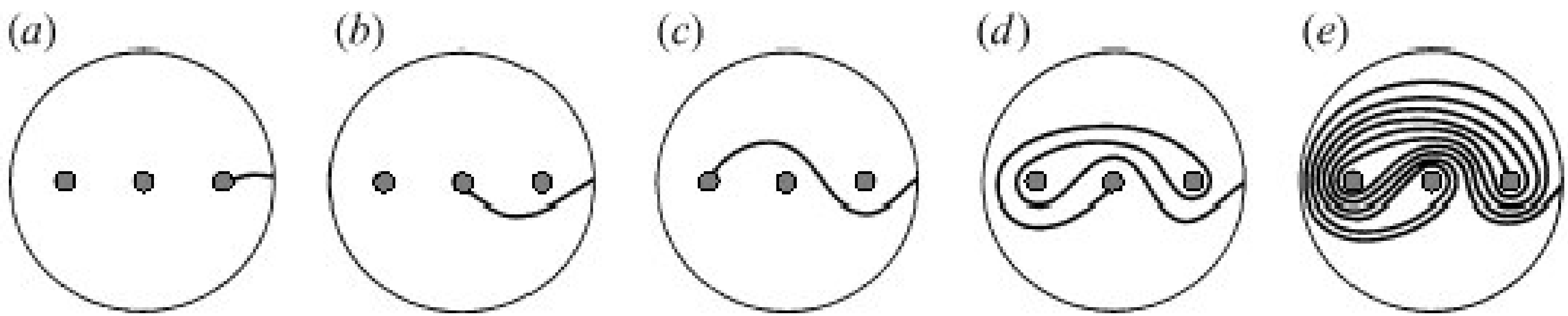
These generators are used to characterise the motion of the rods.

The Two BAS Stirring Protocols

$\sigma_1\sigma_2$ protocol



$\sigma_1^{-1}\sigma_2$ protocol

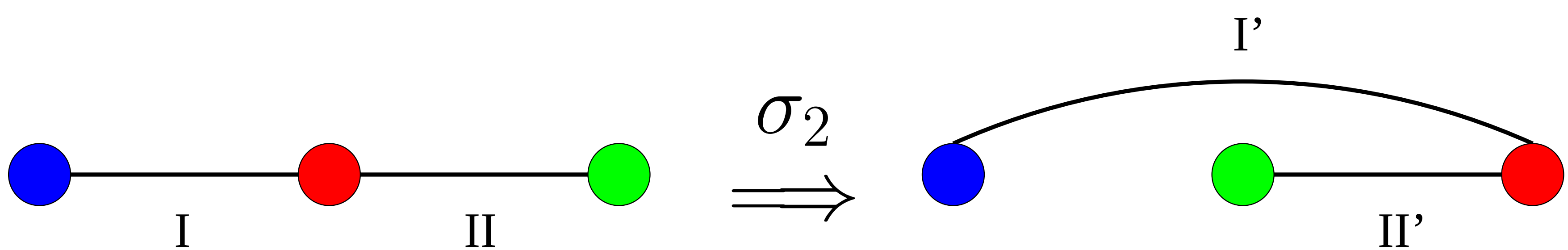


[P. L. Boyland, H. Aref, and M. A. Stremler, *J. Fluid Mech.* **403**, 277 (2000)]

Topological Entropy of a Braid

Practically speaking, the topological entropy of a braid is a lower bound on the line-stretching exponent of the flow!

This is reasonable:



In practice, we compute the topological entropy of a braid using a train-track algorithm due to Bestvina & Handel. The end result is a transition matrix showing the how each edge is mapped under the action of the braid.

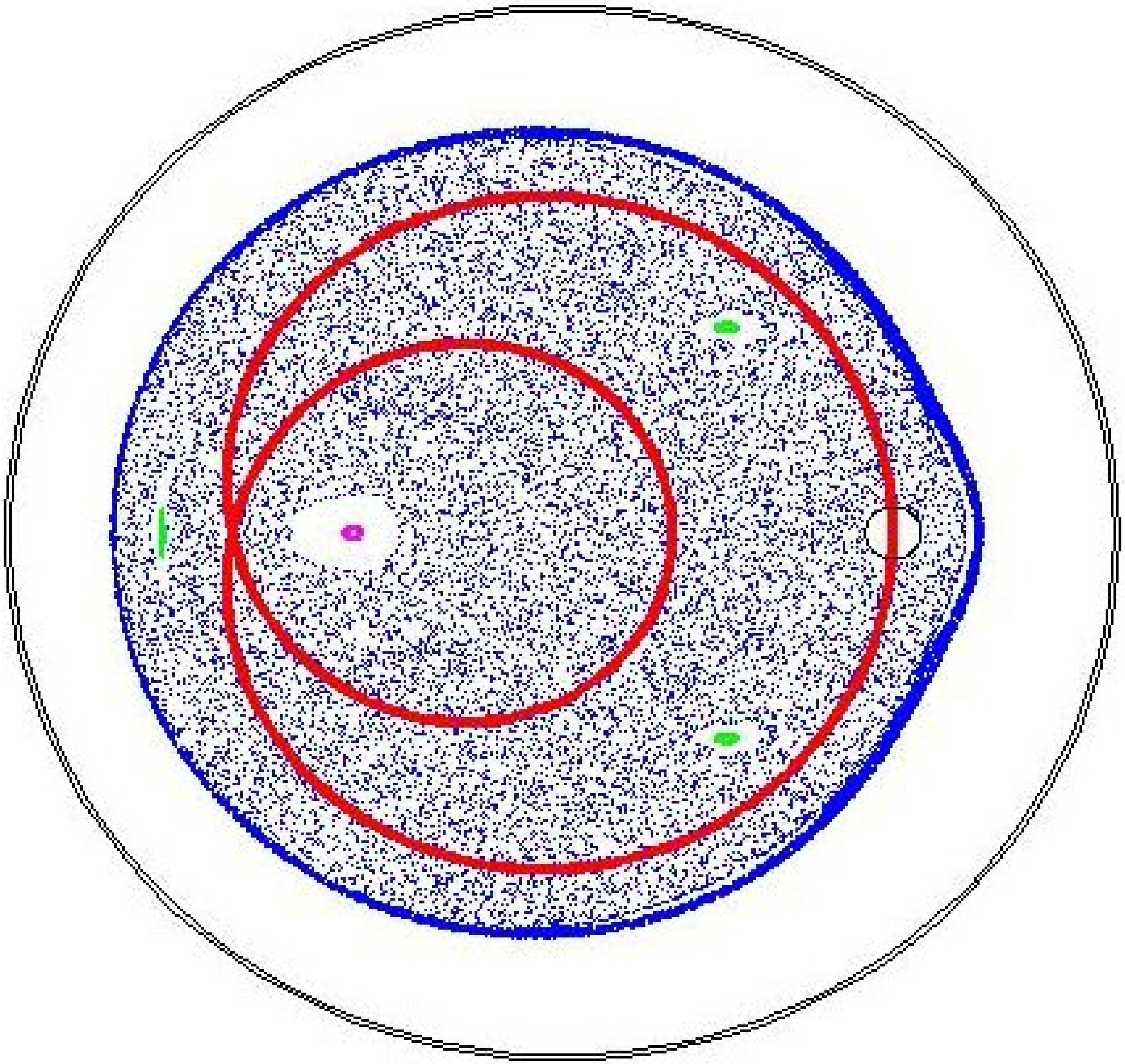
[M. Bestvina and M. Handel, *Topology* **34**, 109 (1995)]

The Difference between BAS's Two Protocols

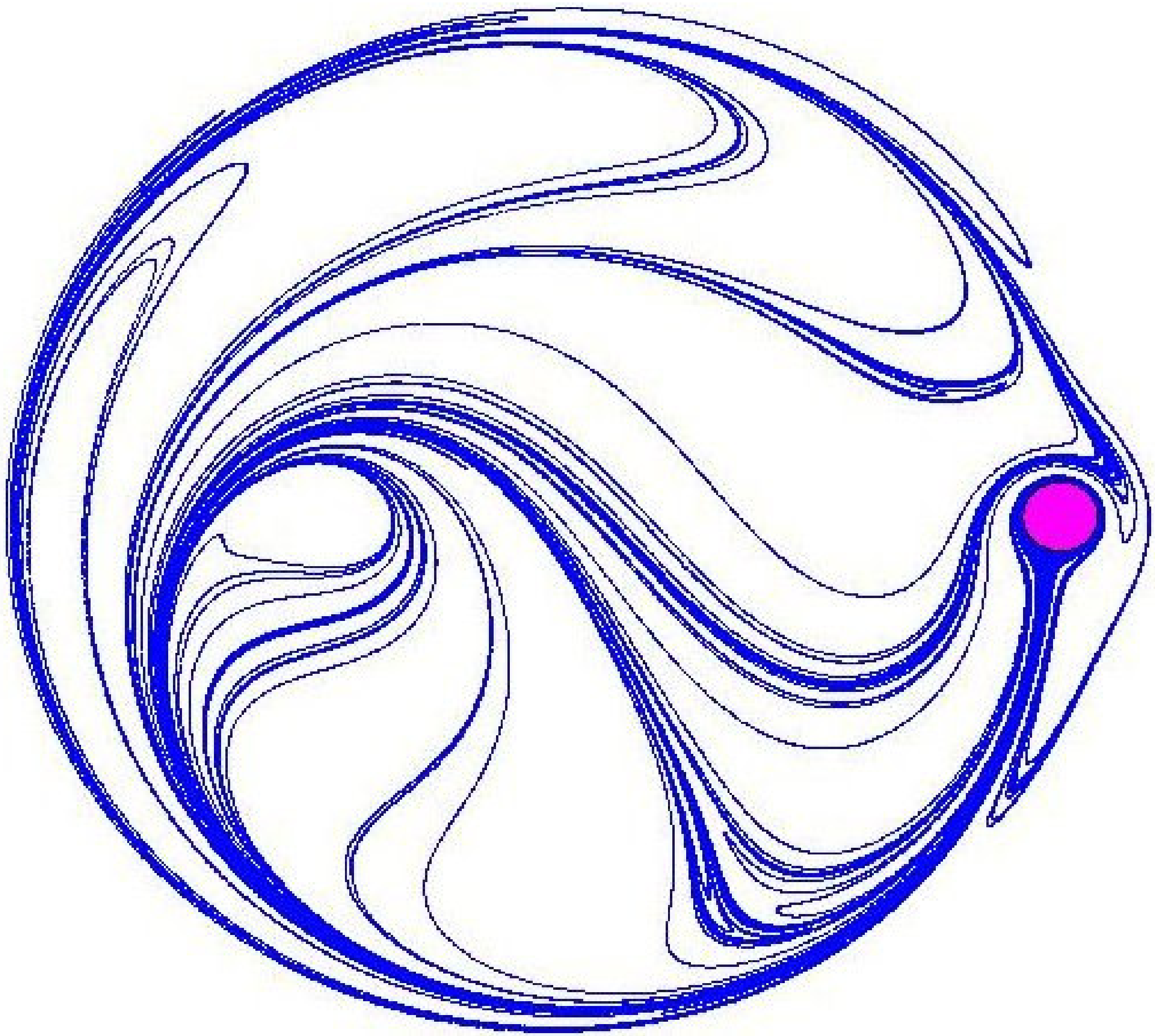
- The matrices associated the generators have eigenvalues on the unit-circle (but their product doesn't necessarily).
- The first (bad) stirring protocol has eigenvalues on the unit circle
- The second (good) protocol has largest eigenvalue $(3+\sqrt{5})/2 = 2.6180$.
- So for the second protocol the length of a line joining the rods grows exponentially!
- That is, material lines have to stretch by at least a factor of 2.6180 each time we execute the protocol $\sigma_1^{-1}\sigma_2$.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).

One Rod Mixer: The Kenwood Chef

Poincaré Section

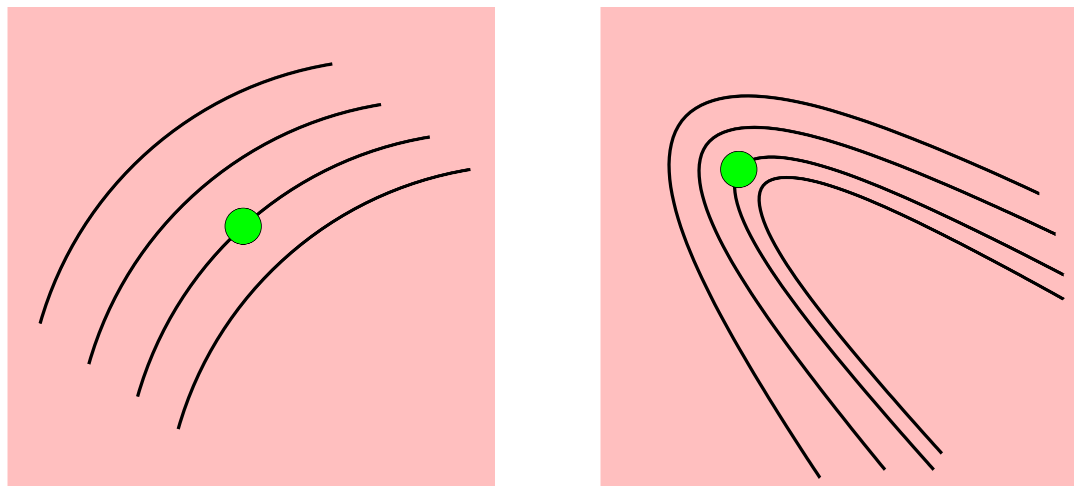


Stretching of Lines



Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).



Material lines must bend around the orbit: it acts just like a “rod”!

[J-LT, *Phys. Rev. Lett.* **94**, 084502 (2005)]

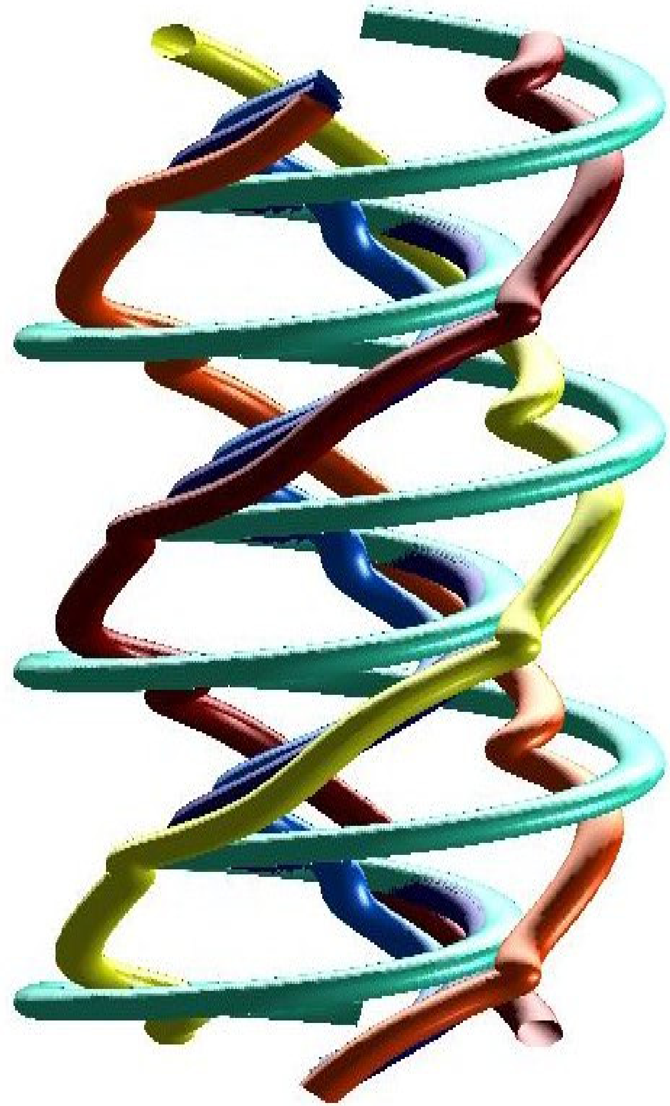
Today: focus on periodic orbits.

How do they braid around each other?

Motion of Islands

Make a braid from the motion of the rod and the periodic islands.

Most (74%) of the topological entropy is accounted for by this braid.



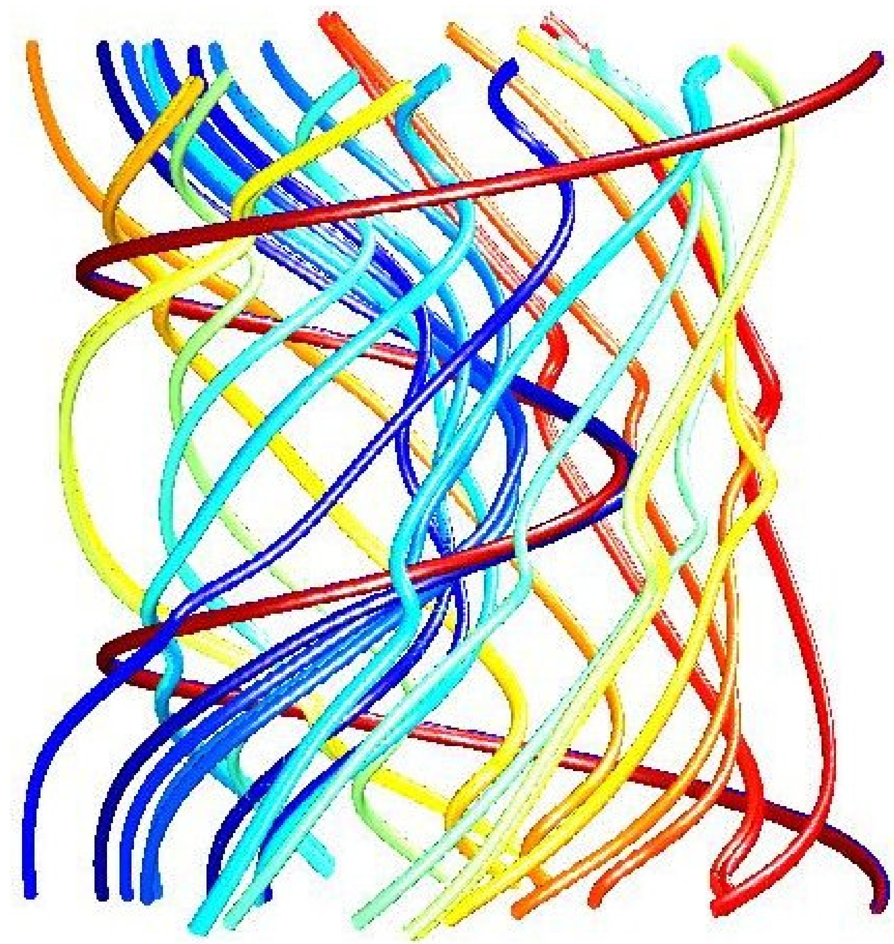
Motion of Islands and Unstable Periodic Orbits

Now we also include unstable periods orbits as well as the stable ones (islands).

Almost all (99%) of the topological entropy is accounted for by this braid.

But are the periodic orbits really “ghost rods”?

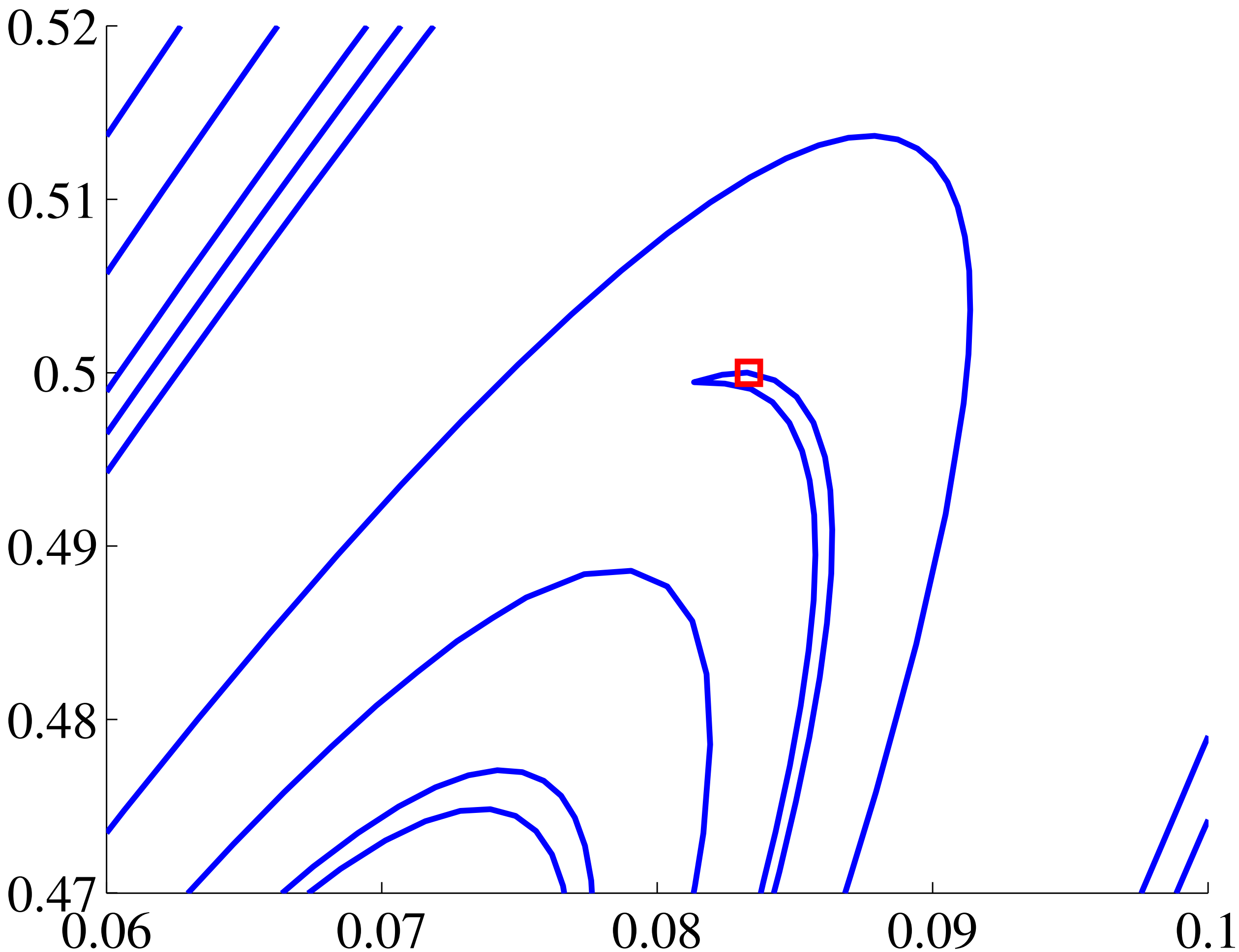
That is, do material lines really get out of their way?



Periodic Orbits as Ghost Rods

[movie: sf_periodic.avi]

Blowup of the Tip



So Which Orbits Make Good Rods?

A good ghost rod should "look" like a real rod: material lines wrap around it.

Look at linearisation of the period-1 map:

Two eigenvalues Λ and Λ^{-1} (Floquet multipliers), with $|\Lambda| \geq |\Lambda|^{-1}$.

Λ complex	Elliptic	Good rod (obvious).
$\|\Lambda\| > 1$	Hyperbolic	Crap rod.
$\Lambda = 1$	Parabolic	Good rod?

The parabolic points are the most interesting: they are associated with the crucial property of folding.

Normal Form near a Parabolic Point

$$X' = X + \text{quadratic terms}$$
$$Y' = Y + \text{quadratic terms}$$

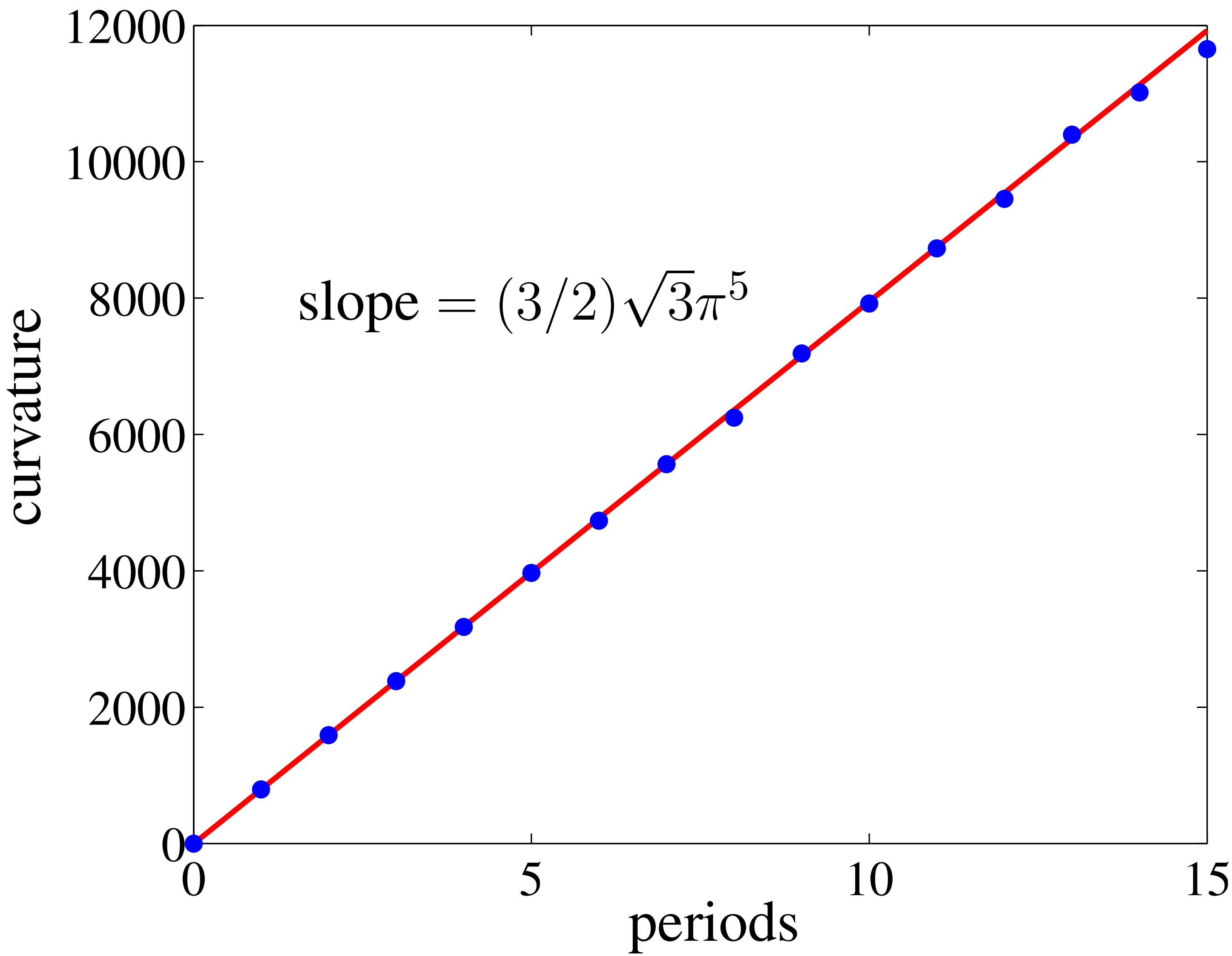
or

$$X' = X + Y + X^2$$
$$Y' = Y + X^2$$

The first of these does not lead to folding.

The coefficient of X^2 determines the “size” of the rod, which is a function of time.

Curvature of the Tip at the Periodic Orbit



Conclusions

- Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
- Periodic orbits make perfect obstacles (in periodic flows).
- This is a good way to “explain” the chaos in a flow — accounts for stretching of material lines.
- Islands (elliptic orbits) look just like rods, and parabolic orbits can look a lot like rods.
- No need for infinitesimal separation of trajectories or derivatives of the velocity field — this is an inherently global description.
- The size of the rods is important — for islands this is obvious but for parabolic points the apparent size is a function of time.