Curvature in Chaotic and Turbulent Flows

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Material Lines in Flows

How do material lines embedded in a chaotic flow evolve?

$\Rightarrow$ Stretch, Twist, Fold

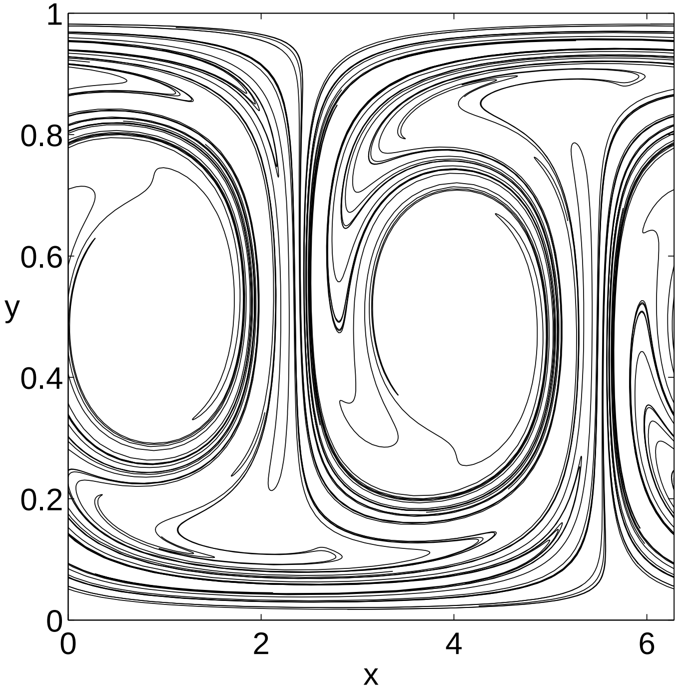
Relevance:

- Magnetic dynamo: evolution of magnetic field in a plasma.
- Chemical and biological mixing: creation of intermaterial contact area.
- Polymer mixing (*i.e.*, DNA): follow material lines closely.
- Much is known about stretching, but less about the bending of material lines (generation of curvature and torsion).

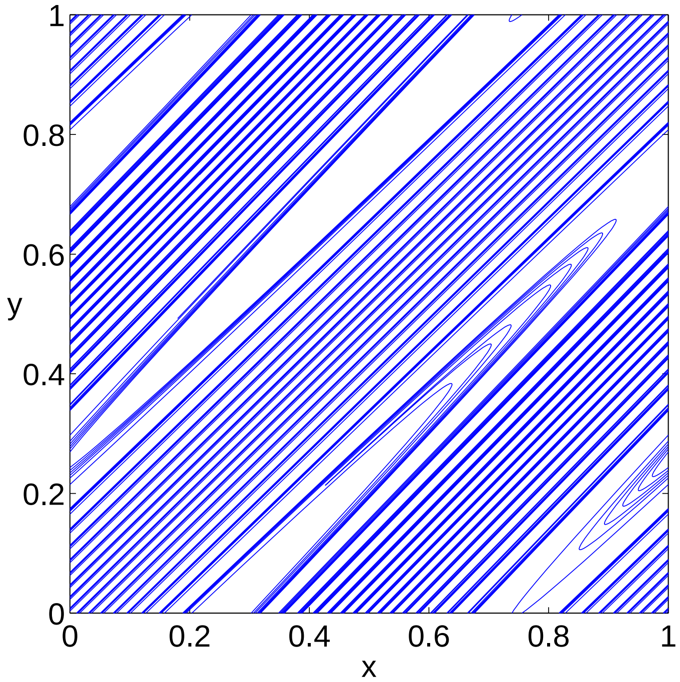
Some interesting regularities, such as a close anticorrelation between stretching and curvature.

Stretching and Folding

Traces out the unstable foliation of the flow.
Note the sharp folds that develop.



Cellular Flow

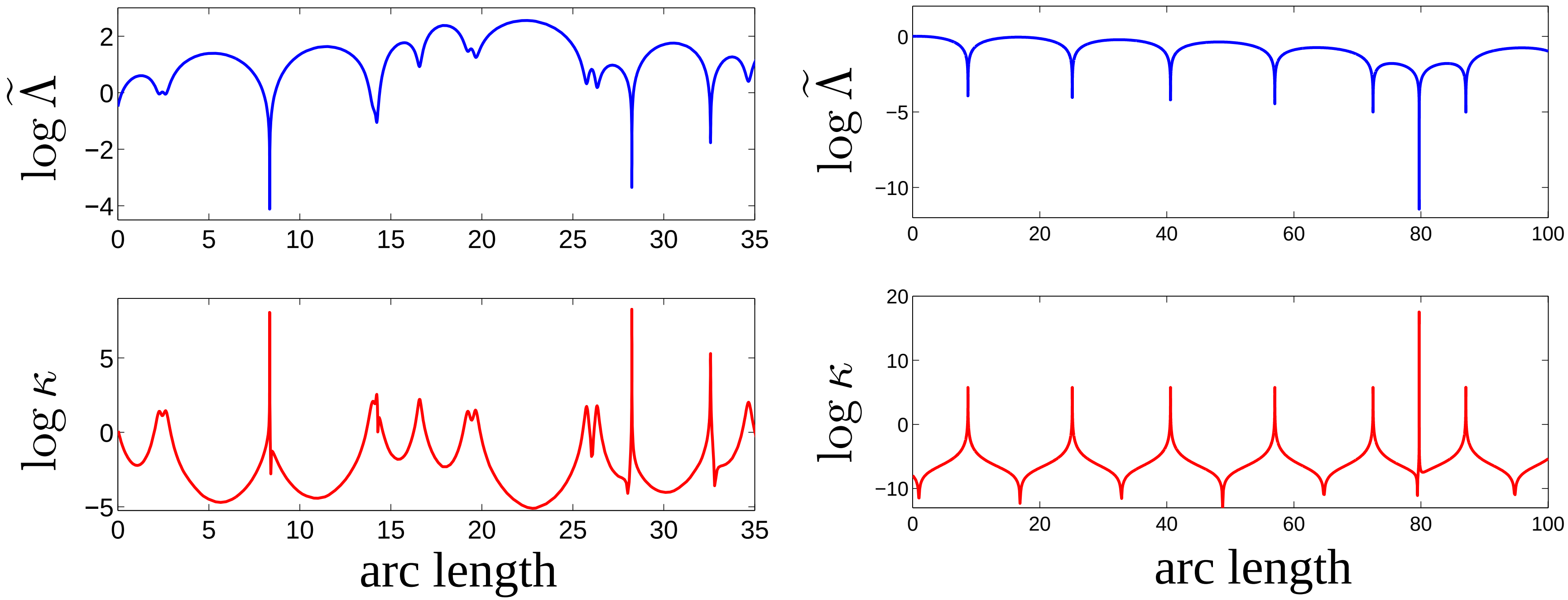


Standard Map

Can look surprisingly regular even in extremely chaotic cases.

Stretching along a Material Line

$\widetilde{\Lambda}$ is the deviation from mean stretching.



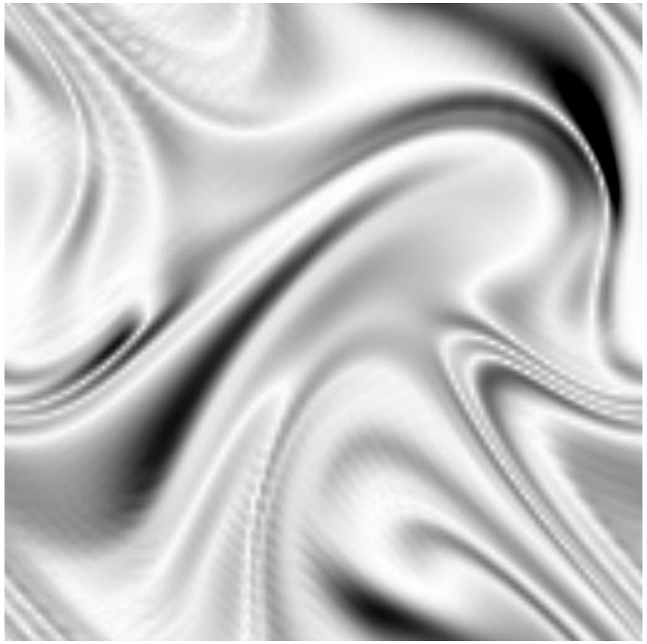
Cellular Flow

Standard Map

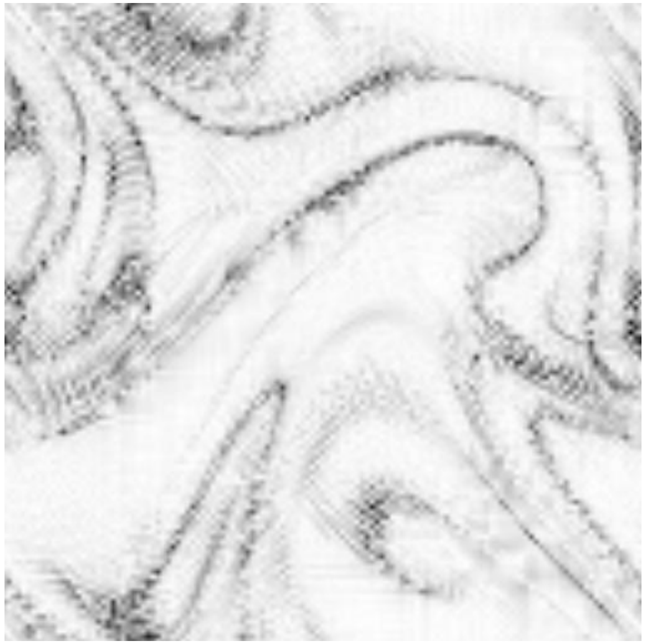
$\Rightarrow$ Suppression of stretching. [Drummond & Münch (1991)]

Stretching and Curvature: the Dynamo

A similar effect was recently observed for the magnetic dynamo.



Magnetic field, B



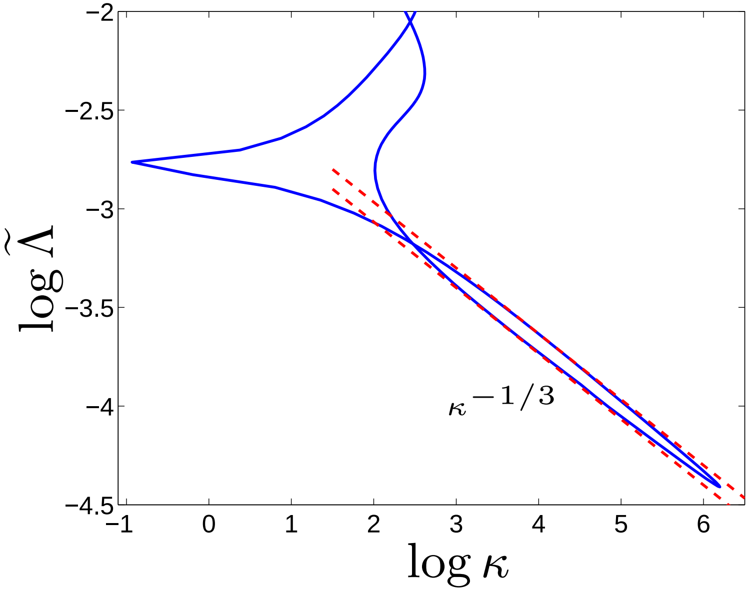
Curvature of B, κ

The magnetic field and its curvature are anticorrelated

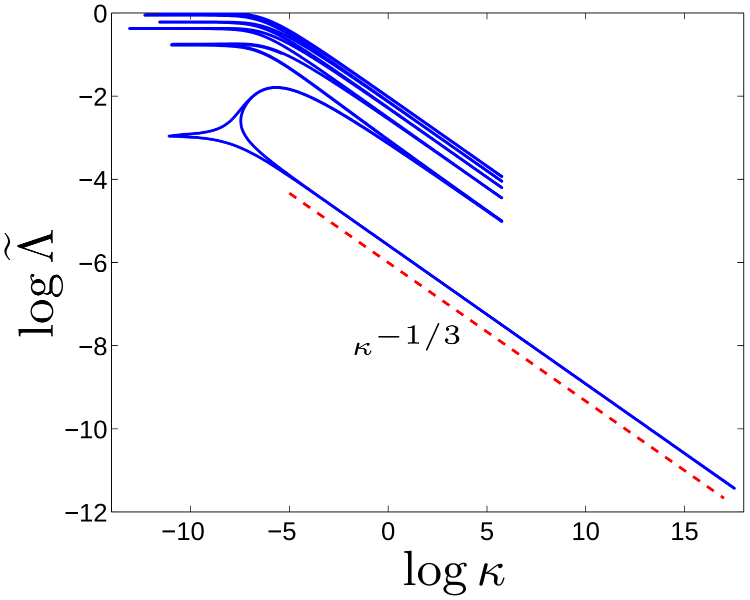
[Schekochihin, Cowley, Maron & Malyshkin, Phys. Rev. E (2002)]

Stretching vs Curvature along a Material Line

Power law relation around sharp folds: The "$-1/3$" law.



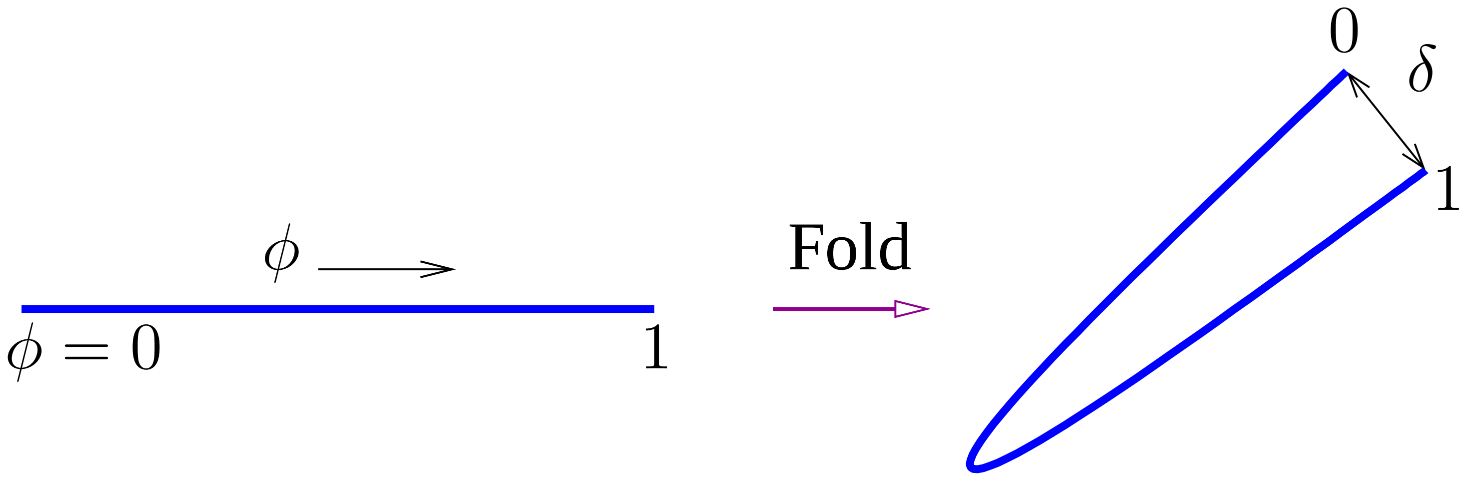
Cellular Flow



Standard Map

The law is very robust even with varying degree of chaos and different flows (2D and 3D).

Enhancement to Gradients by Folding



- Assume linear gradient of ϕ varying from 0 to 1;
- The endpoints of the line are brought to a distance δ;
- Enhancement in $\nabla\phi$ proportional to δ^{-1};
- Fluid elements in the crest of the bend do not benefit.
- Can explain $-1/3$ law with this simple model. [JLT, 2002]

A Simple Model

Very sharp bend in a material line,

$$y = f(x) = \tfrac{1}{2}\kappa_0\, x^2 + \mathrm{O}(x^3)$$

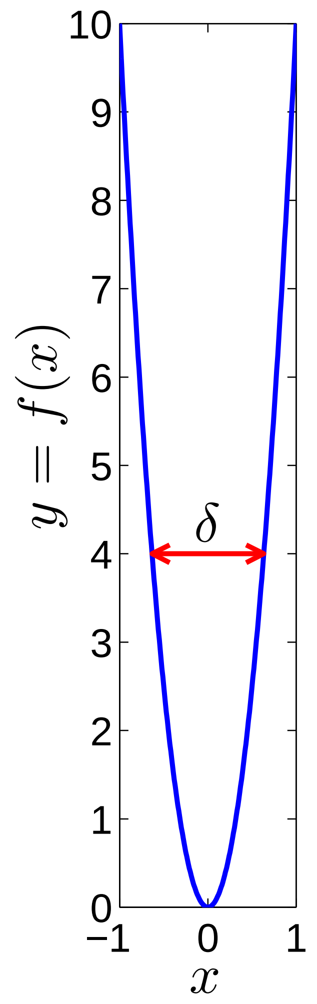
where $\kappa_0 = f''(0)$ is the curvature at the tip. $f(x) \gg x$ away from the tip. Approximate the arc length τ from $(0,0)$ to $(x, f(x))$ by

$$\tau(x) \simeq f(x).$$

Enhancement to gradients:

$$\widetilde{\Lambda}(x) = \tau(x)/x \simeq f(x)/x.$$

$\Rightarrow$ Measure of stretching (incompressible)



The curvature is $\kappa \equiv |(\hat{\mathbf{t}} \cdot \boldsymbol{\nabla})\hat{\mathbf{t}}|$, where $\hat{\mathbf{t}}$ is the unit tangent to f. To leading order this is

$$\kappa(x) = \kappa_0^{-2}\, x^{-3} + \mathrm{O}(x^{-2}), \qquad \widetilde{\Lambda}(x) = \kappa_0\, x + \mathrm{O}(x^2).$$

Solve for x in terms of κ,

$$\boxed{\widetilde{\Lambda} \sim \kappa^{-1/3}}$$

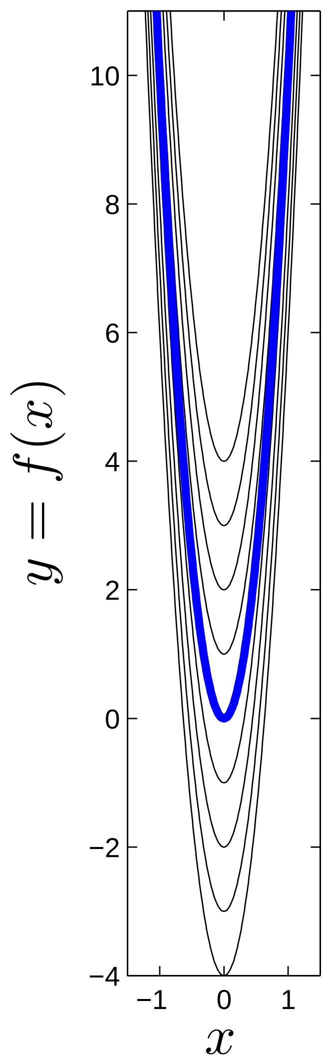
Problem: the $-1/3$ law works much better than predicted by this simple model.

(Predicts breakdown near the tip, works fine in 3D…)

A Foliation of Bends

Some observations:

- Material lines are not isolated objects.
- Continuum of other material lines.
- Standard map resembles a foliation of bends.
- Distance between lines is not constant: Compression is not uniform.
- Curvature is readily computed (geometrical).
- How do we relate to stretching?



Conservation Law for Lyapunov Exponents

The tangent to the material line aligns with the unstable direction of the flow, $\hat{\mathbf{u}}$, the direction of maximum stretching.
That direction satisfies the crucial constraint

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{u}} + \hat{\mathbf{u}}\cdot\boldsymbol{\nabla}\log\widetilde{\Lambda} \longrightarrow 0, \qquad \text{(exponentially)}$$

[JLT, 2002, in press] following earlier work by [Tang & Boozer, 1996] and [JLT & Boozer, 2001].
This is a conservation law on for $\widetilde{\Lambda}$ along the unstable manifold.

$$\frac{\partial}{\partial\tau}\log\widetilde{\Lambda} + \boldsymbol{\nabla}\cdot\hat{\mathbf{u}} = 0, \qquad \tau \equiv \text{arc length along } \hat{\mathbf{u}}$$

Convergence of $\hat{\mathbf{u}}$ $\quad\Rightarrow\quad$ increase in $\widetilde{\Lambda}$.

Assuming a foliation of bends with shape $y = f(x)$, the divergence of $\hat{\mathbf{u}}$ is easily computed,

$$\boldsymbol{\nabla} \cdot \hat{\mathbf{u}} \simeq \boldsymbol{\nabla} \cdot \hat{\mathbf{t}} = \frac{\partial \hat{t}_x}{\partial x} = -\frac{f' f''}{(1 + f'^2)^{3/2}} \,.$$

Derivative of $\widetilde{\Lambda}$ along $\hat{\mathbf{u}}$:

$$\frac{\partial}{\partial \tau} \log \widetilde{\Lambda} = \hat{\mathbf{u}} \cdot \boldsymbol{\nabla} \log \widetilde{\Lambda} = \frac{1}{(1 + f'^2)^{1/2}} \frac{\partial}{\partial x} \log \widetilde{\Lambda} \,,$$

Equate and integrate to yield

$$\widetilde{\Lambda} \sim \left(1 + f'^2\right)^{1/2}.$$

To exhibit the relationship between stretching and curvature, we use

$$\kappa(x) = |f''(x)|/(1 + f'^2)^{3/2}$$

for the magnitude of the curvature and obtain finally

$$\boxed{\widetilde{\Lambda} \sim |f''(x)|^{1/3}\,\kappa^{-1/3}}$$

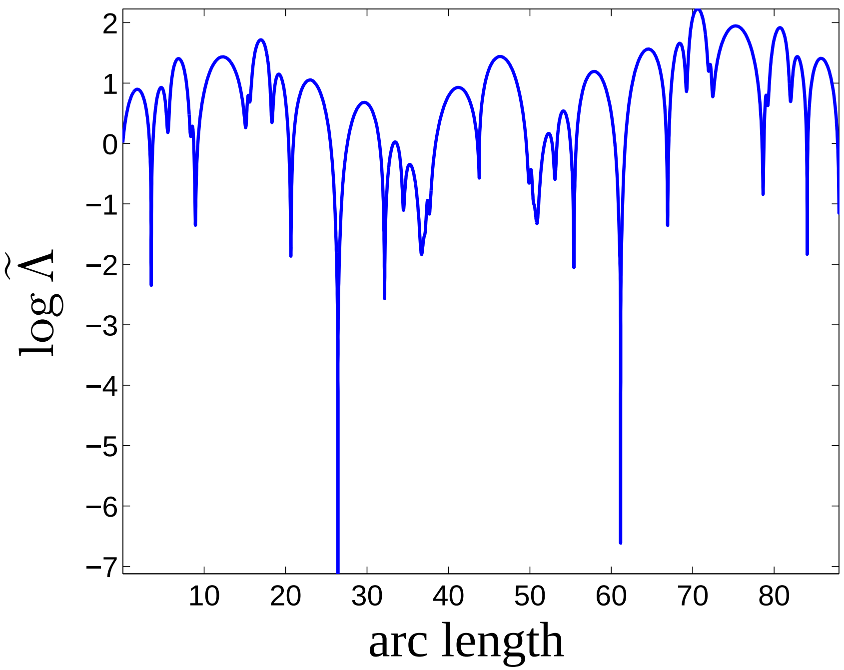
For quadratic f,

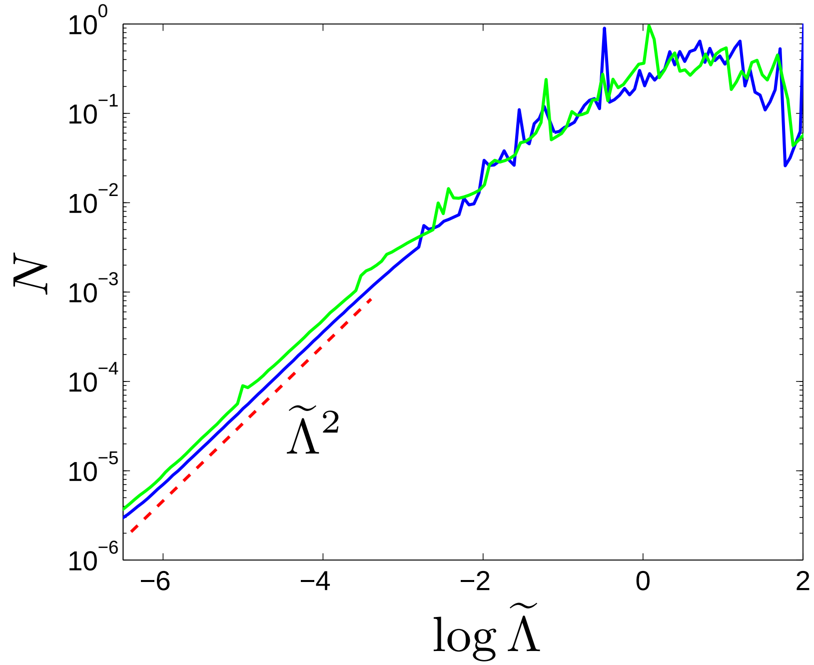
$$\widetilde{\Lambda} \sim (\kappa/\kappa_0)^{-1/3}\,,$$

so that the power-law relation holds exactly.

The shape of the bend and y-dependence of the tangent vector field will cause deviations from the $-1/3$ law.

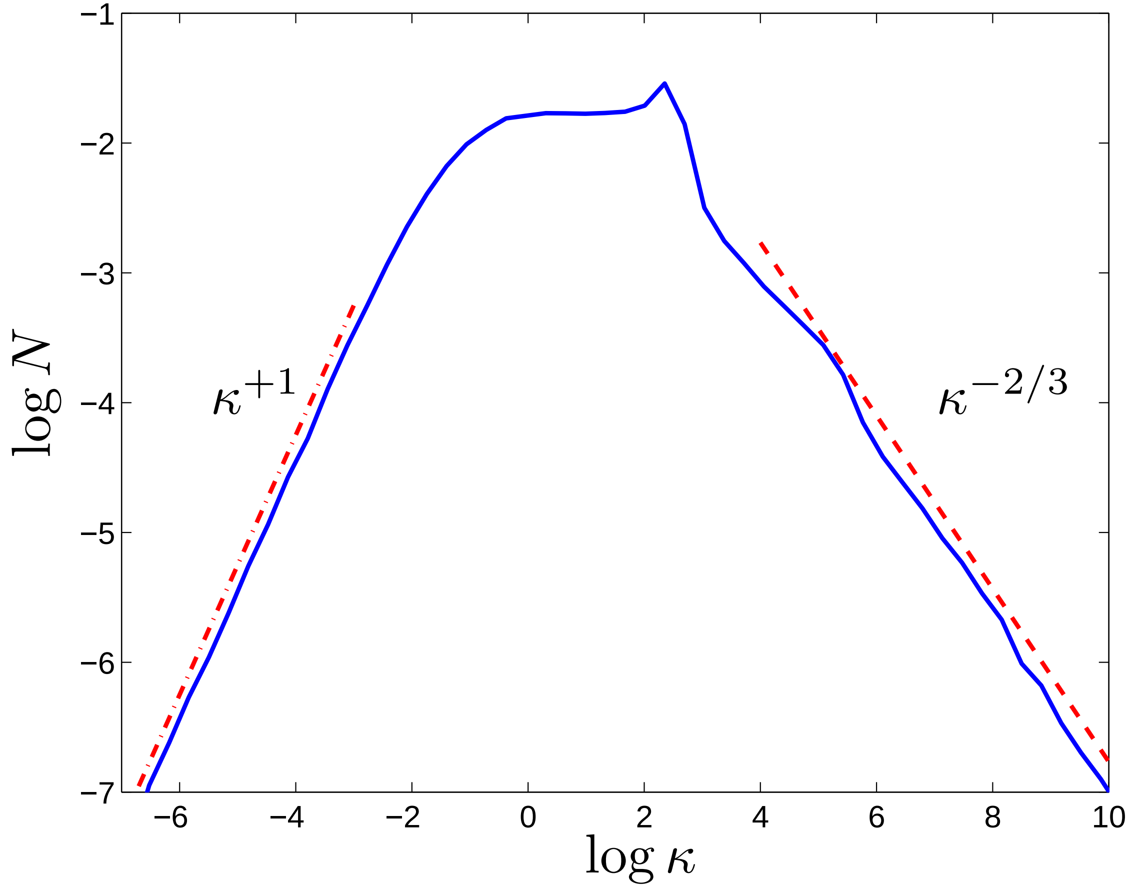
PDF of Stretching along a Material Line





The "folding" model predicts the $\widetilde{\Lambda}^2$ tail of the probability of extremely low stretching events. Exponential ("fat") tail: large fluctuations from the mean stretching.

PDF of Curvature



Stationary distribution. Tails seem independent of specific flow.

Mean moves to the right in less chaotic flows.

Conclusions

- Stretching anticorrelated with curvature.
- Around sharp bends, observe stretching $\sim$ curvature$^{-1/3}$.
- Can be explained using a conservation law for Lyapunov exponents.

Ongoing work:

- The consequences of constraints in physical applications (for the dynamo, with A. Boozer).
- Evolution of torsion. Constrained, like curvature?
- Understand PDF of curvature. 2D special?