Random entanglements

Winding of planar Brownian motions

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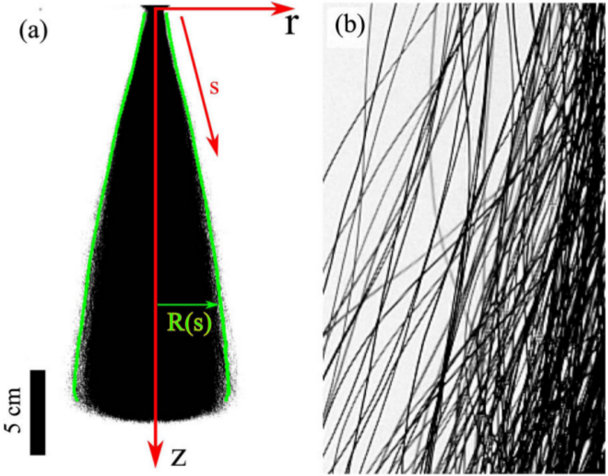
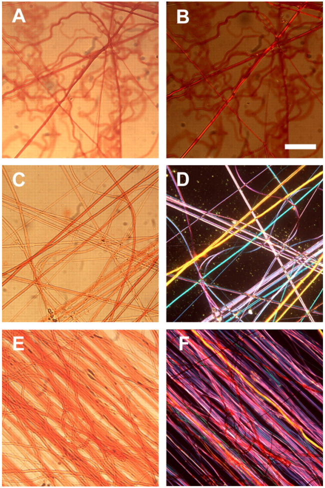
Complex entanglements are everywhere

Tangled hair



[Goldstein, R. E., Warren, P. B., & Ball, R. C. (2012). *Phys. Rev. Lett.* **108**, 078101]

Tangled hagfish slime

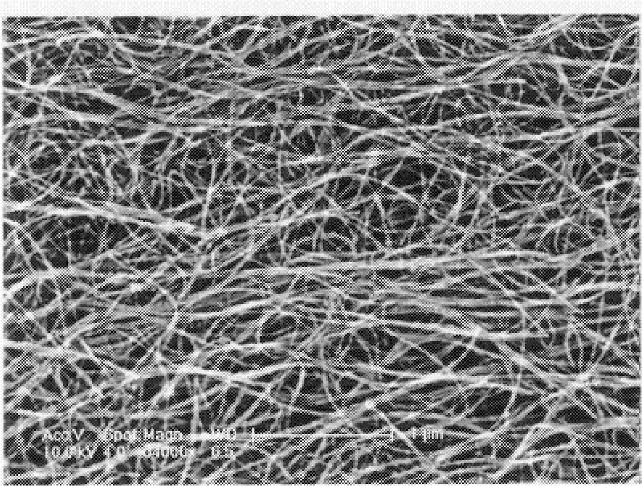


Slime secreted by hagfish is made of microfibers.

The quality of entanglement determines the material properties (rheology) of the slime.

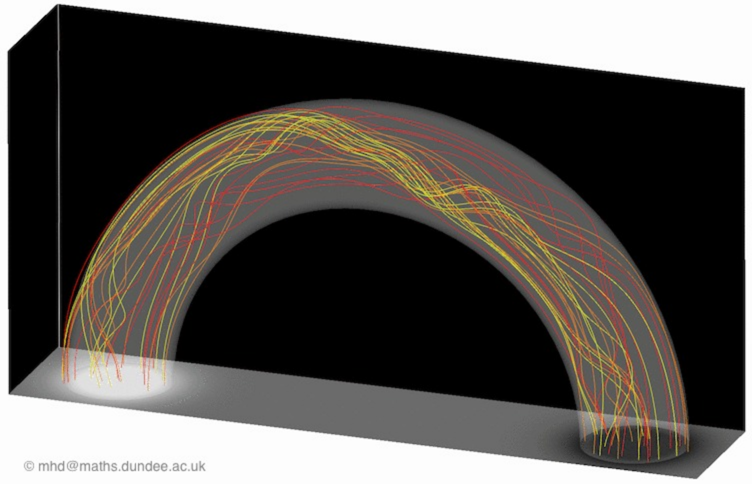
[Fudge, D. S., Levy, N., Chiu, S., & Gosline, J. M. (2005). *J. Exp. Biol.* **208**, 4613–4625]
[Chaudhary, G., Ewoldt, R., & Thiffeault, J.-L. (2019). *J. Roy. Soc. Interface,* **16** (150), 20180710]

Tangled carbon nanotubes



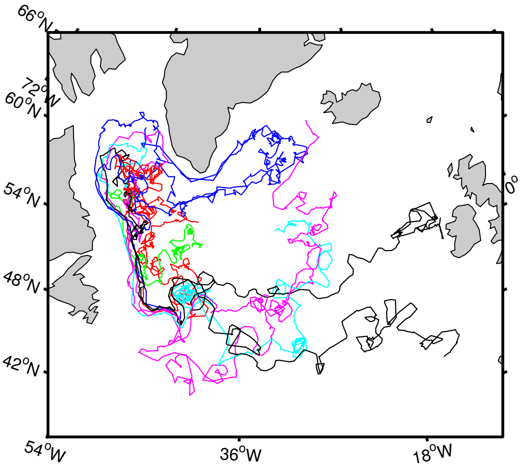
[Source: http://www.ineffableisland.com/2010/04/carbon-nanotubes-used-to-make-smaller.html]

Tangled magnetic fields



[Source: http://www.maths.dundee.ac.uk/mhd/]

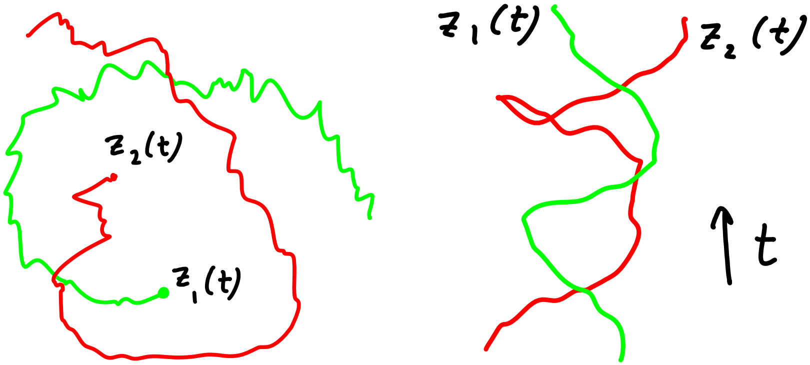
Tangled oceanic float trajectories



[Source: WOCE subsurface float data assembly center, `http://wfdac.whoi.edu`, Thiffeault, J.-L. (2010). *Chaos,* **20**, 017516]

The simplest tangling problem

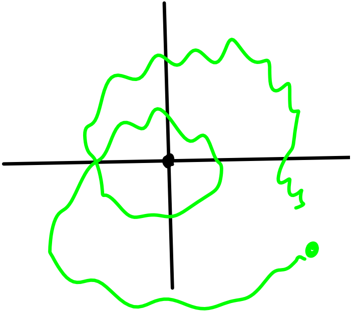
Consider two Brownian motions on the complex plane, each with diffusion constant D:



Viewed as a spacetime plot, these form a ‘braid’ of two strands.

Winding angle

Take the vector $Z(t) = Z_1(t) - Z_2(t)$, which behaves like a Brownian particle of diffusivity $2D$ ($\to D$):



Define $\Theta \in (-\infty, \infty)$ to be the total winding angle of $Z(t)$ around the origin.

Winding angle distribution

Spitzer (1958) found the time-asymptotic distribution of θ to be Cauchy:

$$\frac{\Theta(t)}{\log(2\sqrt{Dt}/r_0)} \xrightarrow{d} X, \qquad p_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}\,.$$

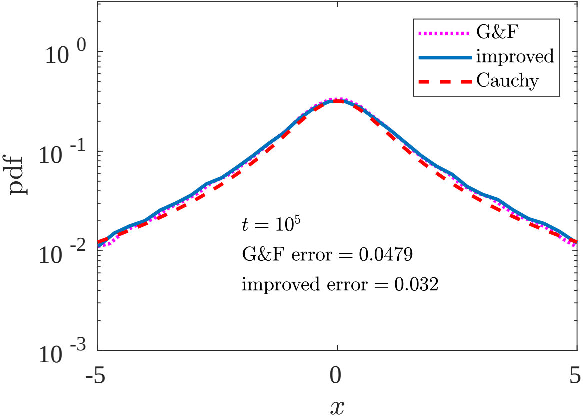
where $r_0 = |Z(0)|$.

The normalized variable is $X \sim \Theta(t)/\log t$.

Note that a Cauchy distribution is a bit strange: the variance is infinite, so large windings are highly probable!

[Spitzer, F. (1958). *Trans. Amer. Math. Soc.* **87**, 187–197]

Winding angle distribution: numerics



The normalized variable is $x = \theta / \log(2\sqrt{Dt}/r_0)$.

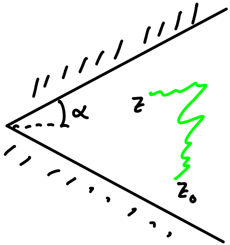
Some care is needed for these simulations (rescale time near the origin) [Wen, H. & Thiffeault, J.-L. (2019). *Philos. Trans. Royal Soc. A*, **377**, 20180347]

Winding angle distribution: derivation

The probability distribution $P(z, t)$ of the Brownian process satisfies the Fokker–Planck PDE (heat equation):

$$\frac{\partial P}{\partial t} = D\Delta P, \qquad P(z, 0) = \delta(z - z_0).$$

Consider the solution in a wedge of half-angle α:



(Reflecting boundary condition at the walls.)

Winding angle distribution: derivation (cont'd)

In polar form, Fokker–Planck PDE for $P(r, \theta, t)$:

$$\frac{\partial P}{\partial t} = D\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 P}{\partial \theta^2}\right), \quad \partial_\theta P(r, \pm\alpha, t) = 0.$$

The solution is a standard eigenfunction expansion, but then take the wedge angle α to ∞ (!):

$$P(z, t) = \frac{1}{2\pi D t}\,\mathrm{e}^{-(r^2 + r_0^2)/4Dt} \int_0^\infty \cos\nu(\theta - \theta_0)\, I_\nu\!\left(\frac{r\, r_0}{2Dt}\right) \mathrm{d}\nu$$

where I_ν is a modified Bessel function of the first kind.

For large t this recovers the Cauchy distribution for the angle.

Key point: by allowing the wedge angle to infinity, we are using Riemann sheets to keep track of the winding angle.

Winding around a finite obstable

So Cauchy distribution is a bit pathological: infinite variance. This is a symptom of the point approximation for the winding center.

Instead of winding around a point, wind around a disk of radius a.

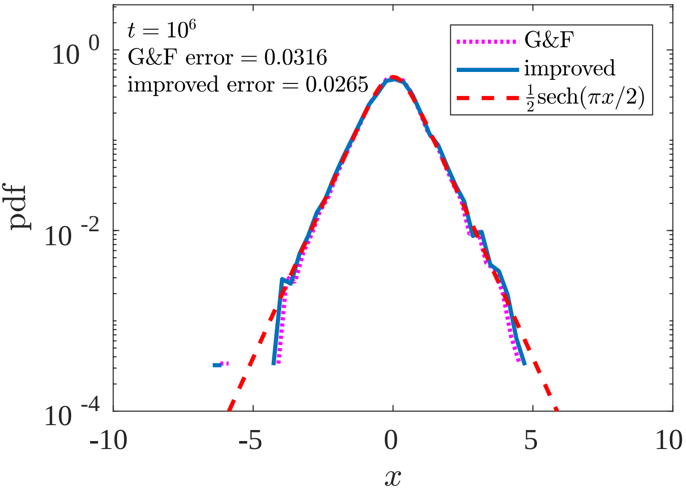
The calculation is quite similar, but now we get convergence to a very different distribution:

$$\frac{\Theta(t)}{\log(2\sqrt{Dt}/a)} \xrightarrow{d} X, \qquad p_X(x) = \tfrac{1}{2}\,\mathrm{sech}(\pi x/2).$$

This has exponential tails: all the moments exist.

[Bélisle, C. (1989). *Ann. Prob.* **17** (4), 1377–1402
Grosberg, A. & Frisch, H. (2003). *J. Phys. A,* **36** (34), 8955–8981
Wen, H. & Thiffeault, J.-L. (2019). *Philos. Trans. Royal Soc. A,* **377**, 20180347]

Finite obstacle: numerics



The normalized variable is $x = \theta / \log(2\sqrt{Dt}/a)$.

[Wen, H. & Thiffeault, J.-L. (2019). *Philos. Trans. Royal Soc. A*, **377**, 20180347]

Let's add drift!

So far the planar motion was pure Brownian motion. One natural extension is to add a tangential drift, which leads to the PDE

$$\frac{\partial p}{\partial t} + \Omega(r,t)\,\frac{\partial p}{\partial \theta} = D\,\Delta p.$$

In general, we cannot solve this equation analytically or even asymptotically in time.

Constant Ω is uninteresting: it simply "shifts" the pdf in time by Ωt.

Fortunately, a tractable case is the point vortex of fluid dynamics:

$$\Omega(r) = \beta/r^2.$$

The flow promotes winding, but falls off if the particle wanders too far.

Why does it work?

The reason why the point vortex allows analytical treatment is that the eigenvalue problem arising from the boundary value problem is

$$\rho'' + \frac{1}{r}\rho' + \left(\lambda^2 - \frac{k_\mu^2}{r^2}\right)\rho = 0, \qquad k_\mu = \sqrt{\mu^2 + \mathrm{i}\beta\mu}$$

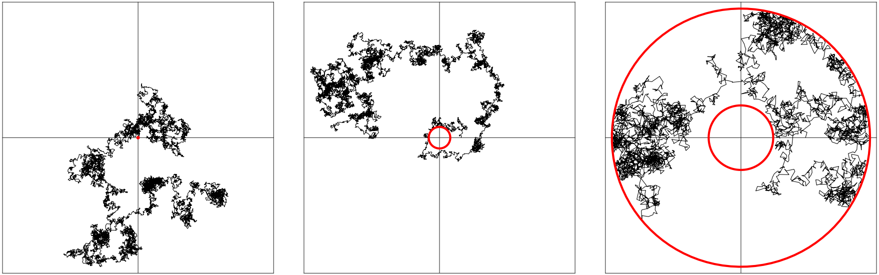
which is still a Bessel equation, though the drift makes the parameter k_μ complex. The asymptotic analysis is thus considerably more challenging.

I spare you the details, which are in
[Wen, H. & Thiffeault, J.-L. (2019). *Philos. Trans. Royal Soc. A*, **377**, 20180347].

Let's examine the limiting distributions in a few cases.

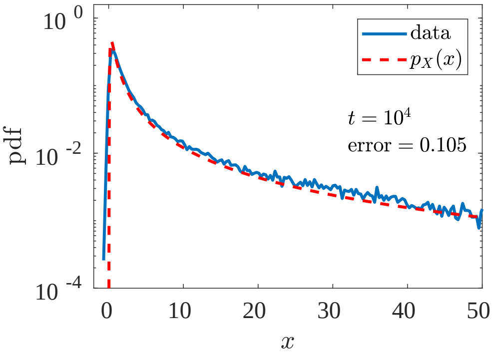
Winding with drift: The three cases

Point, disk, and annulus:



Notice that the particle now winds preferentially counterclockwise, because of the drift $\Omega = \beta/r^2$, $\beta > 0$.

Winding with drift around a point

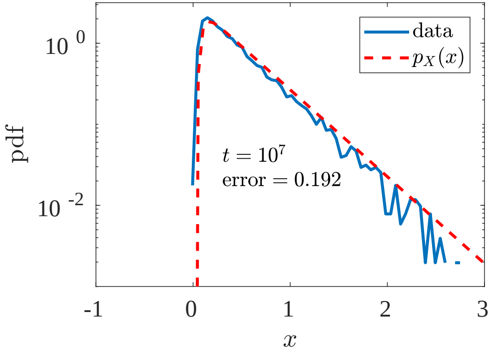


X^{-1} converges to a $\mathrm{Gamma}(\frac{1}{2}, \frac{1}{2})$ distribution:

$$\frac{8\Theta(t)}{\beta \log^2(4t/r_0^2)} \xrightarrow{d} X, \qquad p_X(x) = \frac{1}{\sqrt{2\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/2x}\, \chi_{(x>0)}\,.$$

χ is the indicator function: angle is non-negative.

Winding with drift around a disk of radius a

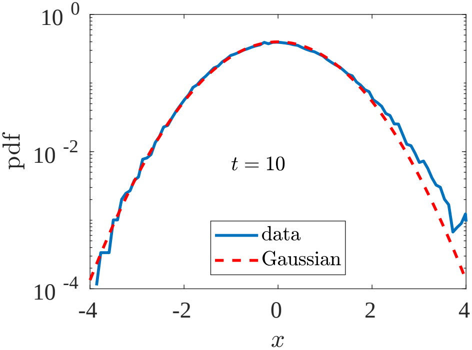


Now the asymptotic distribution involves a second elliptic theta function:

$$\frac{4\Theta(t)}{\beta\,\log^2(4t/a^2)} \xrightarrow{d} X, \qquad p_X(x) = -\tfrac{\pi}{2}\,\vartheta_2'\big(\tfrac{\pi}{2}\,,\,\mathrm{e}^{-\pi^2 x}\big)\,\chi_{(x>0)}\,.$$

Angle is again non-negative.

Winding with drift around an annulus $a < r < b$



The bounded region is strongly recurrent and leads to a Gaussian form:

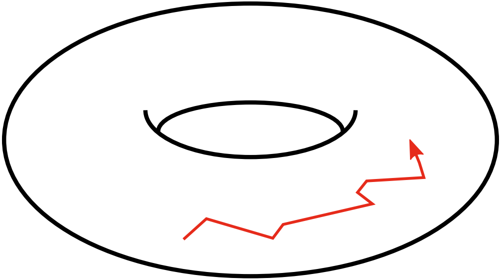
$$\frac{\Theta(t) - A(t)\beta}{\sqrt{2A(t)}} \xrightarrow{d} N(0,1), \qquad A(t) = \frac{2t}{b^2 - a^2} \log(b/a).$$

Now the mean angle increases linearly with time.

The Gaussian form is generic in bounded regions [Geng, X. & Iyer, G. (2018)]

Related example: Brownian motion on the torus

A Brownian motion on a torus can wind around the two periodic directions:

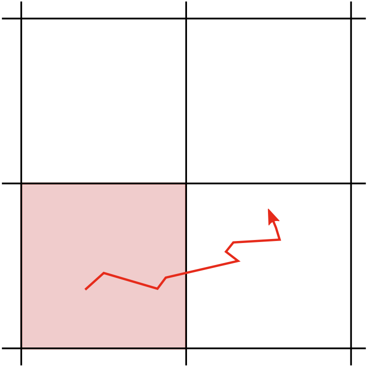


What is the asymptotic distribution of windings?

Mathematically, we are asking what is the homology class of the motion?

Torus: universal cover

We pass to the universal cover of the torus, which is the plane:



The universal cover records the windings as paths on the plane. The original 'copy' is called the fundamental domain.

On the plane the probability distribution is the usual Gaussian heat kernel:

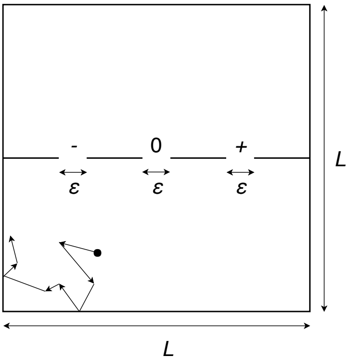
$$P(x, y, t) = \frac{1}{4\pi Dt}\, \mathrm{e}^{-(x^2+y^2)/4Dt}$$

So here $m = \lfloor x \rfloor$ and $n = \lfloor y \rfloor$ will give the homology class: the number of windings of the walk in each direction.

We can think of the motion as entangling with the space itself.

The N-slit problem (w. G. Bonner and B. Valko)

Brownian motion in a square with $N = 3$ narrow "slits" of width ε:



Similar to a particle winding around two obstacles.

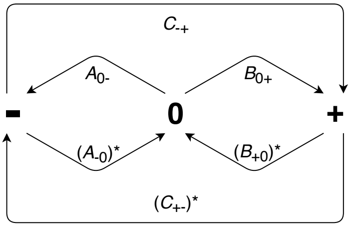
Problem is now non-Abelian: order matters. $\pi_1(D_N)$ instead of homology.

Angle is no longer as relevant as a measure of entanglement.

Narrow slit approximation is crucial: the particle hits a slit uniformly, with expected time $\sim (L^2/D)\log \varepsilon^{-1}$ between hits.

Groupoid description

Write the history of the Brownian motion as a sequence of symbols.



These are groupoid elements: not all mutliplications make sense.

$$A_{0-}C_{-+}(B_{+0})^*(A_{0-})^*\dots$$

The $*$ denotes the lower-half plane.

Whenever the particle returns to slit 0 we have an element of $\pi_1(D_N)$.

The difficulty lies in keeping track of cancellations.

Each return corresponds to one or two letters: $C_{-+} = A_{-0}B_{0+}$.

An explicit formula for the growth

The key quantity is the growth of reduced word length in the letters. Related to growth in regular languages.

Key is derive a certain generating function for last passage times for N slits:

$$R(\lambda) = \frac{1}{2(N-2)\lambda}\left(2(N-1)^2(N-2) - (N-1)\lambda - \sqrt{D(N,\lambda)}\right)$$

with

$$D(N,\lambda) = (N-1)\left((N-1)(\lambda - 2N(N-3)-4)^2 - 4(N-2)^2\lambda^2\right)$$

The growth is then $R(1)/R'(1)$.

[Gajrat, A., Malyshev, V. A., & Menshikov, M. V. (1993). Research Report RR-2202 INRIA

Gilch, L. A. (2008). In: *Proceedings to 5th Colloquium on Mathematics and Computer Science* pp. 2544–2560,]

Some references

Many people have worked on aspects of this problem:

- [Spitzer (1958); Durrett (1982); Messulam & Yor (1982); Berger (1987); Shi (1998)] winding of Brownian motion around a point in $\mathbb{R}^2$.
- [Berger & Roberts (1988); Bélisle (1989); Bélisle & Faraway (1991); Rudnick & Hu (1987)] winding of random walk around a point.
- [Drossel & Kardar (1996); Grosberg & Frisch (2003)] finite obstacle, closed domain.
- [Itô & McKean (1974); McKean (1969); Lyons & McKean (1984)] doubly-punctured plane.
- [McKean & Sullivan (1984)] three-punctured sphere.
- [Pitman & Yor (1986, 1989)] more points.
- [Watanabe (2000)] Riemann surfaces.
- [Nechaev (1988)] lattice of obstacles.
- [Nechaev (1996); Revuz & Yor (1999)] comprehensive books.

Conclusions & outlook

- Entanglement at confluence of dynamics, probability, topology, and combinatorics.
- Instead of Brownian motion, can use orbits from a dynamical system. This yields dynamical information.
- More generally, study random processes on configuration spaces of sets of points (also finite size objects).
- Other applications: Crowd dynamics (Ali, 2013), granular media (Puckett *et al.*, 2012).
- With Michael Allshouse: develop tools for analyzing orbit data from this topological viewpoint (Allshouse & Thiffeault, 2012).
- With Tom Peacock, Marko Budišić, and Margaux Filippi: apply to orbits in a fluid dynamics experiments.

References I

Ali, S. (2013). In: *IEEE International Conference on Computer Vision (ICCV)* pp. 1097–1104, : IEEE.

Allshouse, M. R. & Thiffeault, J.-L. (2012). *Physica D,* **241** (2), 95–105.

Bélisle, C. (1989). *Ann. Prob.* **17** (4), 1377–1402.

Bélisle, C. & Faraway, J. (1991). *J. Appl. Prob.* **28** (4), 717–726.

Berger, M. A. (1987). *J. Phys. A,* **20**, 5949–5960.

Berger, M. A. & Roberts, P. H. (1988). *Adv. Appl. Prob.* **20** (2), 261–274.

Buck, G. & Scharein, R. (2014). preprint.

Chaudhary, G., Ewoldt, R., & Thiffeault, J.-L. (2019). *J. Roy. Soc. Interface,* **16** (150), 20180710.

Chavel, I. (1984). *Eigenvalues in Riemannian Geometry*. Orlando: Academic Press.

Drossel, B. & Kardar, M. (1996). *Phys. Rev. E,* **53** (6), 5861–5871.

Durrett, R. (1982). *Ann. Prob.* **10** (1), 244–246.

Dynnikov, I. A. (2002). *Russian Math. Surveys,* **57** (3), 592–594.

Dynnikov, I. A. & Wiest, B. (2007). *Journal of the European Mathematical Society,* **9** (4), 801–840.

Finn, M. D. & Thiffeault, J.-L. (2011). *SIAM Rev.* **53** (4), 723–743.

References II

Fudge, D. S., Levy, N., Chiu, S., & Gosline, J. M. (2005). *J. Exp. Biol.* **208**, 4613–4625.

Gajrat, A., Malyshev, V. A., & Menshikov, M. V. (1993). Research Report RR-2202 INRIA.

Geng, X. & Iyer, G. (2018).

Gilch, L. A. (2008). In: *Proceedings to 5th Colloquium on Mathematics and Computer Science* pp. 2544–2560,.

Goldstein, R. E., Warren, P. B., & Ball, R. C. (2012). *Phys. Rev. Lett.* **108**, 078101.

Grosberg, A. & Frisch, H. (2003). *J. Phys. A,* **36** (34), 8955–8981.

Itô, K. & McKean, H. P. (1974). *Diffusion processes and their sample paths*. Berlin: Springer.

Lyons, T. J. & McKean, H. P. (1984). *Adv. Math.* **51**, 212–225.

McKean, H. P. (1969). *Stochastic Integrals*. New York: Academic Press.

McKean, H. P. & Sullivan, D. (1984). *Adv. Math.* **51**, 203–211.

Messulam, P. & Yor, M. (1982). *J. London Math. Soc. (2),* **26**, 348–364.

Moussafir, J.-O. (2006). *Func. Anal. and Other Math.* **1** (1), 37–46.

Nechaev, S. K. (1988). *J. Phys. A,* **21**, 3659–3671.

Nechaev, S. K. (1996). *Statistics of Knots and Entangled Random Walks*. Singapore; London: World Scientific.

Pitman, J. & Yor, M. (1986). *Ann. Prob.* **14** (3), 733–779.

Pitman, J. & Yor, M. (1989). *Ann. Prob.* **17** (3), 965–1011.

References III

Puckett, J. G., Lechenault, F., Daniels, K. E., & Thiffeault, J.-L. (2012). *Journal of Statistical Mechanics: Theory and Experiment,* **2012** (6), P06008.

Revuz, D. & Yor, M. (1999). *Continuous Martingales and Brownian motion*. Berlin: Springer, third edition.

Rudnick, J. & Hu, Y. (1987). *J. Phys. A,* **20**, 4421–4438.

Shi, Z. (1998). *Ann. Prob.* **26** (1), 112–131.

Spitzer, F. (1958). *Trans. Amer. Math. Soc.* **87**, 187–197.

Thiffeault, J.-L. (2005). *Phys. Rev. Lett.* **94** (8), 084502.

Thiffeault, J.-L. (2010). *Chaos,* **20**, 017516.

Thiffeault, J.-L. & Finn, M. D. (2006). *Philos. Trans. Royal Soc. Lond. A,* **364**, 3251–3266.

Watanabe, S. (2000). *Acta Applicandae Mathematicae,* **63**, 441–464.

Wen, H. & Thiffeault, J.-L. (2019). *Philos. Trans. Royal Soc. A,* **377**, 20180347.