

# Differential Geometry and Chaotic Advection

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## Setup

The equation we are interested in is the much-discussed advection-diffusion equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \frac{1}{\rho} \nabla \cdot (\rho D \nabla \phi)$$

where the Eulerian velocity field  $\mathbf{v}(\mathbf{x}, t)$  is some **prescribed** time-dependent flow, which may or may not be chaotic. The quantity  $\phi$  represents the concentration of some passive scalar,  $\rho$  is the density, and  $D$  is the diffusion coefficient.

We assume that the **Lagrangian** dynamics are chaotic.

## Stirring and Mixing!

As we were told earlier this summer, the stretching due to positive Lyapunov exponents is known as [stirring](#). The stirring phase creates large gradients, allowing diffusion to take over in the phase known to the cognoscenti as [mixing](#).

Lyapunov exponents are defined in the limit  $t \rightarrow \infty$ , and converge very slowly.

But mixing happens in finite time. So what really matters are the [finite-time Lyapunov exponents](#), which depend on position and time.

## The Goal

We try and find some **generic constraints** on the spatiotemporal dependence of FT Lyapunov exponents, and see how these could affect stirring and mixing.

We shall bring the fancy tools of **differential geometry** to bear on this problem.

**Results:** a generic functional form for the time evolution of finite-time Lyapunov exponents, and a relation between their spatial dependence and the shape of the stable manifolds.

## Time Scales

There are at least two relevant times in the system:

- The diffusion time,  $\tau_D \equiv L^2/D$ .
- The Lyapunov time,  $\tau_\lambda \equiv 1/\lambda$ .

$L$  is some characteristic length scale.

Their ratio is a dimensionless parameter,

$$\Omega \equiv \tau_D/\tau_\lambda = \lambda L^2/D$$

We consider the typical stirring-dominated case,  $\Omega \gg 1$ .

## Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates  $\mathbf{x}$  satisfies

$$\frac{d\mathbf{x}}{dt}(\boldsymbol{\xi}, t) = \mathbf{v}(\mathbf{x}(\boldsymbol{\xi}, t), t),$$

where  $\boldsymbol{\xi}$  are Lagrangian coordinates which label fluid elements. The usual choice is to take as initial condition  $\mathbf{x}(\boldsymbol{\xi}, t = 0) = \boldsymbol{\xi}$ , which says that fluid elements are labeled by their initial position.

$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$  is thus the transformation from Lagrangian ( $\boldsymbol{\xi}$ ) to Eulerian ( $\mathbf{x}$ ) coordinates.

This transformation gets horrendously complicated as time evolves.

## Jacobian

The Jacobian of this transformation is

$$J^i_j \equiv \frac{\partial x^i}{\partial \xi^j}$$

The Jacobian tells us how **tensors** transform:

- Covariant:

$$\tilde{V}_j = J^k_j V_k,$$

- Contravariant:

$$\tilde{W}^i = J^i_k W^k.$$

## Measuring distances

The distance between two infinitesimally separated points in Eulerian space is given by

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \delta_{ij} dx^i dx^j .$$

We assume repeated indices are summed.

(If our Eulerian description was in terms of non-cartesian coordinates, or lived on a curved manifold, like the surface of the earth, then another tensor than  $\delta_{ij}$  would enter.)

Therefore, in Lagrangian coordinates distances are given by

$$ds^2 = \delta_{ij} \left( \frac{dx^i}{d\xi^k} d\xi^k \right) \left( \frac{dx^j}{d\xi^\ell} d\xi^\ell \right) = (J^i_k \delta_{ij} J^j_\ell) d\xi^k d\xi^\ell .$$



## The Metric Tensor

The tensor  $\delta_{ij}$  is a **metric** in the Eulerian (Euclidean) space. The tensor

$$g_{k\ell}(\boldsymbol{\xi}, t) \equiv \sum_i J^i_k J^i_\ell = (J^T J)_{k\ell}$$

is the same metric tensor but in the Lagrangian coordinate system.

Since the metric tells us about the distance between two neighbouring Lagrangian trajectories, its eigenvalues are related to the finite-time Lyapunov exponents.

## 2-D Incompressible Flow

We will now restrict ourselves to a 2-D, incompressible velocity field  $\mathbf{v}$ . This means that

$$\det g = (\det J)^2 = 1.$$

Now,  $g$  is a positive-definite symmetric matrix, which implies that it has real positive eigenvalues,  $\Lambda(\boldsymbol{\xi}, t) \geq 1$  and  $\Lambda^{-1}(\boldsymbol{\xi}, t) \leq 1$ , and orthonormal eigenvectors  $\hat{\mathbf{e}}(\boldsymbol{\xi}, t)$  and  $\hat{\mathbf{s}}(\boldsymbol{\xi}, t)$ :

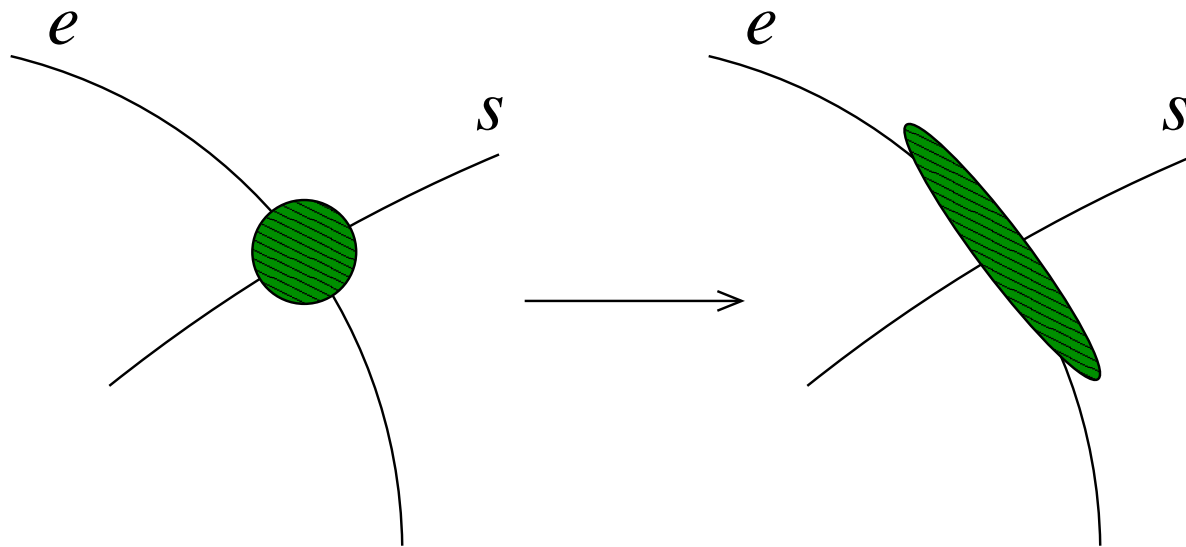
$$g_{kl}(\boldsymbol{\xi}, t) = \Lambda e_k e_l + \Lambda^{-1} s_k s_l$$

The finite-time Lyapunov exponents are given by

$$\lambda(\boldsymbol{\xi}, t) = \ln \Lambda(\boldsymbol{\xi}, t) / 2t$$

## Stable and Unstable Directions

At a fixed coordinate  $\xi$ :



The stable and unstable manifolds  $\hat{e}(\xi, t)$  and  $\hat{s}(\xi, t)$  converge exponentially to their asymptotic values  $\hat{e}_\infty(\xi)$  and  $\hat{s}_\infty(\xi)$ , whereas Lyapunov exponents converge logarithmically.

## The Advection-diffusion Equation

Under the coordinate change to Lagrangian variables the diffusion term becomes

$$\nabla \cdot (D\nabla\phi) = \frac{\partial}{\partial x^i} (D\delta^{ij} \frac{\partial\phi}{\partial x^j}) = \frac{\partial}{\partial \xi^i} (Dg^{ij} \frac{\partial\phi}{\partial \xi^j}).$$

In Lagrangian coordinates the diffusivity becomes  $Dg^{ij}$ : it is no longer **isotropic**.

The advection-diffusion equation is thus just the diffusion equation,

$$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial \xi^i} (Dg^{ij} \frac{\partial\phi}{\partial \xi^j}),$$

because the advection term drops out by construction.

### Diffusion along $\hat{s}_\infty$ and $\hat{e}_\infty$

The diffusion coefficients along the  $\hat{s}_\infty$  and  $\hat{e}_\infty$  lines are

$$D^{ss} = s_{\infty i} (Dg^{ij}) s_{\infty j} = D \exp(2\lambda t),$$

$$D^{ee} = e_{\infty i} (Dg^{ij}) e_{\infty j} = D \exp(-2\lambda t).$$

We see that  $D^{ee}$  goes to zero exponentially quickly. Hence, *essentially all the diffusion occurs along the  $\hat{s}_\infty$  line.*

## Riemann Curvature Tensor

Are there constraints to the form the metric can take?

The **Riemann curvature tensor** is defined as

$$R_{\alpha\beta\mu\nu} \equiv \frac{1}{2} \left( \frac{\partial^2 g_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 g_{\beta\mu}}{\partial x^\alpha \partial x^\nu} - \frac{\partial^2 g_{\beta\nu}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 g_{\alpha\mu}}{\partial x^\beta \partial x^\nu} \right) + g_{\rho\sigma} \left( \Gamma_{\alpha\nu}^\rho \Gamma_{\beta\mu}^\sigma - \Gamma_{\alpha\mu}^\rho \Gamma_{\beta\nu}^\sigma \right),$$

where the **Christoffel symbols** are

$$\Gamma_{\mu\nu}^\rho \equiv \frac{1}{2} g^{\rho\tau} \left( \frac{\partial g_{\mu\tau}}{\partial x^\nu} + \frac{\partial g_{\nu\tau}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\tau} \right).$$

The Riemann tensor and the Christoffel symbols satisfy a bunch of symmetries. In two dimensions there is only one independent component of  $R$ , which we take to be  $R_{1212}$ .

## Who Cares?

What is the meaning of this tensor? For some vector  $V(x)$ ,

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = V_{\sigma} g^{\sigma\rho} R_{\rho\mu\nu\kappa}$$

The semicolon indicates covariant differentiation, which is simply a derivative which takes into account the fact that basis vectors may depend on position.

In flat space, derivatives commute, so the Riemann tensor must vanish identically. It must do so in all coordinate systems—in particular in Lagrangian coordinates.

**Work out the beast...**

After a tedious calculation, we find the Riemann tensor is

$$R_{1212} = -A \exp(2\lambda t) + B \exp(-2\lambda t)$$

where

$$A \equiv 2(\hat{\mathbf{s}} \cdot \nabla_0 \lambda t)^2 + \hat{\mathbf{s}} \cdot \nabla_0 (\hat{\mathbf{s}} \cdot \nabla_0 \lambda t) + 3(\nabla_0 \cdot \hat{\mathbf{s}})(\hat{\mathbf{s}} \cdot \nabla_0 \lambda t) \\ + \hat{\mathbf{s}} \cdot \nabla_0 (\nabla_0 \cdot \hat{\mathbf{s}}) + (\nabla_0 \cdot \hat{\mathbf{s}})^2$$

and  $B$  is a similar expression involving  $\hat{\mathbf{e}}$ .

(The  $\nabla_0$  are gradients with respect to Lagrangian coordinates.)



We require that  $R_{1212}$  vanishes identically:

$$R_{1212} = 0 \quad \Longrightarrow \quad \frac{A}{B} = \exp(-4\lambda t).$$

We now take  $t$  large enough for the  $\hat{\mathbf{s}}$  direction to converge to its asymptotic value,  $\hat{\mathbf{s}}_\infty$ . For such large time (which is not that large if the flow is reasonably chaotic), we must have  $A \rightarrow 0$ , i.e.,

$$\hat{\mathbf{s}}_\infty \cdot \nabla_0 \Upsilon + \Upsilon(2\Upsilon - \nabla_0 \cdot \hat{\mathbf{s}}_\infty) = 0,$$

where

$$\hat{\mathbf{s}}_\infty \cdot \nabla_0 \tilde{\lambda}(\xi) \equiv \lim_{t \rightarrow \infty} \hat{\mathbf{s}} \cdot \nabla_0 \lambda(\xi, t) t$$

and

$$\Upsilon \equiv \hat{\mathbf{s}}_\infty \cdot \nabla_0 \tilde{\lambda}(\xi) + \nabla_0 \cdot \hat{\mathbf{s}}_\infty.$$

The definition of  $\tilde{\lambda}$  allows addition of an arbitrary function  $f(\xi, t)$  satisfying

$$\hat{\mathbf{s}}_\infty \cdot \nabla_0 f(\xi, t) = 0,$$

so that

$$\lambda(\xi, t) = \frac{\tilde{\lambda}(\xi)}{t} + \frac{f(\xi, t)}{\sqrt{t}} + \lambda_\infty,$$

where  $f(\xi, t)$  is bounded by  $\sqrt{t}$ , i.e.

$$\lim_{t \rightarrow \infty} \frac{f(\xi, t)}{\sqrt{t}} = 0.$$

The reason we write the arbitrary term with a  $1/\sqrt{t}$  factor will become clear in a minute.

## Evolution of the Distribution

Antonsen *et al* have shown that the time evolution of the probability distribution function of finite-time Lyapunov exponents is given by

$$P(\lambda, t) = \sqrt{\frac{t G''(\lambda)}{2\pi}} \exp(-t G(\lambda)),$$

where  $G(\lambda_\infty) = G'(\lambda_\infty) = 0$ . This is the probability distribution for a random variable that is the average of many independent, identically distributed variables.

The width of the distribution sharpens as time evolves, and becomes a delta function as  $t \rightarrow \infty$ , peaked at  $\lambda_\infty$ .

If the range of  $\lambda$  of interest is small compared to the standard deviation, we can approximate  $G$  by expanding around  $\lambda_\infty$ ,

$$G(\lambda) \simeq \frac{1}{2} G''(\lambda_\infty)(\lambda - \lambda_\infty)^2.$$

so that the probability distribution becomes Gaussian:

$$P(\lambda, t) = \sqrt{\frac{t G''(\lambda_\infty)}{2\pi}} \exp\left(-\frac{1}{2} t G''(\lambda_\infty)(\lambda - \lambda_\infty)^2\right),$$

Note the the standard deviation is  $\sigma = 1/\sqrt{G''(\lambda_\infty) t}$ .

(The Gaussian approximation works best for strongly chaotic flows.)

If we take the spatial average  $\langle \cdot \rangle$  of our expression for the finite-time Lyapunov exponents,

$$\langle \lambda \rangle (t) = \frac{\langle \tilde{\lambda} \rangle}{t} + \frac{\langle f(\xi, t) \rangle}{\sqrt{t}} + \lambda_\infty,$$

we find that the dominant contribution to the standard deviation for large  $t$  is

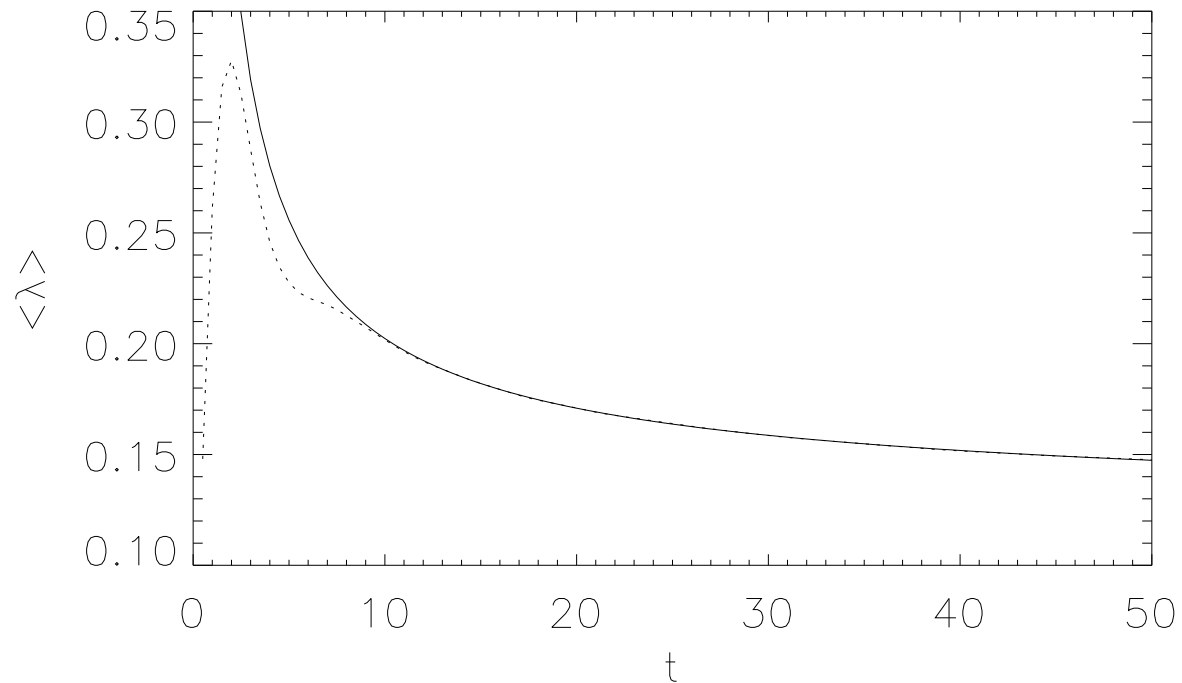
$$\sigma = \frac{\sqrt{\langle \lambda^2 \rangle - \langle \lambda \rangle^2}}{\langle \lambda \rangle} \sim \frac{\sqrt{\langle f(\xi, t)^2 \rangle - \langle f(\xi, t) \rangle^2}}{\lambda_\infty \sqrt{t}}.$$

To agree with the Gaussian result of  $\sigma \sim 1/\sqrt{t}$ , we require

$$\lim_{t \rightarrow \infty} \langle f(\xi, t) \rangle = f_0, \quad \lim_{t \rightarrow \infty} \langle f(\xi, t)^2 \rangle = f_1^2,$$

i.e., the first two moments of  $f$  become independent of time for large  $t$ .

Example:



Dotted: Numerical

Solid:  $0.305/t + 0.175/\sqrt{t} + 0.117$

Allows us to determine  $\lambda_\infty = 0.117$  rapidly and accurately.

## Relationship between $\tilde{\lambda}$ and the $\hat{s}$ line

The curvature vector is defined as

$$\boldsymbol{\kappa} \equiv \hat{\mathbf{s}}_{\infty} \cdot \nabla_0 \hat{\mathbf{s}}_{\infty}$$

It can be shown that the condition that was derived earlier from the vanishing of the Riemann tensor implies

$$\hat{\mathbf{s}}_{\infty} \cdot \left( \nabla_0 (\tilde{\lambda} - \ln \|\boldsymbol{\kappa}\|) + \boldsymbol{\kappa} \times \nabla_0 \times \boldsymbol{\kappa} / \|\boldsymbol{\kappa}\|^2 \right) = 0$$

if the curvature is nonvanishing.

Tang and Boozer have demonstrated that the relationship between the  $\hat{s}$  line and  $\lambda$  can be expressed as

$$\lambda(\xi, t) = -\frac{c_0}{t} \ln \left\| 1 + \frac{\kappa}{\bar{\kappa}} \right\| + \tilde{\lambda}_0(\xi)/t + f(\xi)/\sqrt{t} + \lambda_\infty$$

where  $\bar{\kappa}$  is the average curvature along the  $\hat{s}$  line, and  $\tilde{\lambda}_0$  is slowly varying along the  $\hat{s}$  line. This is well confirmed numerically.

Note that the change in  $\lambda$  during sharp bends of the  $\hat{s}$  line becomes less pronounced as time evolves and the finite-time Lyapunov exponents converge to their uniform, infinite-time value.



Figure 3-5, 3-6, 3-11, 3-10

## Conclusions

- Diffusion occurs overwhelmingly along the stable direction. Differential geometry gives an elegant description.
- The spatial dependence of Lyapunov exponents **along**  $\hat{s}$  lines is contained in the function  $\tilde{\lambda}(\xi)$ , which decays as  $1/t$ .
- The notoriously slow convergence of Lyapunov exponents is embodied in the function  $f(\xi, t)$ , which is **constant** on  $\hat{s}$  lines and decays as  $1/\sqrt{t}$ .
- The new metric tensor obtained after transforming to Lagrangian coordinates has to have a vanishing Riemann tensor, which gives a relation between  $\hat{s}_\infty(\xi)$  and  $\tilde{\lambda}(\xi)$ .
- Sharp bends in the  $\hat{s}$  line lead to locally small finite-time Lyapunov exponents. These are “sticky” regions where gradients don’t build up as quickly, so diffusion is hindered.