Tugging at Polymers in Turbulent Flow

The Polymers Tug Back

Jean-Luc Thiffeault

http://plasma.ap.columbia.edu/˜jeanluc

Department of Applied Physics and Applied Mathematics

Columbia University

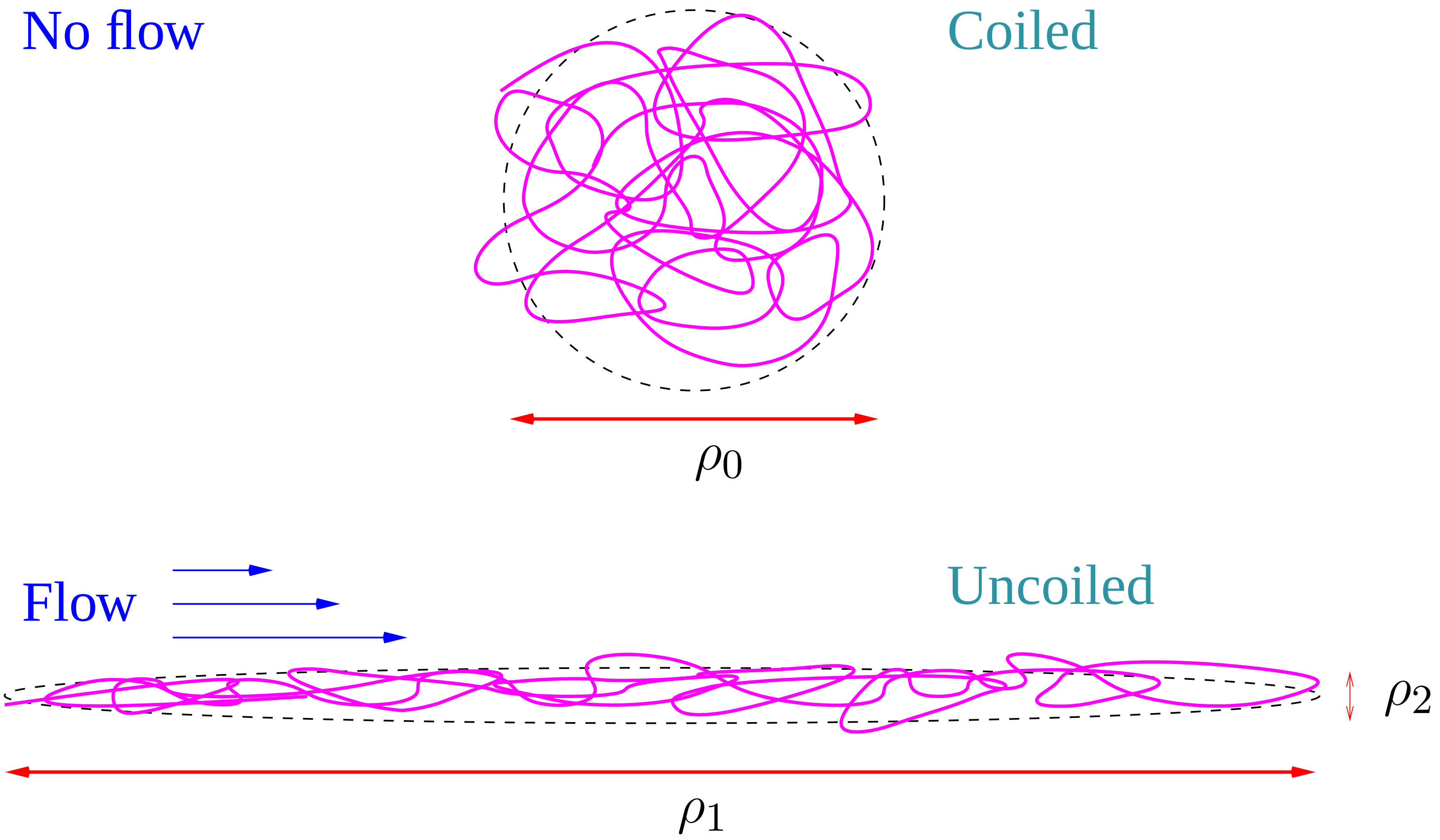
Drag Reduction

Experimental facts:

- Toms (1948) observed that the addition of ~ 10 ppm polymers to turbulent pipe flow reduced the pressure drop substantially.
- Typical example: polyethilene oxide in water: 18 ppm (by weight) reduces drag by 33%!!
- Qualitative understanding: classic review of Lumley (1969) uses dimensional analysis (polymer size, viscosity, etc.) to predict magnitude and onset of effect.
- Molecular scales matter! Mystery?
- Stretched state: Einstein's effective viscosity [Hinch (1977)].

Polymers

Polymers are long chains of molecules. Random walk at rest.



No flow

Coiled

ρ_0

Flow

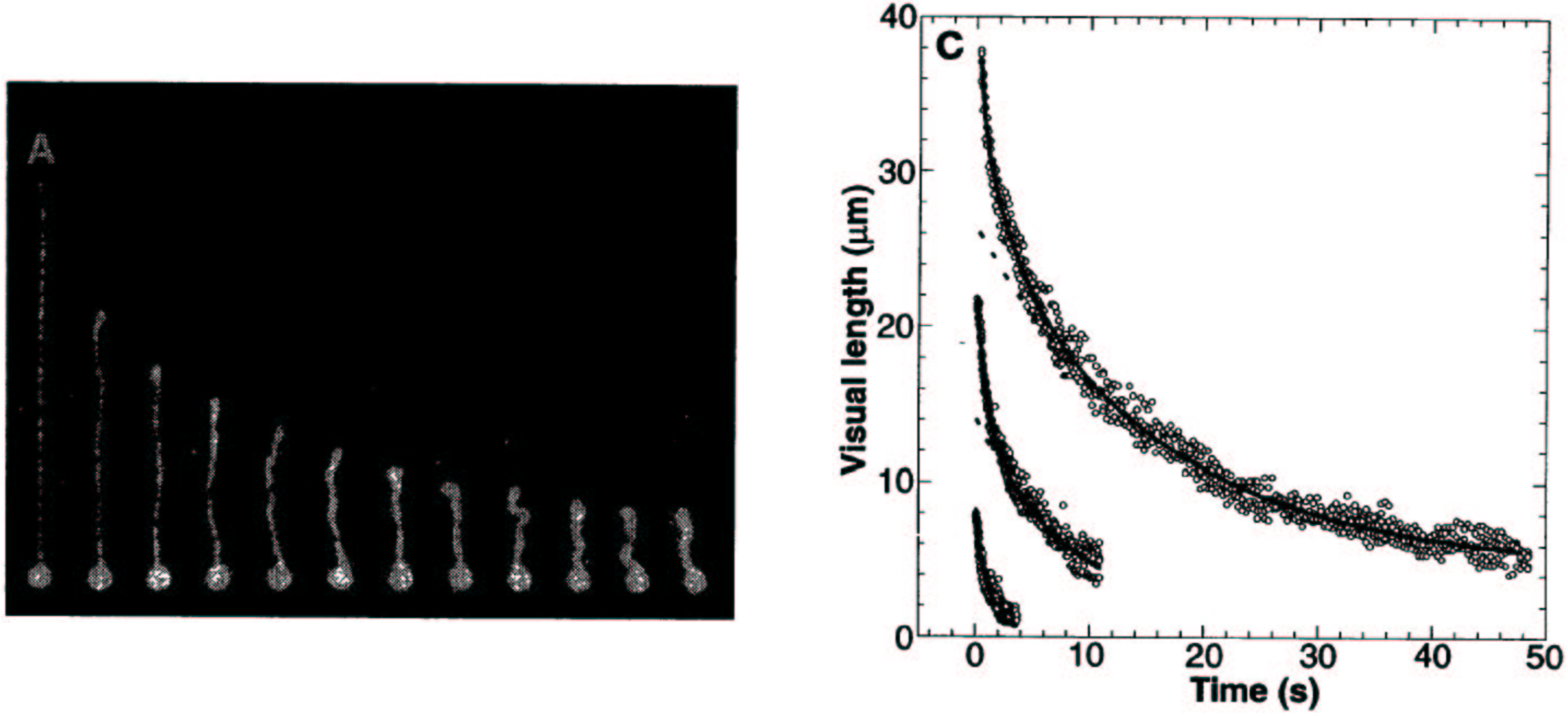
Uncoiled

ρ_2

ρ_1

Relaxation of Polymers

Pretty close to exponential…



From Perkins *et al.*, *Nature* (1994)

Constitutive Models

Model polymers in dilute solution as a continuum:
Stress on the fluid.

How does the stress tensor $\mathbb{T}$ depend on the state of the fluid?

- Elastic:

$$T_{ij} = \eta\,\gamma_{ij}\,, \qquad \gamma \equiv \text{strain (deformation) tensor.}$$

- Viscous or Newtonian:

$$T_{ij} = \mu\,\dot{\gamma}_{ij}\,, \qquad \dot{\gamma} \equiv \boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T$$

- Viscoelastic:

$$T_{ij} = \int_{-\infty}^{t} G(t-t')\,\dot{\gamma}_{ij}(t')\,\mathrm{d}t'$$

Maxwell Model

Exponentially-decaying memory:

$$T_{ij} = (\mu/\tau)\int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau}\,\dot{\gamma}_{ij}(t')\,\mathrm{d}t'$$

Reformulate as differential equation for $\mathbb{T}$:

$$\tau\,\dot{T}_{ij} = \mu\,\dot{\gamma}_{ij} - T_{ij}$$

Problem: not frame-indifferent!

Not good as a fluid relation. Remedied by introducing a frame-independent (Oldroyd) derivative

$$\dot{\mathbb{T}} \quad \Longrightarrow \quad \mathcal{D}\mathbb{T} \equiv \frac{\partial \mathbb{T}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\mathbb{T} - \left(\mathbb{T}\cdot\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{T}\cdot\mathbb{T}\right)$$

The Equations of Motion

Couple stress to Navier–Stokes for an incompressible fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p = \nu \nabla^2 \boldsymbol{u} + \frac{s}{\tau} \boldsymbol{\nabla} \cdot \mathbb{A} \,;$$

$$\mathcal{D}\mathbb{A} = -\frac{2}{\tau} \left(\mathbb{A} - \rho_0^2 \, \mathbb{I} \right) \;; \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

where $\mathbb{A}$ is equal to $\mathbb{T}$ up to constants, and can be regarded as the local deformation of the polymers, with $\mathbb{A} = \rho_0^2 \, \mathbb{I}$ at rest.

Can be derived from a kinetic model of “Hookean dumbbells.”

More generally: allow nonlinear saturation of the length of polymers (FENE-type models)

$$\mathcal{D}\mathbb{A} = -\frac{1}{\tau} \left(f(A) \, \mathbb{A} - \rho_0^2 \, \mathbb{I} \right) .$$

Evolution of the Principal Axes

The polymer conformation tensor $\mathbb{A}$ can be diagonalized, with orthonormal eigenvectors $\boldsymbol{e}_\alpha$ and eigenvalues $(\rho_\alpha)^2$ that evolve according to

$$\frac{d\rho_\alpha}{dt} = \lambda_\alpha\,\rho_\alpha - \frac{1}{\tau}\left(f(\|\rho\|^2)\rho_\alpha - \rho_0^2/\rho_\alpha\right),$$

$$\lambda_\alpha(t,\boldsymbol{x}) \equiv \boldsymbol{e}_\alpha \cdot \boldsymbol{\nabla u} \cdot \boldsymbol{e}_\alpha\,, \qquad d/dt = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\,.$$

The ρ_α are the lengths of the principal axes of the ellipsoid delineating the deformation of the polymer.

If the flow is smooth, the polymers tend to align with the dominant stretching direction, so we consider only the major axis:

$$\boxed{\frac{d\rho}{dt} = \lambda\,\rho - \frac{1}{\tau}\left(f(\rho^2)\rho - \rho_0^2/\rho\right)}$$

Turbulence! (well…sort of)

For turbulence, a locally smooth regime is achieved when the viscous scale is much longer that the polymer length.

In that case, model the velocity field as a Gaussian random variable representing a smooth straining field $\lambda(t)$ that changes rapidly; λ satisfies

$$\langle\lambda(t)\lambda(t')\rangle - \bar{\lambda}^2 = \delta(t-t')\,\Delta\,; \qquad \langle\lambda(t)\rangle = \bar{\lambda}\,,$$

where the angle brackets denote an average over λ.

The variable $\lambda(t)$ is δ-correlated in time, which means that it forgets about its previous state immediately. It has mean $\bar{\lambda}$ and standard deviation Δ.

This “slightly” artificial situation has great analytical advantages.

Fokker–Planck Equation

Because the distribution of $\lambda(t)$ is Gaussian and δ-correlated, can obtain a Fokker–Planck equation for the distribution $\mathcal{P}(t,\rho)$ of the major axis:

$$\mathcal{Z}(t;\mu) = \langle \exp\left(\mathrm{i}\,\mu\,\rho\right)\rangle$$

We can then derive an equation of motion for $\mathcal{Z}$ and average. Gaussian integration by parts allows evaluation of terms of the form $\langle\lambda\,\mathcal{Z}\rangle$.

Inverse Fourier transformation of $\mathcal{Z}$ with respect to μ then gives the equation of motion for $\mathcal{P}(t,\rho)$, the PDF of ρ, (Chertkov, 2000)

$$\boxed{\partial_t\mathcal{P} = \tfrac{1}{2}\Delta\,\partial_\rho\rho\,\partial_\rho\rho\,\mathcal{P} - \bar{\lambda}\,\partial_\rho\,\rho\,\mathcal{P} + \frac{1}{\tau}\,\partial_\rho\left(f(\rho^2)\rho - \rho_0^2/\rho\right)\mathcal{P}}$$

Stationary Distribution: Coiled State

Natural thing to do is to look for stationary solutions of the F–P equation. Assume first that the polymers are uncoiled. Neglect nonlinear relaxation: $f = 1$ (Hookean springs).

$$\boxed{\mathcal{P}_{\rm c}(\rho) \sim \rho^{-1-2(\xi-\zeta)}\ \exp(-\xi\,\rho_0^2/\rho^2)}$$

where ρ is normalized by ρ_0, and

$$\xi \equiv 1/\Delta\tau\,, \qquad \zeta \equiv \bar{\lambda}/\Delta$$

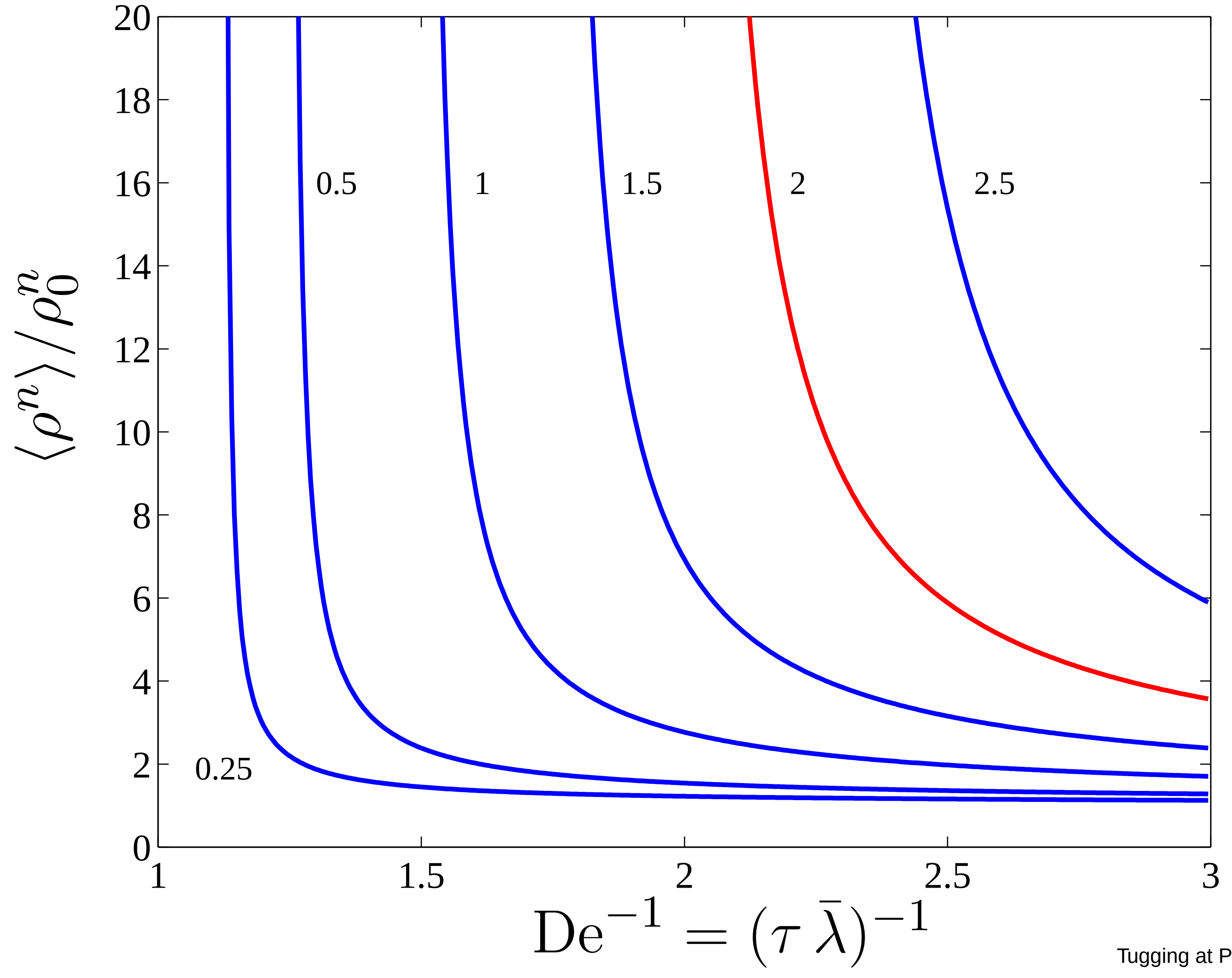
are dimensionless numbers.

$\zeta/\xi = \mathrm{De}$, the Deborah number, the ratio of the polymer relaxation timescale, τ, over the advection timescale, $1/\bar{\lambda}$.

Large $\mathrm{De} \Longrightarrow$ polymers more affected.

Moments of the Distribution

Below the coil-stretch transition at $\mathrm{De} = 1$ (with $\bar{\lambda}/\Delta = 1$).



Stationary Distribution: Stretched State

Assuming a FENE-type (Finite Extension Nonlinear Elastic) model which limits the length of the polymers to ρ_{m},

$$f(\rho^2) = \frac{\rho_{\mathrm{m}}^2 - \rho_0^2}{\rho_{\mathrm{m}}^2 - \rho^2}\,.$$

Can find equilibrium distribution

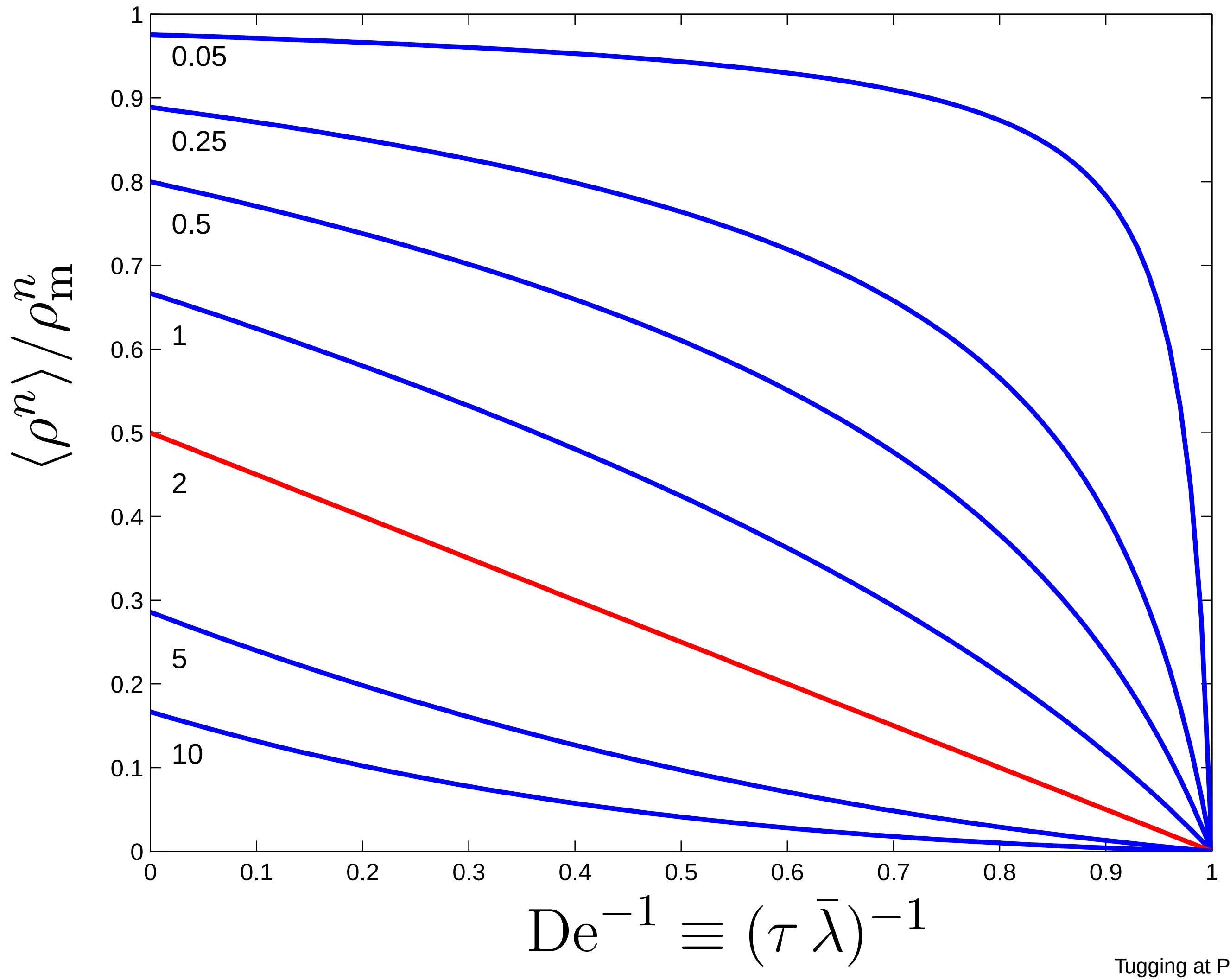
$$\boxed{\mathcal{P}_{\mathrm{s}}(\rho) \sim \rho^{-1+2(\bar{\lambda}/\Delta-\xi)}(1-\rho^2)^{\xi}}$$

where ρ is normalized by ρ_{m}, and $\xi \equiv 1/\tau\Delta$.

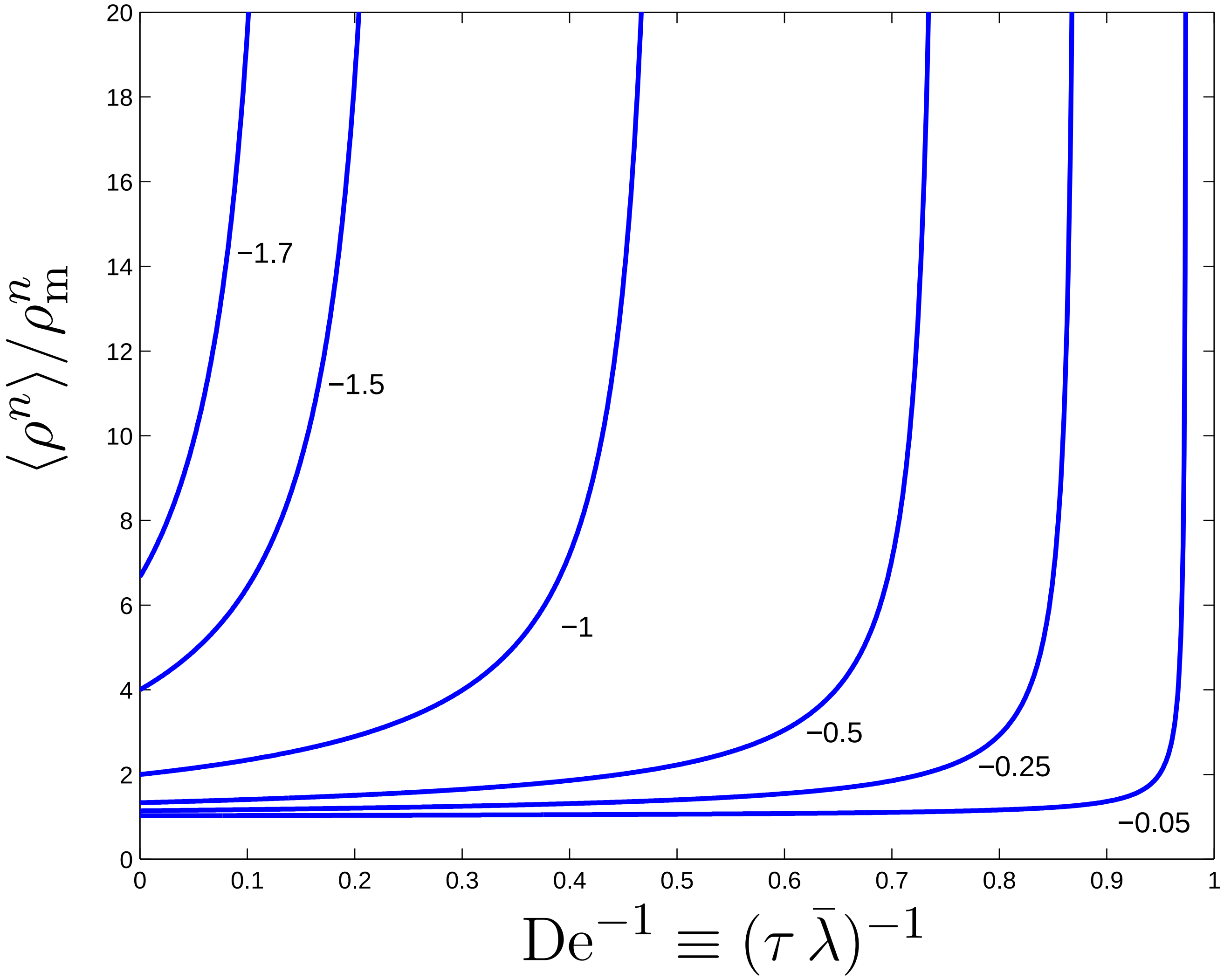
$\Longrightarrow$ Cutoff in PDF for $\rho > \rho_{\mathrm{m}}$.

Moments of the Distribution

Coil-stretch transition at $\mathrm{De} = 1$ (with $\bar{\lambda}/\Delta = 1$).



Negative moments



Breaking Point

The PDF of length can also be used to estimate the fraction of polymers that break.

Assume a polymer breaks if the tension $F(\rho^2) = f(\rho^2)\rho/\tau$ exceeds a critical value, F_c.

To that critical tension corresponds a critical length, ρ_c, obtained by solving

$$F_\mathrm{c} = \frac{1}{\tau} f(\rho_\mathrm{c}^2)\rho_\mathrm{c} = \frac{1}{\tau}\frac{\rho_\mathrm{m}^2}{\rho_\mathrm{m}^2 - \rho_\mathrm{c}^2}\rho_\mathrm{c},$$

$$\Longrightarrow \qquad \rho_\mathrm{c} = \frac{\rho_\mathrm{m}^2}{2\tau F_\mathrm{c}}\left(\sqrt{1 + (2\tau F_\mathrm{c}/\rho_\mathrm{m})^2} - 1\right).$$

The Survivors

The fraction of polymers that survive is equal to the fraction shorter than $\rho_{\rm c}$, obtained by integrating the PDF from 0 to $\rho_{\rm c}$,

$$\text{Prob}(\rho < \rho_{\rm c}) = \frac{B(\rho_{\rm c}^2/\rho_{\rm m}^2\,;\,\zeta - \xi\,,\,\xi + 1)}{B(\zeta - \xi\,,\,\xi + 1)},$$

where

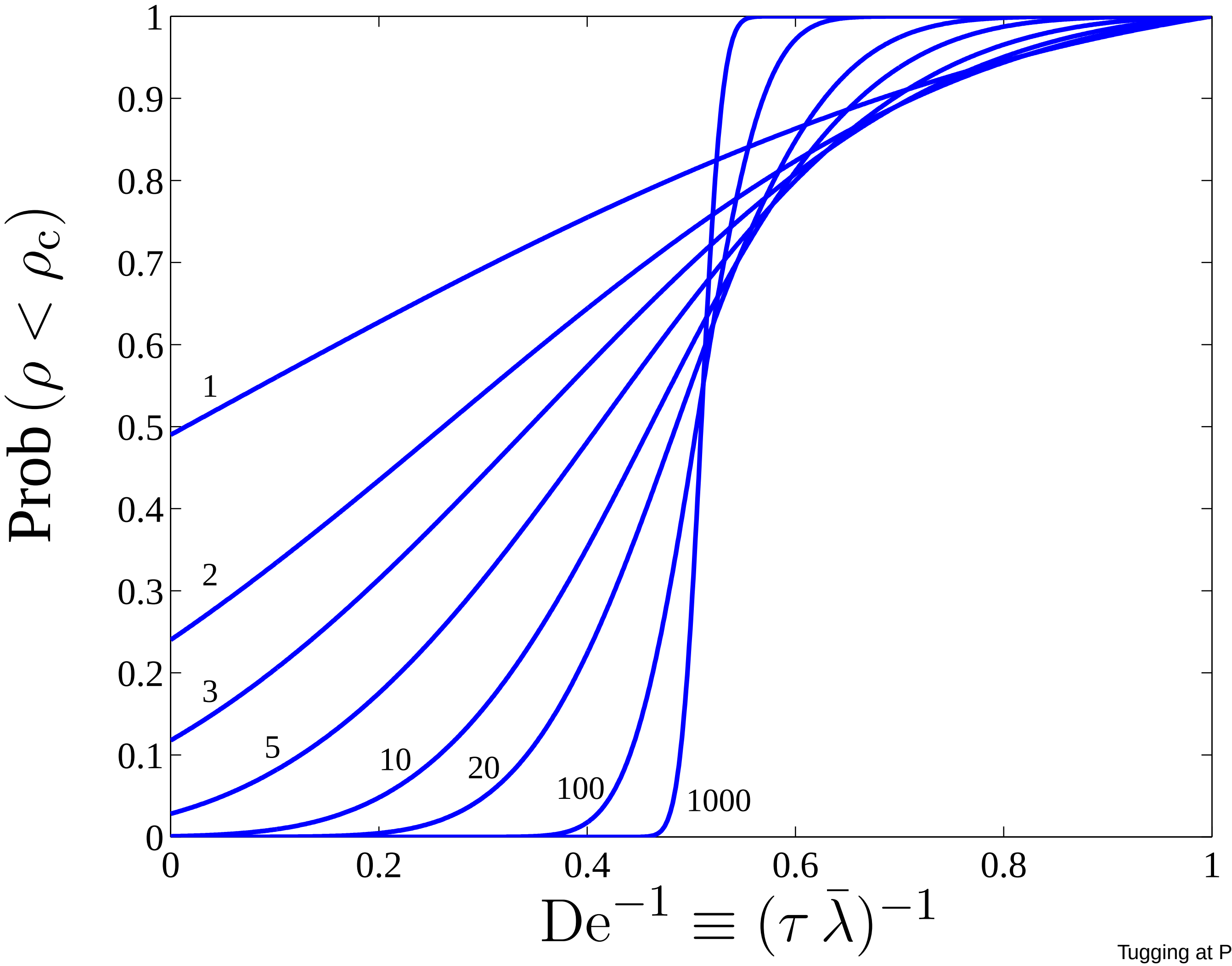
$$B(a, b) \equiv \Gamma(a)\Gamma(b)/\Gamma(a + b)$$

is the beta function, and $B(z; a, b)$ is the incomplete beta function.

Since $B(1; a, b) = B(a, b)$, for $\rho_{\rm c} = \rho_{\rm m}$ none of the polymers break.

Surviving Fraction as a function of De^{-1}

For different values of $\zeta \equiv \bar{\lambda}/\Delta$ with $\rho_c/\rho_m = 0.7$.



The Survivors: Strong Turbulence

For large Deborah number,

$$\mathrm{Prob}(\rho < \rho_{\mathrm{c}}) \simeq (\rho_{\mathrm{c}}/\rho_{\mathrm{m}})^{2\,\zeta},$$

which is not small if ζ is not too large.

$\Longrightarrow$ Some polymers survive breakage for large $\bar{\lambda}\,\tau$ if the fluctuations in $\bar{\lambda}$ (given by Δ) are also large.

Of course, this treatment is for one correlation time of the turbulence. In reality a given polymer is exposed to many different random strains.

Survival prob. decays roughly as $(\rho_{\mathrm{c}}/\rho_{\mathrm{m}})^{2\zeta N}$, where $N = t/t_{\mathrm{corr}}$ with t_{corr} the correlation time.

Ongoing Research

- Model polymer as flexible chains instead of rods.
- Self-consistency: backreaction of the polymers on the fluid.
- Incorporate Einstein model of effective viscosity.
- Non-Gaussian statistics: Path integral formalism.
- Compressibility.
- Magnetic dynamo.
- Statistics of curvature.