Symmetries and Invariants in Euler Systems

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Motivation

Lie–Poisson brackets are a type of noncanonical bracket and are ubiquitous in the reduction of canonical Hamiltonian systems with symmetry. Finite-dimensional examples include the heavy top and the moment reduction of the Kida vortex, while for infinite dimension we have the 2-D ideal fluid, reduced MHD, and the 1-D Vlasov equation. Our goal is to examine first a specific origin of Lie–Poisson brackets as coming from reductions, and to interpret the invariants obtained. This is to motivate the introduction of such brackets.

Having done that we turn to building Lie–Poisson brackets directly from Lie groups. We will show that physically relevant systems are obtained this way.

The invariants of the bracket determine the manifold on which the system evolves. It is thus important to understand what sort of constraints they impose on a system.

Overview

- Review of *reduction* of a Lagrangian system to an Eulerian system using relabeling symmetry.

- Two prototypical examples: the *rigid body* (finite dimensional) and the *2–D ideal fluid* (infinite dimensional).

- We will introduce the *semidirect product* defined by the *action* of a Lie group on a manifold. We illustrate this by two physical examples, the *heavy top* and *low-β reduced MHD*.

- Finally we look at a nonsemidirect example and discuss work in progress.

Reduction for the Free Rigid Body

Hamiltonian for the free rigid body in terms of Euler angles:

$$H(p_\phi, p_\psi, p_\theta, \phi, \psi, \theta) =$$
$$\frac{1}{2}\left\{\frac{[(p_\phi - p_\psi \cos\theta)\sin\psi + p_\theta \sin\theta\cos\psi]^2}{I_1\,\sin^2\theta}\right.$$
$$\left. + \frac{[(p_\theta - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta\sin\psi]^2}{I_2\,\sin^2\theta} + \frac{p_\psi^2}{I_3}\right\}$$

Equations of motion are generated using the canonical bracket:

$$\{f\,,g\} = \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial p_\phi} + \frac{\partial f}{\partial \psi}\frac{\partial g}{\partial p_\psi} + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial p_\theta} - (f \longleftrightarrow g)$$

Here we have 3 degrees of freedom (6 coordinates). The configuration space is the rotation group $SO(3)$, the phase space is $T^*SO(3)$.

However, there is a possible reduction for this system. In terms of angular momenta,

$$H(p_\phi, p_\psi, p_\theta, \phi, \psi, \theta) \longrightarrow H(\ell_1, \ell_2, \ell_3) = \sum_{i=1}^{3} \frac{\ell_i^2}{2I_i}$$

Under this *noncanonical* mapping, the bracket becomes of the *Lie–Poisson* form

$$\{f\,, g\} = \ell \cdot \frac{\partial f}{\partial \ell} \times \frac{\partial g}{\partial \ell}$$

The equations of motion generated by the bracket are permutations of

$$\dot{\ell}_1 = \frac{I_2 - I_3}{I_2\, I_3}\, \ell_2\, \ell_3$$

These are Euler's equations for the rigid body. The Hamiltonian is conserved, and so is the quantity

$$C = \sum_{i=1}^{3} \ell_i^2$$

which commutes with any f. We call C a *Casimir invariant*.

Reduction for the 2–D Ideal Fluid

Hamiltonian functional:

$$H[q;\pi] = \int_D \left(\frac{\pi^2}{2\rho_0} + \rho_0\, U \right)\, d^2a$$

This together with the canonical bracket

$$\{F\,, G\} = \int_D \left[\frac{\delta F}{\delta q}\,\frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta q}\,\frac{\delta F}{\delta \pi} \right]\, d^2a$$

generates the equations of motion for a Lagrangian fluid. The information about the position of every fluid element at any time is contained in the model. There is a *relabeling* symmetry of the initial condition labels, a.

We introduce the streamfunction ϕ

$$v(\mathbf{x},t) = (-\partial_y \phi, \partial_x \phi)$$

so that $\nabla \cdot v = 0$ is automatically satisfied, and the vorticity

$$\omega(\mathbf{x},t) = \hat{z} \cdot \nabla \times v \,.$$

The noncanonical transformation from Lagrangian to Eulerian variables is

$$v(\mathbf{x},t) = \int_D \frac{\pi(a,t)}{\rho_0} \, \delta(\mathbf{x} - q(a,t)) \, d^2a \,.$$

Then, after some manipulation involving integration by parts we get the bracket

$$\{F\,,G\} = \int_D \omega \, \left[\frac{\delta F}{\delta \omega} \,, \frac{\delta G}{\delta \omega} \right] \, d^2x$$

where

$$[f\,,g] := \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \,.$$

The equation of motion generated by the bracket and the transformed Hamiltonian

$$H[\omega] = -\frac{1}{2}\int_D \phi\,\omega\; d^2x = \frac{1}{2}\int_D |\nabla\phi|^2\; d^2x$$

is just Euler's equation for an the ideal fluid

$$\dot{\omega}(\mathbf{x}) = -[\phi\,,\omega]\,.$$

This has a Casimir given by

$$C[\omega] = \int_D f\left(\omega(\mathbf{x})\right)\; d^2x, \quad f \text{ arbitrary.}$$

Semidirect Product

Given two elements A, A' of a Lie group G and two elements $\mathbf{x}, \mathbf{x}'$ of a vector space V we can make a new Lie group called the *semidirect product* of G and the (Abelian) group V with an operation defined by

$$(A, \mathbf{x}) \cdot (A', \mathbf{x}') := (AA', \mathbf{x} + A\,\mathbf{x}')$$

The term $A\,\mathbf{x}'$ implies that G *acts* on V in some way. The simplest example is when G is the rotation group $SO(3)$ and V is R^3. Then the matrix representation of G just acts by matrix multiplication. In that case we have the 6-parameter *Galilean group* of rotations and translations.

We can form a Lie algebra g and thus a Lie–Poisson bracket from the semidirect product of G and V.

The Heavy Top

The Lie–Poisson bracket for the semidirect product of the rotation group and R^3 is

$$\{f\,,g\} = \ell\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \ell}\right)+\alpha\cdot\left(\frac{\partial f}{\partial \ell}\times\frac{\partial g}{\partial \alpha}+\frac{\partial f}{\partial \alpha}\times\frac{\partial g}{\partial \ell}\right)$$

where α denote a 3-vector. The Casimirs for this bracket are

$$C_1 = \alpha^2\,,\quad C_2 = \ell\cdot\alpha\ .$$

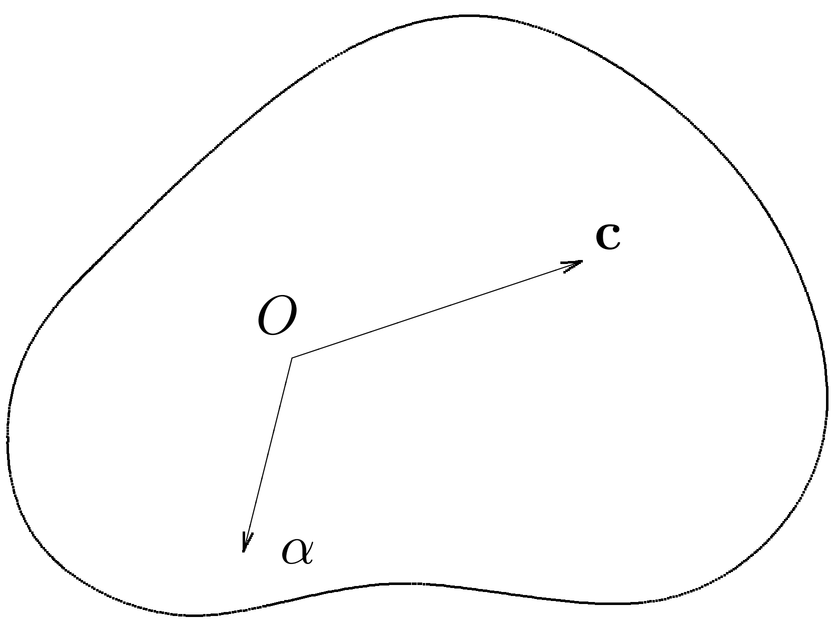
For a Hamiltonian quadratic in ℓ the vector α rotates rigidly with the body. The Casimir C_2 tells us there is still an ambiguity to the orientation of the body: we can rotate it about α.

By using

$$H(\ell, \alpha) = \sum_{i=1}^{3} \frac{\ell_i^2}{2I_i} + \alpha \cdot \mathbf{c}$$

we get the prototypical example of a semidirect product system, the heavy rigid body (in the body frame):

$$\begin{aligned} \dot{\ell}_1 &= \frac{I_2 - I_3}{I_2 I_3} \ell_2 \ell_3 - \alpha_2 c_3 + \alpha_3 c_2 \\ \dot{\alpha}_1 &= \frac{\ell_3 \alpha_2}{I_3} - \frac{\ell_2 \alpha_3}{I_2} \end{aligned}$$



Low–β Reduced MHD

The semidirect product bracket for two fields is

$$
\begin{aligned}
\{F\,,G\} &= \int_D \left(\omega\left[\frac{\delta F}{\delta\omega},\frac{\delta G}{\delta\omega}\right]\right. \\
&\quad \left. +\,\psi\left(\left[\frac{\delta F}{\delta\omega},\frac{\delta G}{\delta\psi}\right]+\left[\frac{\delta F}{\delta\psi},\frac{\delta G}{\delta\omega}\right]\right)\right)d^2x
\end{aligned}
$$

If $\omega = \nabla^2\phi$, where ϕ is the electric potential, ψ is the magnetic flux, and $J = \nabla^2\psi$ is the current, then the Hamiltonian

$$
H[\omega;\psi] = \frac{1}{2}\int_D \left(|\nabla\phi|^2 + |\nabla\psi|^2\right)\, d^2x
$$

with the above bracket gives us

$$
\begin{aligned}
\dot{\omega} &= [\psi, J] + [\omega, \phi]\ , \\
\dot{\psi} &= [\psi, \phi]\ ,
\end{aligned}
$$

a model for low-β reduced MHD derived by Morrison and Hazeltine.

The bracket has Casimir invariants

$$C_1[\psi] = \int_D f(\psi)\, d^2x, \qquad C_2[\omega;\psi] = \int_D \omega\, g(\psi)\, d^2x.$$

The first has the same form as the one for 2–D Euler and has the same interpretation. To make sense of the second one let $g(\psi) = \delta(\psi - \psi_0)$.

$$C_2 \longrightarrow \oint_D \omega(\psi_0, \theta)\, d\theta$$

The "projection" of ω on any contour remains constant. However on a given contour the fluid elements may still move around: there is still a relabeling symmetry, for each contour.

Putting Labels on a Rigid-body

Remember that taking a semidirect product restricted the symmetry group of the body to rotations about α. If we take another semidirect product to get

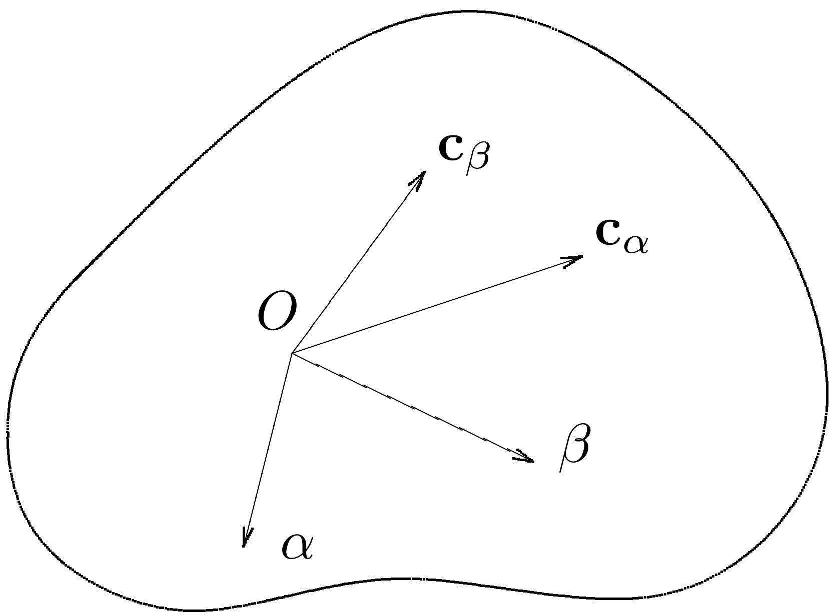
$$
\begin{aligned}
\{f\,,g\} \;\; = \;\; & \ell \cdot \left(\frac{\partial f}{\partial \ell} \times \frac{\partial g}{\partial \ell} \right) \\
& + \alpha \cdot \left(\frac{\partial f}{\partial \ell} \times \frac{\partial g}{\partial \alpha} + \frac{\partial f}{\partial \alpha} \times \frac{\partial g}{\partial \ell} \right) \\
& + \beta \cdot \left(\frac{\partial f}{\partial \ell} \times \frac{\partial g}{\partial \beta} + \frac{\partial f}{\partial \beta} \times \frac{\partial g}{\partial \ell} \right)
\end{aligned}
$$

where β is a 3-vector, the new bracket has Casimirs

$$
C_1 = \alpha^2\,, \quad C_2 = \beta^2, \quad C_3 = \alpha \cdot \beta\,.
$$

The angular momentum ℓ has disappeared from the Casimirs.

This can model a rigid body with two forces acting on it.



Note that knowing α and β completely specifies the orientation of the rigid body. In other words, by taking semidirect products we have reintroduced the Lagrangian information into the bracket.

Passive Scalars in an Ideal Fluid

For the ideal fluid, say low-β MHD with a second advected quantity, χ, the Casimir is

$$C[\psi;\chi] = \int_D f(\psi,\chi)\, d^2x, \quad f \text{ arbitrary}.$$

This Casimir amounts to being able to label two contours. Locally this permits a unique labeling of the fluid elements as long as χ and ψ are not constant in some region. However, globally there is still some ambiguity. Thus, in the infinite-dimensional case the semidirect product is not equivalent to recovering the full Lagrangian information, unless the contours do not close and are monotonic.

Beyond Semidirect: Cocycles

There are other ways to extend Lie algebras than the semidirect product. We have investigated brackets of the form

$$[\,\alpha\,,\beta\,]_\lambda = W_\lambda{}^{\mu\nu}\;[\,\alpha_\mu\,,\beta_\nu\,]$$

where λ is a component of an n-vector.

One example is the bracket derived by Morrison for 2–D compressible reduced MHD, which has four fields. The Hamiltonian is

$$H[\omega;v;p;\phi] = \frac{1}{2}\left\langle |\nabla\phi|^2 + v^2 + \frac{(p-2\beta\,x)^2}{\beta} + |\nabla\psi|^2 \right\rangle.$$

The bracket is rather large,

$$
\begin{aligned}
\{A\,,B\} \;\; = \;\; & \left\langle \omega\,,\; \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta\omega}\right]\right\rangle \\
& + \left\langle v\,,\; \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta v}\right] + \left[\frac{\delta A}{\delta v},\frac{\delta B}{\delta\omega}\right]\right\rangle \\
& + \left\langle p\,,\; \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta p}\right] + \left[\frac{\delta A}{\delta p},\frac{\delta B}{\delta\omega}\right]\right\rangle \\
& + \Bigg\langle \psi\,,\; \left[\frac{\delta A}{\delta\omega},\frac{\delta B}{\delta\psi}\right] + \left[\frac{\delta A}{\delta\psi},\frac{\delta B}{\delta\omega}\right] \\
& \qquad - \beta\left[\frac{\delta A}{\delta p},\frac{\delta B}{\delta v}\right] - \beta\left[\frac{\delta A}{\delta v},\frac{\delta B}{\delta p}\right]\Bigg\rangle
\end{aligned}
$$

The term proportional to β is an obstruction to the semidirect product structure, and it cannot be removed by a coordinate transformation. In the language of Lie algebra cohomology it is a *cocycle*.

Its Casimirs are

$$
\begin{aligned}
C_1 &= \int_D f(\psi)\, d^2x \\
C_2 &= \int_D p\, g(\psi)\, d^2x \\
C_3 &= \int_D v\, h(\psi)\, d^2x \\
C_4 &= \int_D \left(\omega k(\psi) + \frac{v\, p}{\beta}\, k'(\psi)\right)\, d^2x
\end{aligned}
$$

These do not allow a labeling of the fluid elements. We are still investigating the consequences of extensions of this kind.

Conclusions

- We gave an introduction to the reduction of physical systems based on their symmetries.

- The prototypical examples were shown, the rigid body and the 2–D ideal fluid.

- The semidirect product allows us to describe the group acting on larger systems. This led to the recovery of some or all of the Lagrangian information.

- For general extensions things are very different: we do not recover the Lagrangian information and the Casimir represent constraints on the system which do not have obvious physical significance. This is the focus of much of our current work.